

Encyclopedia of
Complexity and Systems Science Series
Editor-in-Chief: Robert A. Meyers

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Mohamed Atef Helal
Editor

Solitons

A Volume in the Encyclopedia of
Complexity and Systems Science,
Second Edition

Encyclopedia of Complexity and Systems Science Series

Editor-in-Chief

Robert A. Meyers

The Encyclopedia of Complexity and Systems Science series of topical volumes provides an authoritative source for understanding and applying the concepts of complexity theory, together with the tools and measures for analyzing complex systems in all fields of science and engineering. Many phenomena at all scales in science and engineering have the characteristics of complex systems and can be fully understood only through the transdisciplinary perspectives, theories, and tools of self-organization, synergetics, dynamical systems, turbulence, catastrophes, instabilities, nonlinearity, stochastic processes, chaos, neural networks, cellular automata, adaptive systems, genetic algorithms, and so on. Examples of near-term problems and major unknowns that can be approached through complexity and systems science include: The structure, history, and future of the universe; the biological basis of consciousness; the integration of genomics, proteomics, and bioinformatics as systems biology; human longevity limits; the limits of computing; sustainability of human societies and life on earth; predictability, dynamics, and extent of earthquakes, hurricanes, tsunamis, and other natural disasters; the dynamics of turbulent flows; lasers or fluids in physics; microprocessor design; macromolecular assembly in chemistry and biophysics; brain functions in cognitive neuroscience; climate change; ecosystem management; traffic management; and business cycles. All these seemingly diverse kinds of phenomena and structure formation have a number of important features and underlying structures in common. These deep structural similarities can be exploited to transfer analytical methods and understanding from one field to another. This unique work will extend the influence of complexity and system science to a much wider audience than has been possible to date.

Mohamed Atef Helal
Editor

Solitons

A Volume in the Encyclopedia of Complexity
and Systems Science, Second Edition

With 151 Figures and 12 Tables

 Springer

Editor

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To my family

Series Preface

The Encyclopedia of Complexity and System Science Series is a multivolume authoritative source for understanding and applying the basic tenets of complexity and systems theory as well as the tools and measures for analyzing complex systems in science, engineering, and many areas of social, financial, and business interactions. It is written for an audience of advanced university undergraduate and graduate students, professors, and professionals in a wide range of fields who must manage complexity on scales ranging from the atomic and molecular to the societal and global.

Complex systems are systems that comprise many interacting parts with the ability to generate a new quality of collective behavior through self-organization, e.g., the spontaneous formation of temporal, spatial, or functional structures. They are therefore adaptive as they evolve and may contain self-driving feedback loops. Thus, complex systems are much more than a sum of their parts. Complex systems are often characterized as having extreme sensitivity to initial conditions as well as emergent behavior that are not readily predictable or even completely deterministic. The conclusion is that a reductionist (bottom-up) approach is often an incomplete description of a phenomenon. This recognition that the collective behavior of the whole system cannot be simply inferred from the understanding of the behavior of the individual components has led to many new concepts and sophisticated mathematical and modeling tools for application to many scientific, engineering, and societal issues that can be adequately described only in terms of complexity and complex systems.

Examples of Grand Scientific Challenges which can be approached through complexity and systems science include: the structure, history, and future of the universe; the biological basis of consciousness; the true complexity of the genetic makeup and molecular functioning of humans (genetics and epigenetics) and other life forms; human longevity limits; unification of the laws of physics; the dynamics and extent of climate change and the effects of climate change; extending the boundaries of and understanding the theoretical limits of computing; sustainability of life on the earth; workings of the interior of the earth; predictability, dynamics, and extent of earthquakes, tsunamis, and other natural disasters; dynamics of turbulent flows and the motion of granular materials; the structure of atoms as expressed in the Standard Model and the formulation of the Standard Model and gravity into a Unified Theory; the structure of water; control of global infectious diseases; and also evolution and quantification of (ultimately) human cooperative behavior in politics,

economics, business systems, and social interactions. In fact, most of these issues have identified nonlinearities and are beginning to be addressed with nonlinear techniques, e.g., human longevity limits, the Standard Model, climate change, earthquake prediction, workings of the earth's interior, natural disaster prediction, etc.

The individual complex systems mathematical and modeling tools and scientific and engineering applications that comprised the *Encyclopedia of Complexity and Systems Science* are being completely updated and the majority will be published as individual books edited by experts in each field who are eminent university faculty members.

The topics are as follows:

Agent-Based Modeling and Simulation
 Applications of Physics and Mathematics to Social Science
 Cellular Automata, Mathematical Basis of
 Chaos and Complexity in Astrophysics
 Climate Modeling, Global Warming, and Weather Prediction
 Complex Networks and Graph Theory
 Complexity and Nonlinearity in Autonomous Robotics
 Complexity in Computational Chemistry
 Complexity in Earthquakes, Tsunamis, and Volcanoes, and Forecasting and
 Early Warning of Their Hazards
 Computational and Theoretical Nanoscience
 Control and Dynamical Systems
 Data Mining and Knowledge Discovery
 Ecological Complexity
 Ergodic Theory
 Finance and Econometrics
 Fractals and Multifractals
 Game Theory
 Granular Computing
 Intelligent Systems
 Nonlinear Ordinary Differential Equations and Dynamical Systems
 Nonlinear Partial Differential Equations
 Percolation
 Perturbation Theory
 Probability and Statistics in Complex Systems
 Quantum Information Science
 Social Network Analysis
 Soft Computing
 Solitons
 Statistical and Nonlinear Physics
 Synergetics
 System Dynamics
 Systems Biology

Each entry in each of the Series books was selected and peer reviews organized by one of our university-based book Editors with advice and consultation provided by our eminent Board Members and the Editor-in-Chief.

This level of coordination assures that the reader can have a level of confidence in the relevance and accuracy of the information far exceeding than that generally found on the World Wide Web. Accessibility is also a priority, and for this reason each entry includes a glossary of important terms and a concise definition of the subject. In addition, we are pleased that the mathematical portions of our Encyclopedia have been selected by Math Reviews for indexing in MathSciNet. Also, ACM, the world's largest educational and scientific computing society, recognized our *Computational Complexity: Theory, Techniques, and Applications* book, which contains content taken exclusively from the *Encyclopedia of Complexity and Systems Science*, with an award as one of the notable Computer Science publications. Clearly, we have achieved prominence at a level beyond our expectations, but consistent with the high quality of the content!

Palm Desert, CA, USA
November 2022

Robert A. Meyers
Editor-in-Chief

Volume Preface

The Solitons is a subject that comes from physical oceanography since the nineteenth century. This field is considered a very interesting subject for both physicists and mathematicians. In both domains, Solitons play an important role in addressing a special phenomenon of water waves that was discovered by Scott Russell in August 1834. It started with following the hump wave in the river and how this wave kept shape until it dissipated. Since then, the fascination for Solitons was born, and scientists began to investigate this phenomenon. Solitons come from solitary waves. Mathematicians tried to formulate the governing equation to explain this phenomenon (solitary waves), while physicians tried to give their vision of its behavior and possible reasons. Also, solid mechanics and optics researchers have an interest in the subject. They tried to give the mathematical formulation of Solitons to solve some physical problems in their fields.

However, the mathematical formulation of Solitons is complicated due to its nonlinearity behavior. One of the well-known formulations of this nonlinear problem is the Korteweg de Vries (KdV) equation. A lot of research was done using this equation since the early twentieth century. After the disastrous effects of the Tsunami in Sumatra in 2004, Solitons have gained more interest to describe the Tsunami phenomenon. Many other well-known equations are also used to try to describe Solitons problems, such as modified KdV. There are many types of solutions such as semi-analytical solutions (e.g., Adomian Decomposition method, homotopy perturbation, etc.) and numerical solutions to solve KdV equations. Till now, this kind of problem does not have a definitive solution. Due to the open nature of this problem, there is still open research in different research areas. Also, the behavior of solitons is still fascinating for further research. There are various types of equations and their extensions that researchers try to adapt to identify and explain the Solitons' behavior and nature.

This book is composed of some of the research work in the field of Solitons. It starts with base concepts and the history of Solitons. Then, most research is concerned with solving hydrodynamical problems. Some research papers have an interest in shallow water problems, with Tsunamis being one of their problems. Quantum field theory, optics, and nano-hydrodynamics are also some of the research domains that benefit from solving Solitons problems. Another subfield of Solitons called the compacton, which is a special type of Soliton, has its own research interest. Finally, Solitons have helped in solving wave propagation problems in solids.

To produce this book, we gathered various research papers from distinguishable researchers from different backgrounds and research institutes. They put their best work to describe the most recent solutions for a variety of Solitons problems. I would like to express my sincere gratitude to all the participants who helped in generating this book in its best form.

Giza, Egypt
November 2022

Mohamed Atef Helal
Volume Editor

Acknowledgment

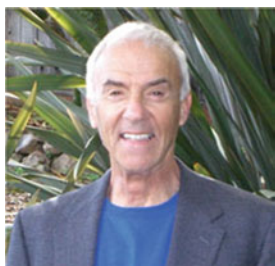
I would like to thank my family (my wife and my children) for their continuous support during my scientific life.

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About the Editor-in-Chief



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Dr. Meyers was manager of Energy and Environmental Projects at TRW (now Northrop Grumman) in Redondo Beach, CA, and is now president of RAMTECH Limited. He is coinventor of the Gravimelt process for desulfurization and demineralization of coal for air pollution and water pollution control and was manager of the Department of Energy project, leading to the construction and successful operation of a first-of-a-kind Gravimelt Process Integrated Test Plant. Dr. Meyers is the inventor of and was project manager for the DOE-sponsored Magnetohydrodynamics Seed Regeneration Project which has resulted in the construction and successful operation of a pilot plant for production of potassium formate, a chemical utilized for plasma electricity generation and air pollution control. He also managed TRW efforts in magnetohydrodynamics electricity generating combustor and plasma channel development.

Dr. Meyers managed the pilot-scale DoE project for determining the hydrodynamics of synthetic fuels. He is a coinventor of several thermooxidative stable polymers which have achieved commercial success as the GE PEI, Upjohn Polyimides, and Rhone-Poulenc bismaleimide resins. He has also managed projects for photochemistry, chemical

lasers, flue gas scrubbing, oil shale analysis and refining, petroleum analysis and refining, global change measurement from space satellites, analysis and mitigation (carbon dioxide and ozone), hydrometallurgical refining, soil and hazardous waste remediation, novel polymers synthesis, modeling of the economics of space transportation systems, space rigidizable structures, and chemiluminescence-based devices.

He is a senior member of the American Institute of Chemical Engineers, member of the American Physical Society, and member of the American Chemical Society and has served on the UCLA Chemistry Department Advisory Board. He was a member of the joint USA-Russia working group on air pollution control and the EPA-sponsored Waste Reduction Institute for Scientists and Engineers.

Dr. Meyers has more than 20 patents and 50 technical papers in the fields of photochemistry, pollution control, inorganic reactions, organic reactions, luminescence phenomena, and polymers. He has published in primary literature journals including *Science* and the *Journal of the American Chemical Society* and is listed in *Who's Who in America* and *Who's Who in the World*.

Dr. Meyers' scientific achievements have been reviewed in feature articles in the popular press in publications such as *The New York Times Science Supplement* and *The Wall Street Journal* as well as more specialized publications such as *Chemical Engineering* and *Coal Age*. A public service film was produced by the Environmental Protection Agency on Dr. Meyers' chemical desulfurization invention for air pollution control.

Dr. Meyers is the author or editor-in-chief of a wide range of technical books including the *Handbook of Chemical Production Processes*; the *Handbook of Synfuels Technology*; the *Handbook of Petroleum Refining Processes* (now in 4th edition); the *Handbook of Petrochemical Production Processes* (McGraw-Hill) (now in 2nd edition); the *Handbook of Energy Technology and Economics*, published by John Wiley & Sons; *Coal Structure*, published by Academic Press; and *Coal Desulfurization* as well as the *Coal Handbook* published by Marcel Dekker. He served as chairman of the advisory board for *A Guide to Nuclear Power Technology*, published by John Wiley & Sons, which won

the Association of American Publishers Award as the best book in technology and engineering. He also served as editor-in-chief of three editions of the *Elsevier Encyclopedia of Physical Science and Technology*. Most recently, Dr. Meyers serves as editor-in-chief of the *Encyclopedia of Analytical Chemistry* as well as *Reviews in Cell Biology and Molecular Medicine* and a book series of the same name both published by John Wiley & Sons. In addition, Dr. Meyers currently serves as editor-in-chief of two Springer Nature book series, *Encyclopedia of Complexity and Systems Science* and *Encyclopedia of Sustainability Science and Technology*.

About the Volume Editor



Mohamed Atef Helal received his BSc in Mathematics (with distinction 1st class honor in 1969) from the Faculty of Science, Cairo University, Egypt. He then received his MSc in Applied Mathematics from the Faculty of Science, Cairo University, in 1975. In 1976, he got his DEA from Institute de Mechanique (IMG) Grenoble, France. After that, he received his Doctorat 3ème Cycle in Fluid Mechanics from IMG, Grenoble, France, in 1979. Finally, he got his Ph.D. in Applied Mathematics from Cairo University in 1982.

Mohamed is currently a Professor of Mathematics in the Faculty of Science, Cairo University. He is an active fellow in the Institute of Physics (England), Institute of Mathematics and its Applications (England), Royal Astronomical Society (England), and London Mathematical Society (England). Mohamed is also a member of several scientific societies in the USA and Europe. Mohamed also acted as a president of the Egyptian Mathematical and Physical Society and was a board member of the Egyptian Mathematical Society.

Mohamed has various publications in different fields such as: non-linear partial differential equations and Soliton solutions, Tsunamis, computational fluid mechanics, physical oceanography, fluids in rotating circular basins, shallow water waves in stratified fluids, wavelets, induction in thin sheets, and its applications in oceans, dark energy and golden mean in cosmology and physics. Mohamed participated in writing and editing three books. He has guided many students who took up MSc and PhD in applied mathematics. He also participated in judging MSc and PhD theses from Egypt and other countries. He is an active reviewer in several well-known journals.

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Nonlinear Water Waves and Nonlinear Evolution Equations with Applications

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Article Outline

Introduction
 Variational Principles and the Euler–Lagrange Equations
 The Euler Equation of Motion and Water Wave Problems
 The Variational Principle for Nonlinear Water Waves
 Basic Equations of Nonlinear Water Waves
 Surface Waves on a Running Stream in Water of Arbitrary, But Uniform, Depth
 Nonlinear Theory of Water Waves by a Moving Pressure Distribution at Resonant Conditions
 The Nonlinear Schrödinger Equation and Evolution of Wave Packets
 Higher-Order Nonlinear Schrödinger Equations
 The Davey–Stewartson (DS) Equations in Water of Finite Depth
 The Camassa–Holm (CH) and Degasperis–Procesi (DP) Nonlinear Model Equations
 Periodic and Solitary Waves with Constant Vorticity
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True Laws of Nature cannot be linear.
 Albert Einstein

... the progress of physics will to a large extent depend on the progress of nonlinear mathematics, of methods to solve nonlinear equations ... and

therefore we can learn by comparing different non-linear problems.

Werner Heisenberg

... as Sir Cyril Hinshelwood has observed. ... fluid dynamics were divided into hydraulic engineers who observed things that could not be explained and mathematicians who explained things that could not be observed.

James Lighthill

Introduction

Water waves are the most common observable phenomena in nature. The subject of water waves is most fascinating and highly mathematical and verified of all areas in the study of wave motions in the physical world. The mathematical as well as physical problems deal with water waves and their breaking on beaches, with flood waves in rivers, with ocean waves from storms, with ship waves on water, and with free oscillations of enclosed waters such as lakes and harbors. The study of water waves and their various ramifications remain central to fluid dynamics in general and to the dynamics of oceans in particular. The mathematical theory of water waves is quite interesting in its own merit and intrinsically beautiful. It has provided the solid background and impetus for the development of the theory of nonlinear dispersive waves. Indeed, most of the fundamental ideas and results for nonlinear dispersive waves and solitons originated in the investigation of water waves.

Historically, the problems of water waves in oceans originated from the classical work of Leonhard Euler (1707–1783), A.G. Cauchy (1789–1857), S.D. Poisson (1781–1840), Joseph Boussinesq (1842–1929), George Airy (1801–1892), Lord Kelvin (1824–1907), Lord Rayleigh (1842–1919), George Stokes (1819–1903), Scott Russell (1808–1882), and many others. Indeed, Euler formulated the boundary-value problems to understand water wave phenomena and provided successful mathematical and physical description of inviscid and incompressible fluid motion in general.

Based on Newton's second law of motion, Euler first formulated his celebrated equation of motion of an inviscid fluid about 250 years ago. Euler's equation is still considered the basis of all inviscid fluid flow.

In 1761, Euler first formulated the equations of motion for an inviscid and incompressible fluid of constant density. Combined with the *equation of mass conservation (or the Continuity equation)* in the form

$$\frac{D\mathbf{u}}{Dt} \equiv \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{F}, \quad (1)$$

$$\text{div} \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (2)$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$ is the *total* or *convective derivative*, where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u, v, w)$ is the fluid velocity, $p(\mathbf{x}, t)$ is the pressure field at a point $\mathbf{x} = (x, y, z)$, and the time t , ρ is the constant density, $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$ is the external body force per unit mass, and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the gradient operator.

Equations (1) and (2) form a closed system of four scalar nonlinear partial differential equations with four unknowns: u , v , w , and p . These four equations with four unknowns with appropriate boundary and initial conditions are sufficient to determine u , v , w , and p . These equations are widely used for an understanding of observed phenomena of fluid flows and vortex motion.

Remarkably, this 250-year-old Euler Eq. (1) and the continuity Eq. (2) provided the fundamental basis of modern fluid mechanics. However, there are certain major difficulties associated with the Euler system Eqs. (1) and (2). First, there are no general results for Eqs. (1) and (2) on existence and uniqueness of solutions, regularity, and continuous dependence of solutions on the initial conditions. Second, another difficulty arises from the strong nonlinear convective term, $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in Eq. (1). There are many open questions and problems (see Debnath 2010) which include:

- (i) Behavior of solutions at the finite time.
- (ii) Numerical simulations of the three-dimensional Euler equations indicate unstable behavior of fluid flows.

- (iii) Possibilities of singularities (Stuart 1987) in the solutions in the finite time.
- (iv) Evolution of wave motion in a boundary layer or shear flow.
- (v) Development of turbulent motion (Moffatt 1985) associated with the natural three-dimensional vortex structure in the flow.
- (vi) Vortices aligned parallel to the main stream suffer from convection, stretching, and tilting producing local shear layers which become more and more intense as time $t \rightarrow \infty$.
- (vii) Solutions of the Euler equations in inviscid fluid flows under appropriate initial conditions become singular in a finite time.

Subsequently, considering internal friction between fluid particles that leads to energy dissipation, in 1822, Louis M.H. Navier (1785–1836), and in 1845, George G. Stokes (1891–1903), first formulated the celebrated equations of motion in incompressible viscous fluid, universally known as the *Navier–Stokes equations*,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}, \quad (3)$$

together with the continuity Eq. (2), where $\nu > 0$ represents the kinematic viscosity which is constant at constant temperature. Obviously, the Navier–Stokes Eqs. (2) and (3) is a generalization of the Euler Eqs. (1) and (2) including the effects of viscosity represented by the last term, $\nu \nabla^2 \mathbf{u}$ in Eq. (3). In the limit of zero viscosity ($\nu \rightarrow 0$), the second-order Navier–Stokes equations reduce to the first order Euler Eq. (1). Thus, the Navier–Stokes equations lead to a singular perturbation problem. However, solution of the Euler equations behave very differently from the solutions of the Navier–Stokes equations, even when ν is very small.

Remarkably, this 150-year-old Navier–Stokes equations provided the fundamental basis of modern viscous fluid mechanics. However, there are certain major difficulties with the Navier–Stokes equations (see Debnath 2010). First, there are no general results of solutions, uniqueness, regularity, and continuous dependence of solutions on the

initial conditions. This is a major unsolved problem in the twenty-first century. Second, in the limit of zero viscosity ($\nu \rightarrow 0$), there exists a thin boundary layer near the rigid boundary where the significant changes from a viscosity dominated behavior to an inviscid behavior take place. Third, $\lim_{t \rightarrow \infty} \lim_{\nu \rightarrow 0} u(x, t) \neq \lim_{\nu \rightarrow 0} \lim_{t \rightarrow \infty} u(x, t)$. Fourth, in view of strong nonlinear convective term, $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in Eq. (3), solutions of Eq. (3) can be *singular* at certain places and at certain time in the viscous flow. Fifth, there is *no* general analytical method of the solution for an arbitrary viscosity ν . Sixth, the existence of weak solutions of the Navier–Stokes equations that dissipates energy in the whole space has not yet been established. However, Leray (1906–1998) proved the existence of global weak solutions of the three-dimensional Navier–Stokes equations in 1934. Subsequently, Caffarelli et al. (1982) have found singularities of the Navier–Stokes equations.

Many physical systems of nonlinear wave propagation include effects of dispersion, diffusion, nonlinearity, and their interactions. One of the pronounced effects of nonlinearity is to produce unstable and/or singular solutions. Peaking and Breaking of water waves are the most common and observable examples. The major question is: why water waves break on ocean surfaces and ocean beaches? Answer to this question led to study of analytical, experimental, and computational studies of instability and stability of nonlinear water waves, topological study of Riemann’s surfaces, algebraic study of Lie groups, and analytical and numerical studies of nonlinear evolution equations. Remarkably, the instability of the Stokes waves on deep water came as a real surprise to researchers in the field in view of the very long and controversial history of the subject. The question of instability went unrecognized for a very long period of time, even though special efforts have been made to prove the existence of a permanent shape for periodic water waves for all amplitudes less than the critical value at which the waves assume a sharp-crested form. However, Whitham’s instability analysis of water waves (1965a, b), Lighthill’s (1965, 1967) mathematical investigation into the elliptic case, and Benjamin and Feir’s (1967)

theoretical and experimental findings have provided conclusive evidence of the instability of nonlinear water waves.

As stated earlier, dispersion and nonlinearity play a fundamental role in wave motions in nature. The nonlinear shallow water equations that neglect dispersion altogether lead to breaking phenomena of the typical hyperbolic kind with the development of a vertical profile. In particular, the linear dispersive term in the Korteweg–de Vries equation prevents this from ever happening in its solution. In general, breaking can be prevented by including dispersive effects in the shallow water theory. The nonlinear theory provides some insight into the question of how nonlinearity affects dispersive wave motions. Another interesting feature is the instability and subsequent modulation of an initially uniform wave profile. To understand these features, considerable attention was given to the Boussinesq and Korteweg–de Vries (KdV) equations and solitons with emphasis on the methods and solutions of these equations that originated from water waves. The reader is referred to Debnath (2009) for details.

The mathematical theory and applications of nonlinear evolution equations have experienced a revolution over the last four decades of the twentieth century. During this revolution many fascinating and unexpected phenomena have been observed in physical, chemical, and biological systems. Many new instability phenomena have been discovered. Considerable attention has been given to these instability and wave-breaking phenomena. Other major achievements of the twentieth century applied mathematics include the remarkable discovery of soliton, soliton interactions, and the inverse scattering transform (IST) for finding the explicit exact solution for several canonical nonlinear partial differential equations.

This research expository and survey article deals with the most general water wave equations in $(3 + 1)$ dimensions (3-space and 1-time dimensions). Included are variational principles and the Euler–Lagrange equations, the Euler equations of motion and water wave problems, and the variational principle for nonlinear water waves. The generalized Fourier transform is used to solve

the linear initial value problem for the generation and propagation of two-dimensional surface waves in a running stream on water of arbitrary and uniform depth. An asymptotic analysis is carried out to obtain both the ultimate steady-state and transient solutions without the need to resort to the use of a radiation condition or equivalent device. Special attention is given to the existence of surface waves at both subcritical and supercritical stream velocities. The critical values and the resonant phenomena are described in some detail. This is followed by the nonlinear theory of water waves by moving pressure distribution near resonant condition. It is shown that the amplitude of the nonlinear water waves satisfies the *forced Korteweg–de Vries (fKdV) equation* and *forced nonlinear Schrödinger (fNLS) equation* near resonant conditions. An asymptotic investigation is carried out to describe the evolution of wave packets which leads to the standard *nonlinear Schrödinger (NLS) equation* and the higher-order NLS equations (or *Dysthe's equations*) for capillary–gravity waves in water of finite and infinite depth. This is followed by derivation of $(2 + 1)$ dimensional coupled *Davey–Stewartson (DS) nonlinear evolution equations* in water of finite depth with NLS equation for the $(1 + 1)$ dimensional case, and the *Kadomtsev–Petviashvili (KP) equation* for the $(2 + 1)$ dimensional case. Special attention is also given to the derivation of the conservation laws for the DS equations. In order to understand major roles of nonlinearity and dispersion in wave peaking and breaking and other physical features of wave phenomena, several new and modern evolution models including *Camassa and Holm (CH)*, *Degasperis–Procesi (DP)* and *EPDiff (Euler–Poincaré (EP) equation* for geodesic motion on the diffeomorphism), *Constantin–Lannes*, and *Harry Dym integrable model* equations have been developed after 1999. Several singular solutions including peakons, solitons, cusp-solitons, and peakon–solitary waves have been found in recent years. Many authors have investigated analytical and numerical solitary wave solutions with constant vorticity generated at the free surface by

wind stress or current or by friction at the rigid bottom boundary of the fluid flows. These solutions are presented in this entry in some details. Many new research papers and standard books have been added to the bibliography to stimulate new interest in future study and research. This entry is written to provide current information that puts the reader at the forefront of the current research and study.

Variational Principles and the Euler–Lagrange Equations

Many physical systems are often characterized by their extremum (minimum, maximum, or saddle point) property of some associated physical quantity that appears as an integral in a domain, known as *functional*. Such a characterization is a variational principle leading to the Euler–Lagrange equation which optimizes the related functional. For example, light rays travel along a path from one point to another in a minimum time. The shortest distance between two points on a plane curve is a straight line. A physical system is in equilibrium if its potential energy is minimum. So the main problem is to optimize a physical quantity (time, distance, or energy) in most real-world problems. These problems belong to the subject of the calculus of variations.

The classical Euler–Lagrange variational problem is to determine the extreme value of the functional

$$I(u) = \int_a^b F(x, u, u') dx, \quad u' = \frac{du}{dx}, \quad (4)$$

with boundary conditions

$$u(a) = \alpha \quad \text{and} \quad u(b) = \beta, \quad (5)$$

where α and β are given numbers and $u(x)$ belongs to the class $C^2([a, b])$ of functions which have continuous derivatives up to the second order in $a \leq x \leq b$ and the integrand F has continuous second derivatives with respect to all of its arguments.

We assume that $I(u)$ has an extremum at some $u \in C^2([a, b])$. Then we consider the set of all variations $u + \varepsilon v$ for finite u and arbitrary v belonging to $C^2([a, b])$ such that $v(a) = 0 = v(b)$. We next consider the variation δI of the functional $I(u)$

$$\begin{aligned}\delta I &= I(u + \varepsilon v) - I(u) \\ &= \int_a^b [F(x, u + \varepsilon v, u' + \varepsilon v') - F(x, u, u')] dx\end{aligned}$$

which, by Taylor expansion is,

$$\begin{aligned}&= \int_a^b \left[F(x, u, u') + \varepsilon \left(v \frac{\partial F}{\partial u} + v' \frac{\partial F}{\partial u'} \right) \right. \\ &\quad \left. + \frac{\varepsilon^2}{2!} \left(v \frac{\partial F}{\partial u} + v' \frac{\partial F}{\partial u'} \right)^2 + \dots - F(x, u, u') \right] dx \\ &= \int_a^b \varepsilon \left(v \frac{\partial F}{\partial u} + v' \frac{\partial F}{\partial u'} \right) dx + O(\varepsilon^2).\end{aligned}\tag{6}$$

Thus, a necessary condition for the functional $I(u)$ to have an extremum (or for $I(u)$ to be stationary) for an arbitrary ε is

$$0 = \delta I = \int_a^b \left(v \frac{\partial F}{\partial u} + v' \frac{\partial F}{\partial u'} \right) dx, \tag{7}$$

which, integrating the second term by parts, is

$$= \int_a^b v \left[\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) \right] dx + \left[v \frac{\partial F}{\partial u'} \right]_a^b. \tag{8}$$

Since v is arbitrary with $v(a) = 0 = v(b)$, the last term of Eq. (8) vanishes, and consequently, the integrand of the integral Eq. (8) must vanish, that is,

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial u'} \right) = 0. \tag{9}$$

This is called the *Euler–Lagrange equation* of the variational problem involving one independent variable. Using the result

$$\begin{aligned}d \left(\frac{\partial F}{\partial u'} \right) &= \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u'} \right) dx + \frac{\partial}{\partial u} \left(\frac{\partial F}{\partial u'} \right) du \\ &\quad + \frac{\partial}{\partial u'} \left(\frac{\partial F}{\partial u'} \right) du',\end{aligned}\tag{10}$$

the Euler–Lagrange Eq. (9) can be written in the form

$$F_u - F_{xu'} - u' F_{uu'} - u'' F_{u'u'} = 0. \tag{11}$$

This is a second-order nonlinear ordinary differential equation for u provided $F_{u'u'} \neq 0$, and, hence, there are two arbitrary constants involved in the solution. However, when F does not depend explicitly on one of its variables x , u , or u' , the Euler–Lagrange equation assumes a simplified form. Equivalently, there are three possible cases:

1. If $F = F(x, u)$, then Eq. (9) reduces to $F_u(x, u) = 0$, which is an algebraic equation.
2. If $F = F(x, u')$, the Eq. (9) becomes

$$\frac{\partial F}{\partial u'} = \text{const.} \tag{12}$$

3. If $F = F(u, u')$, then Eq. (9) takes the form

$$F - u' F_{u'} = \text{const.} \tag{13}$$

This follows from the fact that

$$\begin{aligned}\frac{d}{dx} (F - u' F_{u'}) &= \frac{dF}{dx} - u' \frac{d}{dx} F_{u'} - u'' F_{u'} \\ &= F_x + u' F_u + u'' F_{u'} - u' \frac{d}{dx} F_{u'} - u'' F_{u'} \\ &= u' \left(F_u - \frac{d}{dx} F_{u'} \right) = 0,\end{aligned}$$

by Eq. (9).

The Euler–Lagrange variational problem involving two independent variables is to determine a function $u(x, y)$ in a domain $D \subset \mathbb{R}^2$ satisfying the boundary conditions prescribed on the boundary ∂D and D and extremizing the functional

$$[I(x, y)] = \iint_D F(x, y, u, u_x, u_y) dx dy, \quad (14)$$

where the function F is defined over the domain D and assumed to have continuous second-order partial derivatives.

Similarly, for functionals depending on a function of two variables, the first variation δI of I is defined by

$$\delta I = I(u + \varepsilon v) - I(u). \quad (15)$$

In view of Taylor's expansion theorem, this reduces to

$$\delta I = \iint_D [\varepsilon(vF_u + v_x F_p + v_y F_q) + O(\varepsilon^2)] dx dy, \quad (16)$$

where $u = u(x, y)$ is assumed to vanish on ∂D and $p = u_x$ and $q = u_y$.

A necessary condition for the functional I to have extremum is that the first variation of I vanishes, that is,

$$\begin{aligned} 0 = \delta I &= \iint_D (vF_u + u_x F_p + u_y F_q) dx dy \\ &= \iint_D v \left(F_u - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial y} F_q \right) dx dy \\ &\quad + \iint_D \left[v \left(\frac{\partial}{\partial x} F_p + \frac{\partial}{\partial y} F_q \right) \right. \\ &\quad \left. + (v_x F_p + v_y F_q) \right] dx dy \\ &= \iint_D v \left(F_u - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial y} F_q \right) dx dy \\ &\quad + \iint_D \left[\frac{\partial}{\partial x} (v F_p) + \frac{\partial}{\partial y} (v F_q) \right] dx dy. \end{aligned} \quad (17)$$

We assume that the boundary curve ∂D has a piecewise, continuously moving tangent so that Green's theorem can be applied to the second double integral in Eq. (17).

Consequently Eq. (17) reduces to

$$\begin{aligned} 0 = \delta I &= \iint_D v \left(F_u - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial y} F_q \right) dx dy \\ &\quad + \int_{\partial D} v (F_p dy - F_q dx). \end{aligned} \quad (18)$$

Since $u = 0$ on ∂D , the second integral in Eq. (18) vanishes. Moreover, since v is an arbitrary function, it follows that the integrand of the first integral in Eq. (18) must vanish. Thus, the function $u(x, y)$ extremizing the functional defined by Eq. (14) satisfies the partial differential equation:

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial y} F_q = 0. \quad (19)$$

This is called the *Euler-Lagrange equation* for the variational problem involving two independent variables.

The above variational formulation can be readily be generalized for the functionals depending on functions of three or more variables. Many physical problems require determining a function of several independent variables which will lead to an extremum of such functionals.

The Euler Equation of Motion and Water Wave Problems

The Euler equation of motion and the equation of continuity have provided the fundamental basis of the study of modern theories of water waves, which are the most common observable phenomena in nature. Water wave motions are of great importance as they range from waves generated by wind or solar heating at the surface of oceans to flood waves in rivers, from waves caused by a moving ship in a channel to tsunami (tidal waves) generated by earthquakes, and from solitary waves on the surface of a channel generated by a disturbance to waves generated by underwater explosions, to mention only a few.

Making references to Debnath's book (1994), *Nonlinear Water Waves*, the Euler equation of motion in an inviscid and incompressible fluid of constant density ρ under the action of body force $\mathbf{F} = (0, 0, -g)$ where g is the constant acceleration of gravity and the equation of continuity are given by

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{F}, \quad (20)$$

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (21) \quad \text{where}$$

where $\mathbf{x} = (x, y, z)$ is the rectangular Cartesian coordinates and $\mathbf{u} = (u, v, w)$ is the velocity vector, p is the pressure field, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}. \quad (22)$$

These equations constitute a closed system of four *nonlinear* partial differential equations for four unknowns: u , v , w , and p . So these equations with appropriate initial and boundary conditions are sufficient to determine the velocity field $\mathbf{u}$ and pressure p uniquely.

In the study of water waves, the body force is always the acceleration due to gravity, that is, $\mathbf{F} = (0, 0, -g)$. It is convenient to write the three components of the Euler equation in the form

$$\frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (23)$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y}, \quad (24)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g, \quad (25)$$

and the continuity Eq. (21).

In cylindrical polar coordinates $\mathbf{x} = (r, \theta, z)$ with the velocity vector $\mathbf{u} = (u, v, w)$ equations and the continuity equation are given by

$$\frac{Dw}{Dt} - \frac{v^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}, \quad (26)$$

$$\frac{Dv}{Dt} + \frac{uv}{r} = -\frac{1}{\rho} \frac{1}{r} \frac{\partial p}{\partial \theta}, \quad (27)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g, \quad (28)$$

$$\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \quad (29)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z}. \quad (30)$$

One of the fundamental properties of a fluid flow is called the *vorticity*, which is defined by the *curl* of the velocity field so that $\boldsymbol{\omega} = \text{curl} \mathbf{u} = \nabla \times \mathbf{u}$. The vorticity vector $\boldsymbol{\omega}$ measures the local spin or rotation of individual fluid particles. Evidently, fluid flows in which $\boldsymbol{\omega} = 0$ are called *irrotational*. In the real world, fluid flows are hardly irrotational anywhere; however, for many flows the vorticity is very small almost everywhere and the fluid motion may be treated as irrotational. In problems of water waves, the motion of fluid is considered unsteady and irrotational which implies that the $\boldsymbol{\omega} = \text{curl} \mathbf{u} = \nabla \times \mathbf{u} = 0$. So there exists a single-valued velocity potential ϕ so that $\mathbf{u} = \nabla \phi$. The continuity Eq. (21) then reduces to the Laplace equation:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (31)$$

Using vector identity, $(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla^2 u^2 - \mathbf{u} \times \boldsymbol{\omega}$ combined with $\boldsymbol{\omega} = 0$ and $\mathbf{u} = \nabla \phi$, the Euler Eq. (20) may be written in the form

$$\nabla \left[\phi_t + \frac{1}{2} (\nabla \phi)^2 + \frac{p}{\rho} + gz \right] = 0. \quad (32)$$

This can be integrated with respect to the space variables to obtain the equation

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz = c(t), \quad (33)$$

where $c(t)$ is an arbitrary function of time ($\nabla c = 0$) determined by the pressure imposed at the boundaries of the fluid flow. Since only the pressure gradient affects the flow, a function of t alone added to the pressure field p has virtually no effect on the motion. So without loss of generality, we can set $c(t) = 0$ in Eq. (33).

Consequently, Eq. (33) becomes

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz = 0. \quad (34)$$

This equation is known as the *Bernoulli's equation* (or *pressure equation*), which completely determines the pressure in terms of velocity potential ϕ . Thus, the Laplace Eqs. (31) and (34) are used to determine ϕ and p and, hence, the velocity components u, v, w , and the pressure p .

In cylindrical polar coordinates $\mathbf{x} = (r, \theta, z)$ with velocity field $u = (\phi_r, \frac{1}{r}\phi_\theta, \phi_z)$, the Laplace equation becomes

$$\nabla^2\phi = \frac{1}{r} \frac{\partial}{\partial r} (r\phi_r) + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \quad (35)$$

We assume that the fluid occupies the region $-h \leq z \leq 0$ with the plane $z = -h$ as the rigid *bottom boundary* and the plane $z = 0$ as the *upper (free surface) boundary* in the undisturbed state. We suppose that the upper boundary is the surface exposed to a constant atmospheric pressure p_a , and we have $p = p_a$ on the surface. After the motion is set up, we denote this surface S with the equation $z = \eta(x, y, t)$ where η is an unknown function of x, y , and t that tends to zero as $t \rightarrow 0$. The function $\eta(x, y, t)$ is referred to as the *free-surface elevation*.

The rate of change η_t , following a fluid particle, is equal to the vertical component $\nabla\phi$ at the surface, that is,

$$\eta_t + \mathbf{u} \cdot \nabla\eta = \phi_z \quad \text{on } z = \eta.$$

Or, equivalently, this free-surface condition reads as

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z = 0 \quad \text{on } z = \eta. \quad (36)$$

This is called *kinematic free-surface condition*.

Since $p = p_a$ on S , after absorbing $\frac{p_a}{\rho}$ and $c(t)$ into ϕ_t , Eq. (33) can be rewritten as

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz = 0 \quad \text{on } S \quad \text{for } t \geq 0. \quad (37)$$

Since S is a free boundary surface, it contains the same fluid particles for all times, that is, S is a material surface. Hence, it follows from Eq. (37) that

$$\frac{D}{Dt} \left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz \right] = 0 \quad \text{on } S \quad \text{for } t \geq 0. \quad (38)$$

Or, equivalently, on S for $t \geq 0$,

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \nabla\phi \cdot \nabla \right) \left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz \right] \\ &= \phi_{tt} + 2\nabla\phi \cdot \nabla(\phi_t) + \frac{1}{2}\nabla\phi \cdot \nabla(\nabla\phi)^2 + g\phi_z \\ &= 0. \end{aligned} \quad (39)$$

Since the bottom boundary $z = -h$ is a rigid solid surface at rest, the condition to be satisfied at this boundary is

$$\frac{\partial\phi}{\partial z} = 0 \quad z = -h, \quad t \geq 0. \quad (40)$$

Thus, Laplace's equation (31) together with the free-surface boundary conditions (36) and (39) and the bottom boundary condition (40) determine the velocity potential ϕ and the free-surface elevation η . Because of the presence of the non-linear terms in the free-surface boundary conditions (36) and (39), the determination of ϕ and η in the general case is a difficult task. We restrict our discussion to two particular cases because of the great importance of water wave motions (see Debnath 2012, pp. 176–193).

The Variational Principle for Nonlinear Water Waves

In his pioneering work, Whitham (1965a, b) first developed a general approach to linear and nonlinear dispersive waves using a Lagrangian. It is well known that most of the general ideas about dispersive waves have originated from the classical problems of water waves. So it is important to have a variational principle for water waves. Luke (1967) first explicitly formulated a variational principle for two-dimensional water waves and showed that the basic equations and boundary and free-surface conditions can be derived from the Hamilton principle.

We now formulate the *variational principle* for three-dimensional water waves in the form

$$\delta I = \delta \int \int_D L dx dt = 0, \quad (41)$$

where the *Lagrangian* L is assumed to be equal to the pressure, so that

$$L = -\rho \int_{-h(x,y)}^{\eta(x,y)} \left[\phi_t + \frac{1}{2} (\nabla \phi)^2 + gz \right] dz, \quad (42)$$

where D is an arbitrary region in the $(\mathbf{x}, t)$ space, ρ is the density of water, g is the gravitational acceleration, and $\phi(\mathbf{x}, z, t)$ is the velocity potential in an unbounded fluid lying between the rigid bottom at $z = -h(x, y)$ and the free surface $z = \eta(x, y, t)$ as shown in Fig. 1. The functions $\phi(\mathbf{x}, z, t)$ and $\eta(\mathbf{x}, t)$ are allowed to vary subject to the restrictions $\delta \phi = 0$ and $\delta \eta = 0$ at x_1, x_2, y_1, y_2, t_1 , and t_2 .

Using the standard procedure in the calculus of variations, Eq. (41) becomes

$$\begin{aligned} 0 &= -\delta \int \int_D \frac{L}{\rho} dx dt \\ &= \int \int_D \left\{ \left[\phi_t + \frac{1}{2} (\nabla \phi)^2 + gz \right]_{z=\eta} \delta \eta \right. \\ &\quad + \int_{-h}^{\eta} \left[\phi_x \delta \phi_x - \phi_y \delta \phi_y + \phi_z \delta \phi_z \right. \\ &\quad \left. \left. + \delta \phi_t \right] dz \right\} dx dt, \end{aligned} \quad (43)$$

which, integrating the z -integral by parts, is

$$\begin{aligned} &= \int \int_D \left\{ \left[\phi_t + \frac{1}{2} (\nabla \phi)^2 + gz \right]_{z=\eta} \delta \eta \right. \\ &\quad + \left[\frac{\partial}{\partial t} \int_{-h}^{\eta} \delta \phi dz + \frac{\partial}{\partial x} \int_{-h}^{\eta} \phi_x \delta \phi dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} \phi_y \delta \phi dz \right] \\ &\quad - \int_{-h}^{\eta} (\phi_{xx} + \phi_{yy} + \phi_{zz}) \delta \phi dz \\ &\quad - \left[(\eta_t + \eta_x \phi_x + \eta_y \phi_y - \phi_z) \delta \phi \right]_{z=\eta} \\ &\quad \left. + \left[(\phi_x h_x + \phi_y h_y + \phi_z) \delta \phi \right]_{z=-h} \right\} dx dt. \end{aligned} \quad (44)$$

The second term within the square brackets integrates out to the boundaries ∂D of D and vanishes if $\delta \phi$ is chosen to be zero on ∂D . If we take $\delta \eta = 0$, $[\delta \phi]_{z=\eta} = [\delta \phi]_{z=-\eta} = 0$, since $\delta \phi$ is otherwise arbitrary, it turns out that

$$\nabla^2 \phi = 0, \quad -\infty < x, y < \infty, \quad -h < z < \eta. \quad (45)$$

Since $\delta \eta$, $[\delta \phi]_{z=\eta}$, $[\delta \phi]_{z=-h}$ may be given arbitrary values, it follows that

$$\phi_t + \frac{1}{2} (\nabla \phi)^2 + g\eta = 0 \quad \text{on } z = \eta, \quad (46)$$

$$\eta_t + \eta_x \phi_x + \eta_y \phi_y - \phi_z = 0 \quad \text{on } z = \eta, \quad (47)$$

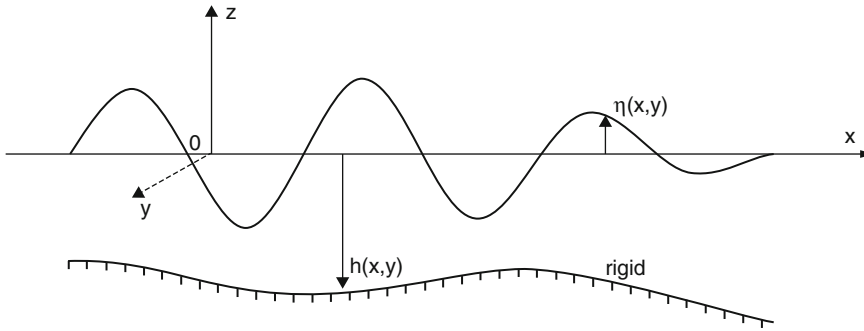
$$\phi_x h_x + \phi_y h_y + \phi_z = 0 \quad \text{on } z = -h. \quad (48)$$

Thus Eqs. (45), (46), (47), and (48) represent the well-known nonlinear system of equations for classical water waves. Finally, this analysis is in perfect agreement with that of Luke (1967) and Whitham (1965a, b, 1974) for two-dimensional waves on water of arbitrary but uniform depth h .

It may be relevant to mention Zakharov's (1968a, b) Hamiltonian formulation. The Hamiltonian of the problem is

$$\mathcal{H} = \frac{1}{2} \int_{-\infty}^{+\infty} \left(g\eta^2 + \int_{-h}^{\eta} (\nabla \phi)^2 dz \right) dx. \quad (49)$$

On the other hand, Benjamin and Olver (1982) have described the Hamiltonian structure, symmetries, and conservation laws for water waves. Olver (1984a, b) has discussed Hamiltonian and



Nonlinear Water Waves and Nonlinear Evolution Equations with Applications, Fig. 1 A general surface gravity wave problem

non-Hamiltonian models for water waves and Hamiltonian perturbation theory and nonlinear water waves.

$$\begin{aligned} R_1 &= \eta_x \left(1 + \eta_x^2 + \eta_y^2 \right)^{-\frac{1}{2}} \quad \text{and} \\ R_2 &= \eta_y \left(1 + \eta_x^2 + \eta_y^2 \right)^{-\frac{1}{2}}, \end{aligned} \quad (54)$$

Basic Equations of Nonlinear Water Waves

We consider an unsteady irrotational motion of a inviscid and incompressible water in a constant gravitational field g which is in the negative z -direction. The uneven bottom is described by $z = b(x, y)$. Including the effects of surface tension T , the basic equations and the free-surface boundary conditions for the velocity potential $\phi = \phi(x, y, z, t)$ and the free-surface elevation $\eta = \eta(x, y, t)$ are given by

$$\begin{aligned} \nabla^2 \phi &\equiv \phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad b(x, y) \leq z \\ &\leq h_0 + a\eta, \quad t > 0 \end{aligned} \quad (50)$$

$$\begin{aligned} \eta_t + \left(\phi_x \eta_x + \phi_y \eta_y \right) - \phi_z &= 0 \quad \text{on } z \\ &= h_0 + a\eta, \end{aligned} \quad (51)$$

$$\begin{aligned} \phi_t + \frac{1}{2} (\nabla \phi)^2 + g\eta - T(R_1 + R_2) &= 0 \quad \text{on } z \\ &= h_0 + a\eta, \end{aligned} \quad (52)$$

$$\phi_z - \left(\phi_x b_x + \phi_y b_y \right) = 0 \quad \text{on } z = b(x, y), \quad (53)$$

where the curvatures R_1 and R_2 are defined by

h_0 is the typical depth of water and a is the typical amplitude of the surface wave.

It is convenient to introduce a typical horizontal length scale l (which may be wavelength λ) and a typical vertical length scale h_0 and a typical velocity scale $c = \sqrt{gh_0}$ (shallow water wave speed). We also introduce two fundamental water wave parameters ε and δ as

$$\varepsilon = \frac{a}{h_0} \quad \text{and} \quad \delta = \frac{h_0^2}{l^2}, \quad (55)$$

where ε is called amplitude parameter and δ is called long wavelength or shallowness parameter.

Making reference to Debnath (1994), the basic water wave equations and the free-surface boundary conditions without the effects of surface tension ($T = 0$) in nondimensional form are

$$\begin{aligned} \delta \left(\phi_{xx} + \phi_{yy} \right) + \phi_{zz} &= 0, \quad b \leq z \leq 1 + \varepsilon\eta, \\ t &> 0 \end{aligned} \quad (56)$$

$$\begin{aligned} \phi_t + \eta + \frac{\varepsilon}{2} \left(\phi_x^2 + \phi_y^2 \right) + \frac{\varepsilon}{2\delta} \phi_z^2 &= 0 \quad \text{on} \\ z &= 1 + \varepsilon\eta, \end{aligned} \quad (57)$$

$$\begin{aligned} \phi_z - \delta \left[\eta_t + \varepsilon \left(\phi_x \eta_x + \phi_y \eta_y \right) \right] &= 0 \quad \text{on } z \\ &= 1 + \varepsilon \eta, \end{aligned} \quad (58)$$

$$\phi_z - \delta \left(\phi_x b_x + \phi_y b_y \right) = 0 \quad \text{on } z = b(x, y). \quad (59)$$

Surface Waves on a Running Stream in Water of Arbitrary, But Uniform, Depth

This section deals with the *initial value problem* (Debnath and Rosenblat 1969) for the generation and propagation of two-dimensional waves at the free surface of a running stream on water of finite depth. The problem will be considered according to standard linearized theory of surface waves, the applied pressure being “switched on” at time $t = 0$. An asymptotic development then leads to the ultimate steady-state solution without the need to resort to the use of a radiation condition or equivalent device; at the same time asymptotic transient solutions can be obtained. Additionally, this method of solution provides an interesting illustration of the applicability of generalized functions (Lighthill 1958; Jones 1966) in problems of this type. Of course, the problem can be treated by standard counter-integral methods, but a comparison of the two approaches suggests that the use of generalized functions simplifies the analysis considerably. The problem is solved under the following assumptions:

- (i) In its undisturbed state the liquid, which is of infinite horizontal extent, has uniform depth h and flows with constant speed U .
- (ii) The wave-generating mechanism is an oscillating pressure applied at the free surface.
- (iii) This applied pressure is two dimensional, so that the wave motion is wholly parallel to the flow.
- (iv) The liquid is taken to be inviscid and has constant density and negligible surface tension.

It is convenient to formulate the problem in a coordinate frame with respect to which the applied pressure at the surface is at rest. We thus take Cartesian axes $Oxyz$, such that the z -axis points vertically upward, the origin is located at the level of the undisturbed free surface, and the stream moves in the Ox -direction with speed U relative to this frame.

The applied pressure may be an arbitrary function of x , but it will be sufficient for the determination of the principal features of the flow to take the simple form

$$p(x, t) = P\delta(x)e^{i\omega t}, \quad (60)$$

where P is constant, $\delta(x)$ is the Dirac delta function, and ω is the given frequency.

Let $\phi(x, z, t)$ be the disturbance velocity potential and $\eta(x, t)$ the elevation of the free surface. Then the problem reduces (Stoker 1957, p. 198) to finding the solution of Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (61)$$

in $-h \leq z \leq 0$, $-\infty < x < \infty$, $t > 0$, subject to the boundary conditions

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = -h, \quad (62)$$

and

$$\left. \begin{aligned} \frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} + g\eta &= -\frac{P}{\rho} \delta(x) e^{i\omega t} \\ \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} &= 0 \end{aligned} \right\} \quad \text{on} \quad z = 0, \quad t > 0, \quad (63a, b)$$

and the initial conditions

$$\phi(x, z, 0) = \eta(x, 0) = 0. \quad (64a, b)$$

To solve the partial differential systems (61), (62), (63a, b), and (64a, b), we take the Fourier transform with respect to x , defined by

Eq. (1.7.1) of Debnath (2012, p. 34). It is thus tacitly assumed that the functions ϕ and η are generalized functions of x in the sense of Lighthill (1958), so that their Fourier transforms exist.

After transformation, Eqs. (61) and (62) are found to have the solution

$$\bar{\phi}(k, z, t) = \bar{A}(k, t) \cosh k(z + h) \quad (65)$$

which is then substituted into the transformed versions of Eq. (63a, b) giving

$$\frac{d\bar{\eta}}{dt} + ikU\bar{\eta} - k\bar{A} \sinh kh = 0, \quad (66)$$

and

$$\left(\frac{d\bar{\eta}}{dt} + ikU\bar{A} \right) \cosh kh + g\bar{\eta} = -\frac{P}{\rho\sqrt{(2\pi)}} e^{i\omega t}, \quad (67)$$

with initial conditions

$$\bar{\eta}(k, 0) = \bar{A}(k, 0) = 0. \quad (68a, b)$$

The solution of Eqs. (67) and (68a, b) for $\bar{\eta}$ is

$$\bar{\eta}(k, t) = \frac{P\sqrt{(k \tanh kh)}}{2\rho\sqrt{(2\pi g)}} \left[\frac{e^{i\omega t} - e^{im_1 t}}{\omega - m_1} - \frac{e^{i\omega t} - e^{im_2 t}}{\omega - m_2} \right], \quad (69)$$

where

$$\left. \begin{matrix} m_1 \\ m_2 \end{matrix} \right\} = -kU \pm \sqrt{(gk \tanh kh)}. \quad (70a, b)$$

Application of the Fourier inversion integral now gives

$$\eta(x, t) = \frac{P}{4\pi\rho\sqrt{(g)}} \int_{-\infty}^{+\infty} \sqrt{(k \tanh kh)} \left[\frac{e^{i\omega t} - e^{im_1 t}}{\omega - m_1} - \frac{e^{i\omega t} - e^{im_2 t}}{\omega - m_2} \right] e^{ikx} dk. \quad (71)$$

The solution for $\phi(x, z, t)$ can be similarly obtained from Eqs. (65), (66), (67), and (68a, b)

To evaluate the integral Eq. (71) for large values of $|x|$ and t , we shall use formulae developed by Lighthill (1958) and Jones (1966) for the asymptotic behavior of Fourier transforms of generalized functions. It is convenient to rewrite Eq. (71) as

$$\eta(x, t) = \frac{P}{4\pi\rho\sqrt{g}} (e^{i\omega t} I - J), \quad (72)$$

where

$$I = \int_{-\infty}^{+\infty} \sqrt{(k \tanh kh)} \left[\frac{1}{\omega - m_1} - \frac{1}{\omega - m_2} \right] e^{ikx} dk \quad (73)$$

and

$$J = \int_{-\infty}^{+\infty} \sqrt{(k \tanh kh)} \left[\frac{e^{im_1 t}}{\omega - m_1} - \frac{e^{im_2 t}}{\omega - m_2} \right] e^{ikx} dk, \quad (74)$$

and to evaluate these integrals separately. The nature both of the ultimate wave system and of the asymptotic approach to it is determined by the singularities of the integrands in Eqs. (73) and (74). The remainder of this section, therefore, is concerned with locating and characterizing these singularities.

The dominant contributions to I , as $x \rightarrow \infty$, come from the poles of the integrand, that is, from points where

$$\omega - m_1 = \omega + kU - \sqrt{(gk \tanh kh)} = 0 \quad (75a)$$

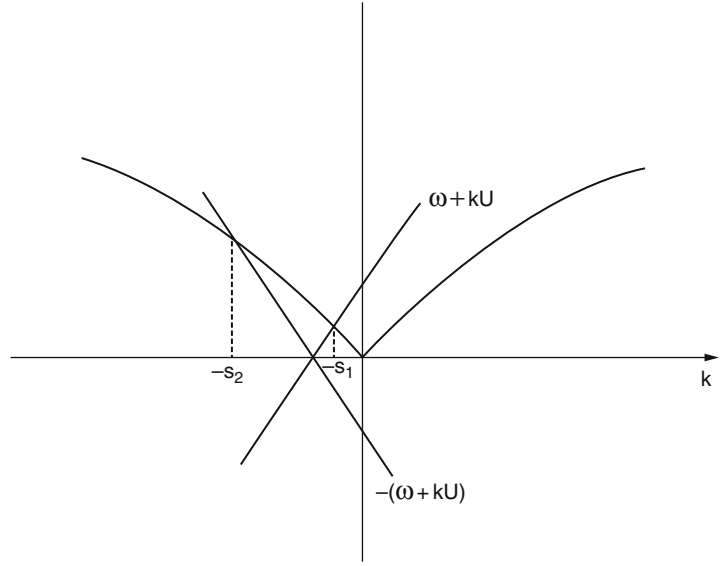
and

$$\omega - m_2 = \omega + kU - \sqrt{(gk \tanh kh)} = 0. \quad (75b)$$

These points are indicated in Figs. 2 and 3, both of which depict the intersections of the curve $\sqrt{(gk \tanh kh)}$ with the straight lines $(\omega + kU)$ and $-(\omega + kU)$ and different values of U , ω and h being used in the two figures, respectively.

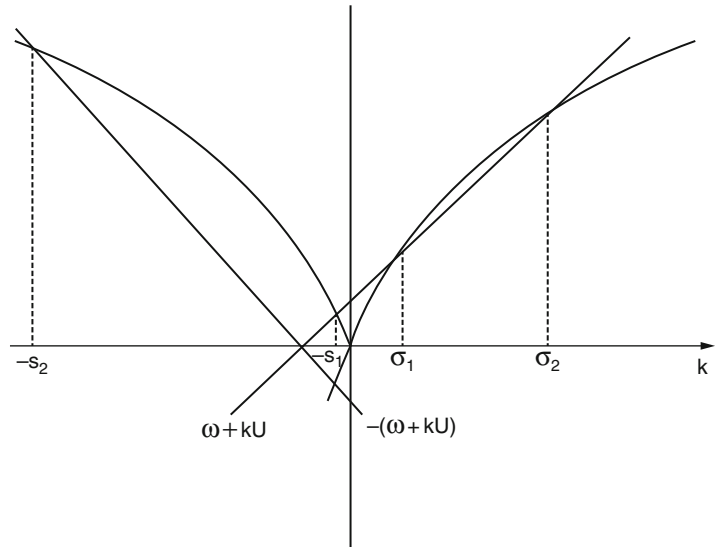
**Nonlinear Water Waves
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Fig. 2 Position of poles
 $k = -s_1, -s_2$ given by
intersection of the curve $(gk \tanh kh)^{1/2}$ with the lines $\pm$
 $(\omega + kU)$



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Fig. 3 Position of poles
 $k = -s_1, -s_2, \sigma_1, \sigma_2$ given
by intersection of the curve
 $(gk \tanh kh)^{1/2}$ with the lines
 $\pm (\omega + kU)$



It is apparent from the diagrams that Eqs. (75a) and (75b) each give rise to one pole on the negative k , ω -axis at locations

$$k = -s_1 \quad \text{and} \quad k = -s_2, \quad (76)$$

respectively. If the values of U , ω and h are such that Fig. 2 is relevant, then those are the only poles. On the other hand, we see from Fig. 3 that there may be two further poles, arising from Eq. (75a), at positive values of k given by, say,

$$k = \sigma_1 \quad \text{and} \quad k = \sigma_2. \quad (77)$$

Depending again on U , ω and h , there will be a delimiting case when σ_1 and σ_2 coalesce into a double pole. This is the *critical case*, whose details will be examined later; meanwhile we shall denote it by

$$U = U_c, \quad (78)$$

where $U_c = g/\omega$ or $\sqrt{gh}$ according to whether $h \rightarrow \infty$ (ω and U are finite) or $h \rightarrow \infty$ ($\omega = 0$ and U are

finite). It is then clear from Figs. 2 and 3 that the poles at σ_1 and σ_2 appear only when

$$U < U_c. \quad (79)$$

Application of the formula (3.4.16) developed by Debnath (1994) on page 95, together with Eqs. (76), (77), and (78), gives immediately

$$\begin{aligned} I \sim \pi_i \cdot \operatorname{sgn} x & \left[\phi(-s_1)e^{-is_1x} - \psi(-s_2)e^{-is_2x} \right. \\ & \left. + H(U_c - U) \{ \phi(\sigma_1)e^{i\sigma_1x} + \phi(\sigma_2)e^{i\sigma_2x} \} \right] \\ & + O(1/|x|), \end{aligned} \quad (80)$$

where H is the Heaviside step function and where

$$\begin{aligned} \phi(\kappa) &= -\sqrt{(\kappa \tanh \kappa h)} / \left(\frac{dm_1}{dk} \right)_{k=\kappa} \quad \text{and} \quad \psi(\kappa) \\ &= -\sqrt{(\kappa \tanh \kappa h)} / \left(\frac{dm_2}{dk} \right)_{k=\kappa}. \end{aligned} \quad (81a, b)$$

The solution Eq. (80) is degenerate at the critical value $U = U_c$. This will be considered in section “Critical Values and Resonance-Type Effect” below.

The integral J will be evaluated asymptotically for large t , using the formula (3.7.16) developed by Debnath (1994, p. 95) and the classical method of stationary phase (Jones 1966).

For $t \rightarrow \infty$ the integrand of J has stationary points when

$$\frac{d}{dk} \left(m_1 + \frac{kx}{t} \right) = 0 \quad \text{and} \quad \frac{d}{dk} \left(m_2 \frac{kx}{t} \right) = 0; \quad (82a, b)$$

by Eq. (70a, b), these relations are equivalent to

$$\frac{d}{dk} \sqrt{(gk \tanh kh)} = U - x/t \quad (83a)$$

and

$$\frac{d}{dk} \sqrt{(gk \tanh kh)} = -(U - x/t), \quad (83b)$$

respectively. However, our principal interest is in finding solutions in the limits $|x| \rightarrow \infty, t \rightarrow \infty$ taken in such a way that $|x| < U t$. Consequently, it will in general be sufficient for our purposes to replace Eqs. (83a) and (83b) by the approximations

$$\frac{d}{dk} \sqrt{(gk \tanh kh)} = U \quad (84a)$$

and

$$\frac{d}{dk} \sqrt{(gk \tanh kh)} = -U. \quad (84b)$$

The locations of the stationary points can easily be inferred from Figs. 2 and 3. The gradient of $\sqrt{(gk \tanh kh)}$ decreases monotonically from $\sqrt{(gk)}$ to 0 as k varies from 0 to $+\infty$, and it decreases monotonically from 0 to $\sqrt{(-gk)}$ as k increases from $-\infty$ to 0. Hence, each of Eqs. (84a) and (84b) has at most one real root, which is positive for Eq. (84a) and negative for Eq. (84b). We designate the positions of these roots by

$$k = \sigma \quad \text{and} \quad k = -\rho, \quad (85)$$

respectively. Moreover, a necessary and sufficient condition for these roots to occur is

$$\sqrt{(gh)} \geq U. \quad (86)$$

The transient part of J , that is, the contribution from the stationary points, can now be written down. The standard formula for the stationary phase approximation (see, e.g., Jones 1966, p. 323) is invoked, and we use the Heaviside function to incorporate the condition (86). It is then found that

$$J_{\text{transient}} \sim H(\sqrt{gh} - U) \left[\left(-\frac{2\pi}{m_1''(\sigma)} \right)^{1/2} \frac{\sqrt{(\sigma \tanh \sigma h)}}{\omega - m_1(\sigma)} \exp \left\{ im_1(\sigma)t + i\sigma x - \frac{1}{4}\pi i \right\} \right. \\ \left. - \left(-\frac{2\pi}{m_2''(-\rho)} \right)^{1/2} \frac{\sqrt{(\rho \tanh \rho h)}}{\omega - m_2(-\rho)} \exp \left\{ im_2(-\rho)t - i\rho x - \frac{1}{4}\pi i \right\} \right] + O(1/t). \quad (87)$$

The result of Eq. (87) includes the case $\sqrt{gh} = U$ in the sense that $\rho = \sigma = 0$ and the contribution to J is $O(1/t)$.

It remains to calculate the contribution to J from its poles, which are the same as the poles of I . At this stage it is important to note that if the positive poles at σ_1 and σ_2 appear, then the stationary point at $k = \sigma$ is necessarily present. In other words, inequality Eq. (79) implies inequality Eq. (86) (though the converse is not true). Moreover, when this happens, we have

$$\sigma_1 < \sigma < \sigma_2. \quad (88)$$

In addition, if there is a stationary point at $k = -\rho$, then

$$\rho < s_2. \quad (89)$$

These observations are readily verified by a glance at Figs. 2 and 3. To evaluate the polar contribution to J , it is convenient to write

$$J_{\text{polar}} = J_1 - J_2, \quad (90)$$

where

$$J_1 = \left[\int_{-\infty}^0 + H(U_c - U) \left\{ \int_0^\sigma + \int_\sigma^\infty \right\} \right] \frac{\sqrt{(k \tanh kh)} e^{im_1 t + ikx}}{\omega - m_1} dk \quad (91)$$

and

$$J_2 = \int_{-\infty}^{-\rho} \frac{\sqrt{(k \tanh kh)} e^{im_2 t + ikx}}{\omega - m_2} dk. \quad (92)$$

Each of the integrals in Eq. (91) now contains one pole; this follows from the inequality Eq. (88). The integral Eq. (92) contains the polar contribution from $k = -s_2$; if there is no stationary point $(-\rho)$, the upper limit of integration can be replaced by any number such that the range includes the pole.

In Eq. (91) we change the variable of integration from k to $m_1 = m_1(k)$, noting from Eq. (70a, b) that $m_1(-\infty) = +\infty$, $m_1(+\infty) = -\infty$, and

$$m_1(\sigma) = -\sigma U + \sqrt{(g\sigma \tanh \sigma)} = M_1, \quad (93)$$

say, where it is important to observe that $M_1 > 0$.

Equation (91) now becomes

$$J_1 = \left[-\int_0^\infty + H(U_c - U) \left\{ \int_0^{M_1} - \int_{-\infty}^{M_1} \right\} \right] \frac{[\phi(k)e^{ikx}]_{k=k^{-1}(m_1)} e^{im_1 t} dm_1}{m_1 - \omega}, \quad (94)$$

where $\phi(k)$ is defined in Eq. (81a, b). The asymptotic value of J_1 as $t \rightarrow \infty$ can now be written down with the aid of formula (3.4.16) developed by Debnath (1994, p. 95), in terms of the contributions from the poles:

$$J_1 = \pi i e^{i\omega t} \left[-\phi(-s_1) e^{-is_1 x} + H(U_c - U) \{ \phi(\sigma_1) e^{i\sigma_1 x} - \phi(\sigma_2) e^{i\sigma_2 x} \} \right]. \quad (95)$$

The multiplicative factor $\text{sgn } t$ is omitted in Eq. (95) because we are concerned only with $t > 0$. The integral J_2 is treated similarly, with the change of variable to $m_2 = m_2(k)$. We find

$$J_2 = - \int_{M_2}^{\infty} \frac{[\psi(k)e^{ikx}]_{k=k^{-1}(m_2)} e^{im_2 t} dm_2}{m_2 - \omega}, \quad (96)$$

where

$$M_2 = \rho U + \sqrt{(g\rho \tanh \rho h)} > 0. \quad (97)$$

Hence, using Eq. (3.4.16) of Debnath (1994), p 95 again, we obtain

$$J_2 = -\pi i e^{i\omega t} \psi(-s_2) e^{-is_2 x}. \quad (98)$$

A simple combination of Eqs. (95) and (98) now gives

$$\begin{aligned} J_{\text{polar}} = \pi i e^{i\omega t} & \left[-\phi(-s_1) e^{-is_1 x} \right. \\ & + \psi(-s_2) e^{-is_2 x} + H(U_c - U) \left\{ \phi(\sigma_1) e^{i\sigma_1 x} \right. \\ & \left. \left. - \phi(\sigma_2) e^{i\sigma_2 x} \right\} \right]. \end{aligned} \quad (99)$$

We have now found all the essential components of the surface elevation $\eta(x, t)$, as given by Eq. (72). It is convenient to write

$$\eta = \eta_{st} + \eta_{tr}, \quad (100)$$

where η_{st} is the *steady-state component* and η_{tr} is the *transient component*.

The former is made up of the polar contributions to I and J given by Eqs. (80) and (99), respectively, and therefore is

$$\begin{aligned} \eta_{st} = -\frac{iP}{2\rho\sqrt{g}} e^{i\omega t} & \left[-\phi(-s_1) e^{-is_1 x} \right. \\ & - \psi(-s_2) e^{-is_2 x} + H(U_c - U) \\ & \left. \cdot \phi(\sigma_2) e^{i\sigma_2 x} \right] \text{ when } x > 0 \end{aligned} \quad (101)$$

and

$$\begin{aligned} \eta_{st} = -\frac{iP}{2\rho\sqrt{g}} H(U_c - U) \cdot \phi(\sigma_1) e^{i\sigma_1 x} \\ \text{when } x < 0. \end{aligned} \quad (102)$$

The fact that Eq. (80) contains the factor $\text{sgn } x$, while Eq. (99) does not, accounts for the difference in the expressions for $x > 0$ and $x < 0$.

The dominant contribution to the transient part η_{tr} is given by Eq. (87) and has the asymptotic form

$$\eta_{tr} \sim H\left(\sqrt{gh} - U\right) \cdot O(1/\sqrt{t}) + O(1/|x|). \quad (103)$$

Equations (101) and (102) indicate that the steady-state wave system has the following characteristics:

- (i) When $U > U_c$, there are *two* surface waves downstream of the origin, traveling with speeds ω/s_1 and ω/s_2 , respectively, in the positive x -direction; there are none upstream.
- (ii) When $U < U_c$, there are, in addition, two more surface waves, which move with speeds ω/σ_1 and ω/σ_2 , both in the negative x -direction. The former of these exists only on the upstream side of the origin and the latter only on the downstream side. This latter wave thus appears to originate at infinity, but this is only when it is viewed from the moving coordinate system. It is easily verified (from a consideration of Eq. (75a)) that the speed ω/σ_2 is always less than U , the speed of the frame, so that relative to axes at rest this wave travels in a positive direction. A similar situation was noted by Kaplan (1957).
- (iii) The wave system is degenerate when $\omega = 0$. For then (see Figs. 2 and 3) $s_1 = s_2 = \sigma_1 = 0$, and the only wave that occurs is the one associated with σ_2 . As before, it exists on the downstream side, when $U > U_c$ only, but is now a steady motion relative to the moving frame. This is the case discussed by Stoker (1957, p. 214), who found a similar behavior.

Critical Values and Resonance-Type Effect

It is necessary now to locate precisely the *critical point* denoted earlier by $U = U_c$ and to derive the solution in its neighborhood.

The critical value occurs when Eq. (75a) has just one positive root — $k = \sigma$, say, attained through the coalescence of the roots σ_1 and σ_2 . This is a double zero of Eq. (75a), which implies that $k = \sigma$ simultaneously satisfies Eq. (84a), and its occurrence demands a certain relationship between the physical quantities involved, which can be combined into velocities U , $\sqrt{gh}$, and g/ω . The nature of this relationship is best seen by writing

$$\lambda = \sigma h$$

for the value of the root, so that, at critical, Eqs. (75a) and (84a) are

$$(gh)(\omega/g) + \lambda U - \sqrt{gh} \sqrt{(\lambda \tanh \lambda)} = 0 \quad (104a)$$

and

$$\sqrt{gh} \frac{\tanh \lambda + \lambda \sec^2 h^2 \lambda}{2\sqrt{(\lambda \tanh \lambda)}} = U. \quad (104b)$$

These two equations constitute, through the parameter λ , a relationship of the Eq. (119) between the velocities mentioned later. Geometrically, they represent a surface in the velocity space, and the critical value is attained when the point $(U, \sqrt{gh}, g/\omega)$ lies on this surface.

Two limiting cases can now be obtained from Eqs. (104a) and (104b) by taking appropriate limits.

- (a) When $h \rightarrow \infty$, Eq. (104a) requires $\lambda \sim gh/4U^2$. Substituting this into Eq. (104b), we have

$$\begin{aligned} (gh)(\omega/g) + (gh/4U) - \sqrt{gh} \sqrt{(gh/4U^2)} \\ \sim 0 \end{aligned} \quad (105)$$

which yields immediately

$$U = g/4\omega \quad (106)$$

in the limit. This is Kaplan's (1957) *critical result*.

- (b) When $\omega \rightarrow 0$, Eq. (104b) has only one real root, $\lambda = 0$. Substituting this into Eq. (104b), we retrieve Stoker's (1957) *critical value*

$$U = \sqrt{gh} \quad (107)$$

A simpler representation of the critical situation is achieved by introducing the dimensionless parameters

$$\alpha = U\omega/g \quad \text{and} \quad \beta = U/\sqrt{gh}. \quad (108)$$

In terms of these parameters, Eqs. (104a) and (104b) may be written as

$$(\alpha/\beta) + \lambda\beta - \sqrt{(\lambda \tanh \lambda)} = 0 \quad (109a)$$

and

$$\beta = \frac{\tanh \lambda + \lambda \sec^2 h^2 \lambda}{4\sqrt{(\lambda \tanh \lambda)}} \quad (109b)$$

though it is convenient to eliminate β and replace Eq. (109a) by

$$\alpha = \frac{\tanh^2 \lambda - \lambda^2 \sec^4 h^2 \lambda}{4 \tanh \lambda}. \quad (109c)$$

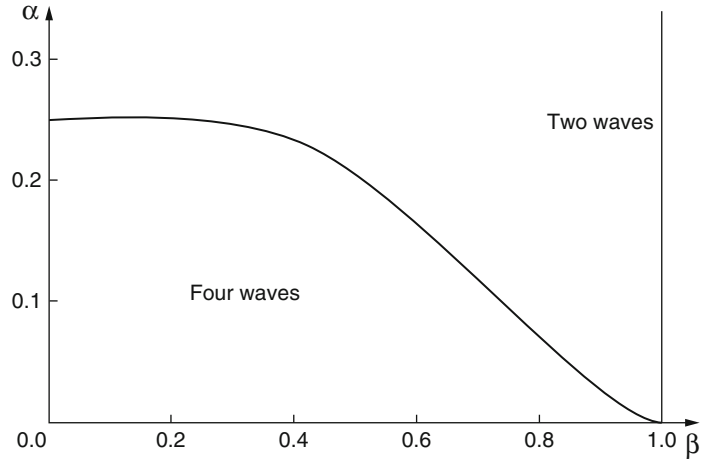
Equations (109b) and (109c) are the parametric equations of a curve in the (α, β) -plane that locates the velocity ratios (108) at critical. This curve is shown in Fig. 4; clearly it intercepts the α -axis at $\alpha = \frac{1}{4}$ and the β -axis at $\beta = 1$. It divides the quadrant into two regions: one, between the curve and the axes, in which there are four waves in the steady state, and the other, between the curve and infinity, in which only two waves occur.

To the left of the line $\beta = 1$ in this diagram, the transient part of the solution is $O(t^{-1/2})$, as given by Eq. (87); to the right, it is $O(1/|x|)$.

It remains to evaluate the solution $\eta(x, t)$ on the critical curve. We again express η in terms of the

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Fig. 4 Critical curve located in the parametric (α, β) -plane



integrals I and J , defined by Eqs. (72), (73), and (74). In the determination of I , given previously by Eq. (80), the contributions from the poles at $-s_1$ and $-s_2$ remain unchanged, while the term

$$H(U_c - U) \{ \phi(\sigma_1) e^{i\sigma_1 x} + \phi(\sigma_2) e^{i\sigma_2 x} \} \quad (110)$$

is replaced by a term giving the contribution from the double pole at $k = \sigma$. From a formula of Lighthill (1958, p. 43) analogous to (3.4.16) developed by Debnath (1994, p. 95), we find this to be

$$2\pi \operatorname{sgn} x \left[\sqrt{(\sigma \tanh \sigma h) / \left(\frac{d^2 m_1}{dk^2} \right)} \right]_{k=\sigma} x e^{i\sigma x}. \quad (111)$$

As regards J , it is clear that the contributions from the poles at $k = -s_1$ and $k = -s_2$ will be unaltered; these are given by the first term of Eq. (95) and by Eq. (98), respectively. Also, there will be the same transient behavior at the stationary point $-\rho$, expressed in the second term of Eq. (87). The remaining contribution to J comes from the neighborhood of $k = \sigma$, which is now simultaneously a pole and a stationary point. The dominant part of this can be written as

$$J_\sigma = \left\{ \int_{\sigma-\varepsilon}^{\sigma} \int_{\sigma}^{\sigma+\varepsilon} \right\} \frac{\sqrt{(k \tanh kh)} e^{ikx + im_1 t} dk}{\omega - m_1}, \quad (112)$$

where $0 < \varepsilon \ll 1$.

A convenient change of variable in these integrals is $s = \omega - m_1(k)$; then we note that $s = 0$ when $k = \sigma$, and $s = \varepsilon_1$, say, > 0 when $k = \sigma + \varepsilon$. This latter follows because near $k = \sigma$,

$$s = -\frac{1}{2}(k - \sigma)^2 m_1''(\sigma) \quad (113)$$

which is positive since $m_1''(\sigma) < 0$. Also, we find

$$\frac{ds}{dk} = |2sm_1''(\sigma)|^{1/2} \operatorname{sgn}(k - \sigma). \quad (114)$$

Substituting these into Eq. (112), we obtain

$$J_\sigma = 2e^{i\omega t} \int_0^{\varepsilon_1} |2m_1''(\sigma)|^{-1/2} \left[\sqrt{(k \tanh kh)} e^{ikx} \right]_{k=k^{-1}(s)} \frac{e^{-ist} ds}{|s|^{3/2}}. \quad (115)$$

The asymptotic value of this integral is unchanged if the upper limit is ∞ and hence, from a result of Lighthill (1958, p. 33), is

$$J_\sigma \sim 2 \left(-\frac{3}{2} \right)! |2m_1''(\sigma)|^{-1/2} \sqrt{(\sigma \tanh \sigma h)} \cdot t^{1/2} e^{i\omega t + i\sigma x + (i\pi/4)} \quad (116)$$

as $t \rightarrow +\infty$

It is now clear that the solution for $\eta(x, t)$ is singular on the critical (α, β) -curve, since the contributions from Eqs. (111) and (116) are both

asymptotically infinitely large; Eq. (111) represents a wave whose amplitude increases like x , while Eq. (116) is a wave whose amplitude is of order $t^{1/2}$ as $t \rightarrow \infty$.

This singular solution is not unexpected and is in accord with the findings of Stoker (1957) and Kaplan (1957). Since the critical curve corresponds to the coalescence of two poles, which in turn is equivalent to the reinforcement by superposition of two like waves, it would be natural to expect a *resonance-type effect*. Physically this situation would reveal itself through wave motions of large amplitude, which cannot come within the scope of linearized theory. Consequently, it would be necessary to include nonlinear terms in the original formulation of the problem in order to achieve a physically reasonable solution. The nonlinear theory of water waves at the resonant frequency will be discussed in the next section.

For certain limiting values of the three quantities U , ω and h , the problem just posed has been extensively discussed. Thus, for example, Stoker (1957, p. 177) presented a solution for the case $U = 0$, $h = \infty$, though an error in his analysis was later found and corrected by Miles (1962). Two other limiting cases, which have interesting features, occur (a) when $h = \infty$, with U and ω both finite, which has been treated by Kaplan (1957) and Cherkesov (1962), and (b) when $\omega = 0$, with U and h finite. This has also been solved by Stoker (1957) and, with the additional restriction $h = \infty$, by Wurtele (1955).

A special property of (a) and (b) is that in both situations, there is a critical value of the speed U , above and below which the wave systems have respectively quite a different character, and that when U assumes its critical value the solution is singular, due presumably to a breakdown of the linearization. Kaplan (1957) found that in case (a) the critical speed was

$$U = U_c = \frac{1}{4}(g/\omega), \quad (117)$$

while in case (b) Stoker (1957) showed it to be

$$U = U_c = \sqrt{(gh)}. \quad (118)$$

Taking the formulae (117) and (118) together, we set that there are three quantities involved having the dimensions of velocity, namely,

$$U, g/\omega, \quad \text{and} \quad \sqrt{(gh)}.$$

It has been proved earlier that in the general case when all three are finite, the critical situation can be expressed as a functional relationship between them of the form

$$F(U, g/\omega, \sqrt{(gh)}) = 0. \quad (119)$$

The function F is found and is seen to yield U as a single-valued function of g/ω , and $\sqrt{(gh)}$, with the formulae (97) and (98) emerging as limiting cases.

It may be appropriate to mention initial value problems (Debnath 1977; Debnath and Basu 1978) for magnetohydrodynamic surface waves on a running stream in the presence of a uniform magnetic field.

Nonlinear Theory of Water Waves by a Moving Pressure Distribution at Resonant Conditions

According to linearized theory (Stoker 1957; Debnath and Rosenblat 1969) of two-dimensional water waves generated by a steady surface pressure moving at a constant speed U , surface waves propagate in the far field only if the pressure travels at speed $U < c = \sqrt{gh}$ and appear behind the pressure distribution and no disturbance ahead. On the other hand, if $U > c$, no disturbance at all exists far from the pressure source, but only transients are generated, and they decay in the far field as time $t \rightarrow \infty$. However, at the *critical (or resonant)* speed $U = c$, the linear response becomes unbounded. The reason for this unbounded growth is that the energy transferred by the moving pressure to the water cannot be radiated away from the disturbance because the group velocity of the generated waves tends to U . Hence, the linearized theory cannot be

valid, however small the pressure field; the excited disturbance eventually attains a finite amplitude, and the linearized theory breaks down. Thus, the neglected nonlinear effect apparently plays a crucial role in the evolution of the response in the finite-amplitude regime. Akylas (1984a) developed a nonlinear theory near the critical speed that shows that the nonlinear response is of bounded amplitude and is governed by a *forced Korteweg–de Vries (fKdV) equation* subject to appropriate asymptotic initial data. His numerical analysis of the KdV equation reveals that a series of solitons are generated in front of the pressure distribution. This prediction is in theoretical agreement with the nonlinear study of Wu and Wu (1982) and also in good agreement with the experimental findings of Huang et al. (1982) who observed solitons in their experiments involving a ship moving in shallow water. We closely follow Akylas's work to discuss the nonlinear theory near the critical speed.

We formulate the initial value problem in water of finite depth h , $0 < z < h$, under the action of a surface pressure $p(x)$ moving at a constant speed U . In a frame of reference such that the pressure is stationary and a uniform stream of velocity U exists in the water, the velocity potential is $\Phi = U_x - \frac{1}{2}U^2t + \phi(x, z, t)$ and the free-surface elevation is $z = h + \eta(x, t)$. It is convenient to introduce nondimensional variables

$$(x^*, z^*) = \frac{1}{h}(x, z), \quad t^* = \frac{ct}{h}, \quad \eta^* = \frac{\eta}{a}, \\ \phi^* = \frac{c}{ag} \phi, \quad p^* = p/agp,$$

where a is the amplitude, g is the acceleration due to gravity, and ρ is the density of water. In terms of the preceding nondimensional variables, the classical nonlinear equations for water waves are, dropping the asterisks,

$$\phi_{xx} + \phi_{zz} = 0, \quad 0 < z < 1 + \varepsilon\eta; \quad (120)$$

$$\eta_t + \frac{U}{c}\eta_x + \varepsilon\phi_x\eta_x - \phi_z = 0, \quad \text{on } z = 1 + \varepsilon\eta; \quad (121)$$

$$\phi_t + \eta + \frac{U}{c}\phi_x + \frac{1}{2}\varepsilon(\phi_x^2 + \phi_z^2) = -p(x) \quad \text{on } z = 1 + \varepsilon\eta; \quad (122)$$

$$\phi_z = 0 \quad \text{on } z = 0. \quad (123)$$

It has already been shown that the amplitude of the linearized solution grows like $\varepsilon t^{1/3}$, while the dispersive effect, which is proportional to the square of the wave number, decays like $t^{-2/3}$; a balance is attained when $t = 0(\frac{1}{\varepsilon})$. Thus, the originally negligible nonlinear terms eventually become significant and the unbounded growth can be modified. An asymptotic theory that accounts for the finite-amplitude effects ($\varepsilon \ll 1$) is developed here. From the asymptotic analysis of the linearized solution, the slow time $T = \varepsilon t$ and the slow spatial variable $X = \varepsilon^{1/3}x$ can be adopted to discuss the nonlinear theory. Moreover, when $t = 0(\frac{1}{\varepsilon})$, $\phi = 0(\varepsilon^{-2/3})$ and $\eta = 0(\varepsilon^{-1/3})$, new rescaled variables, appropriate in the far field, are introduced as follows:

$$\phi = \varepsilon^{-2/3}\Phi(X, z, T) \quad \eta = \varepsilon^{-1/3}A(X, T), \quad (124a, b)$$

where Φ and A are assumed to be $0(1)$.

In terms of new variables, Eqs. (120), (121), (122), and (123) reduce to

$$\varepsilon^{2/3}\Phi_{XX} + \Phi_{zz} = 0, \quad 0 < z < +\varepsilon^{2/3}A, \quad (125)$$

$$\varepsilon^{2/3}A_X + \varepsilon^{4/3}(A_T + \gamma A_X + \Phi_X A_X) = \Phi_z \quad \text{on } z = 1 + \varepsilon^{2/3}A \quad (126)$$

$$A + \Phi_X + \frac{1}{2}\Phi_z^2 + \varepsilon^{2/3}(\Phi_T + \gamma\Phi_X + \Phi_X^2) \\ = -\varepsilon^{1/3}p(\varepsilon^{-1/3}X) \quad \text{on } z = 1 + \varepsilon^{2/3}A \quad (127)$$

$$\Phi_z = 0, \quad z = 0, \quad (128)$$

and the possibility of being slightly off resonant conditions, $U/c = 1 + \gamma\epsilon^{2/3}$, $\gamma = 0(1)$, is included.

In order to determine the asymptotic approximation of Eqs. (125), (126), (127), and (128) in the far field ($X = 0(1)$, $T = 0(1)$, $\epsilon \rightarrow 0$), we assume an expansion of Φ in powers of $\epsilon^{1/3}z$ as

$$\Phi(X, z, T) = F(X, T) - \frac{\epsilon^{2/3}z^2}{2!}F_{XX} + \frac{\epsilon^{4/3}z^4}{4!}F_{XXXX} + \dots, \quad (129)$$

which satisfies the Laplace Eq. (125) and the bottom boundary condition (128). Substituting Eq. (129) into the free-surface conditions (126) and (127) leads to two equations for F and A :

$$\begin{aligned} A_X + F_{XX} \\ + \epsilon^{2/3} \left(A_T + \gamma A_X + A_X F_X + A F_{XX} - \frac{1}{6} F_{XXX} \right) \\ = 0 \left(\epsilon^{4/3} \right) \end{aligned} \quad (130)$$

$$\begin{aligned} A + F_X + \epsilon^{2/3} \left(F_T + \gamma F_X + F_X^2 - \frac{1}{2} F_{XXX} \right) \\ = -\epsilon^{-1/3} p \left(\epsilon^{-1/3} X \right) + 0 \left(\epsilon^{4/3} \right). \end{aligned} \quad (131)$$

Further,

$$\epsilon^{-1/3} p \left(X/\epsilon^{1/3} \right) \rightarrow 2\pi\bar{p}(0)\delta(X) \quad \text{as } \epsilon \rightarrow 0, \quad (132)$$

where $\delta(X)$ is the Dirac delta function and $\bar{p}(k)$ is the Fourier transform of $p(x)$.

Differentiating Eq. (131) with respect to X , and using Eq. (130) with $F_X = -A + 0(\epsilon^{2/3})$, it turns out that A satisfies a single nonhomogeneous equation to leading order as

$$A_T + \gamma A_X - 2AA_X - \frac{1}{6}A_{XXX} = \pi\bar{p}(0)\delta'(X). \quad (133)$$

Clearly, the corresponding homogeneous equation is the Korteweg–de Vries equation. The appropriate initial conditions for the solution of Eq. (133) can be obtained by matching asymptotically the finite-amplitude response to the solution predicted by the linearized theory. At intermediate time scale $t \gg 1$ with $T \ll 1$, the linearized theory seems to be valid. So, according to the linearized theory with $\eta \sim -\phi_x$, it turns out that for $T \ll 1$,

$$A(X, T) \sim 6\pi\bar{p}(0) \left(\frac{T}{2} \right)^{1/3} f^\pm(\xi) \quad (134)$$

where $\xi = x(2/t)^{1/3}$,

$$f^+(\xi) = \int_{-\infty}^{-\xi} ds \int_{-\infty}^s Ai(\tau) d\tau \quad (\xi > 0) \quad (135)$$

$$f^-(\xi) = \int_{-\xi}^{\infty} ds \int_s^{\infty} Ai(\tau) d\tau \quad (\xi > 0) \quad (136)$$

and $Ai(\tau)$ is the standard Airy function.

For $\gamma \neq 0$, the appropriate solution of the linearized nonhomogeneous KdV equation has the form

$$\begin{aligned} A(X, T) = -3\bar{p}(0) \int_{\Gamma^\pm} \frac{\exp \left[is(X - \gamma T) - \frac{1}{6} is^3 T \right]}{s^3 + 6\gamma} ds, \\ X \lesseqgtr 0, \end{aligned} \quad (137)$$

where Γ^+ passes above the poles of the integrand, whereas Γ^- passes below. It follows from asymptotic evaluation of Eq. (137) that for $\gamma > 0$, the disturbance eventually decays as $T \rightarrow \infty$ for any fixed X . On the other hand, for $\gamma < 0$, the disturbance also decays when $X < 0$, and it represents a steady wave solution for $X > 0$ in the form

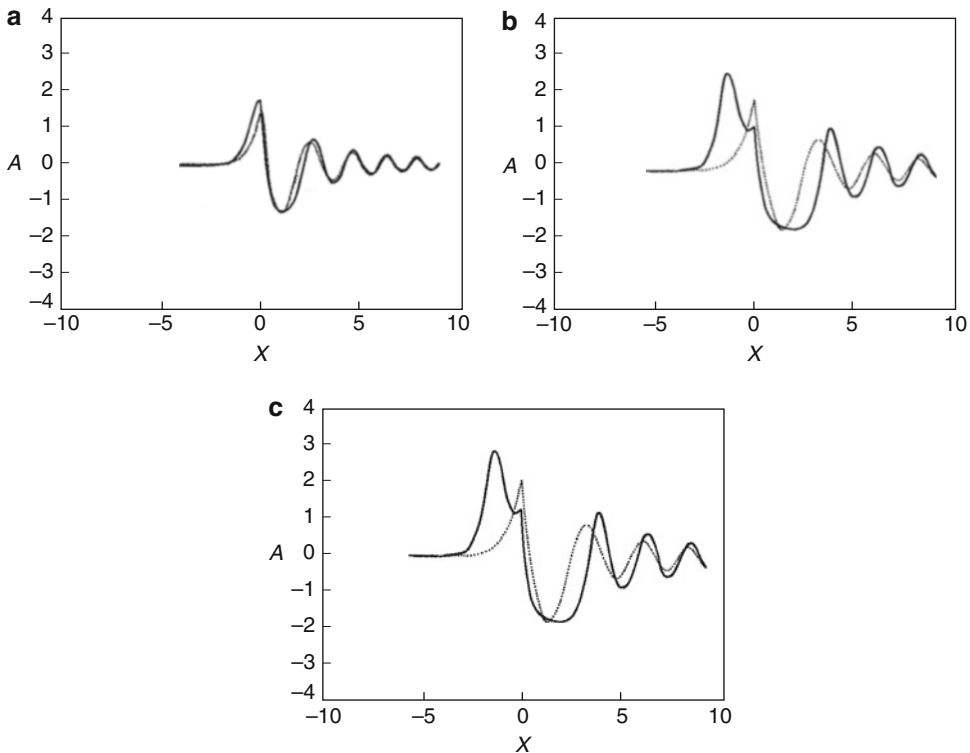
$$A(X, T) \sim 6\pi\bar{p}(0)(6\gamma)^{-1/2} \sinh(6\gamma)^{1/2} X, \quad (138)$$

These predictions are in agreement with those of the linearized theory.

Finally, it follows from the scaling Eq. (124a, b) that the far-field disturbance is of relatively large amplitude, but it remains bounded; the unbounded response obtained by the linearized theory is just the first term in the expansion of the nonlinear response for small T . The evolution of the disturbance for $T = 0(1)$ is governed by the inhomogeneous KdV Eq. (133) with the initial condition (134). The main question whether the nonlinear response evolves to solitons, or disperses out or even gives a cnoidal wave, still remains open. However, it can be shown by a numerical study that Eq. (133) admits a series of solitons that are generated in front of the pressure field. For $\gamma < 0$, the linear solution consists of a periodic sinusoidal wave in $X < 0$. The nonlinear evolution of the wave disturbance at resonant conditions ($\gamma = 0$) is shown in Fig. 5. A comparison with the corresponding linear solution (134) is also made. The major conclusion of this analysis

is that solitons are successively created when $X < 0$ and propagate in front of the pressure field; after each soliton reaches a certain equilibrium amplitude, a new soliton of slightly smaller equilibrium amplitude is generated. Immediately behind the pressure field, a long wave trough appears in order to balance more or less the amount of water needed to form the solitons. At larger distances from the pressure distribution, waves are highly oscillatory with a larger amplitude than predicted by the linearized theory. Thus, theoretical predictions concerning the generated solitons are in excellent agreement with the experimental findings of Huang et al. (1982).

On the other hand, in Debnath and Rosenblat's (1969) study of the transient development of two-dimensional surface waves generated by an oscillatory surface pressure on a running stream of finite depth, there is no linear steady-state response at the boundary curve separating two



Nonlinear Water Waves and Nonlinear Evolution Equations with Applications, Fig. 5 Evolution of wave disturbance at the critical condition ($\gamma = 0$):,

linear response, —, nonlinear response. (a) $T = 0:80$; (b) $T = 1:70$; (c) $T = 3:20$ (From Akylas 1984a)

kinds of possible steady states. This leads clearly to a resonance phenomenon. So, at the critical values separating these two possible steady states, the solution is singular in the sense that the amplitude of the generated waves grows without any bound. Thus, the failure of the linearized theory can be attributed to the fact that at a critical situation the group velocity of the waves, in a frame moving with pressure distribution, vanishes and thus the energy transferred to water cannot be radiated away from the source of disturbance. So it is necessary to develop a nonlinear theory close to resonant conditions in order to determine a physically realistic solution. Akylas (1984b) developed a nonlinear theory that deals with the evolution of surface waves generated by a moving oscillatory pressure near critical (or resonant) conditions.

Following Akylas's work (1984b) closely, we consider a two-dimensional harmonically oscillating pressure distribution of frequency ω traveling at a uniform speed U on the free surface of water of finite depth h , $-h \leq z \leq 0$. It is convenient to use a frame of reference moving with the pressure field so that the pressure is at rest and a uniform stream U exists in the water. In terms of nondimensional (asterisks) variables and parameters,

$$(x^*, z^*, h^*) = \frac{g}{U^2}(x, z, h), \quad t^* = \frac{g}{U}t, \quad \omega^* = \frac{U}{g}\omega \quad \varepsilon = \frac{a}{h},$$

classical water wave equations are, dropping the asterisks,

$$\nabla^2 \phi = 0, \quad -h \leq z \leq 0, \quad t > 0; \quad (139)$$

$$\eta_t + \eta_x + \varepsilon \phi_x \eta_x = \phi_z \quad \text{on} \quad z = \varepsilon \eta; \quad (140)$$

$$\begin{aligned} \phi_t + \eta + \phi_x + \frac{1}{2} \varepsilon (\phi_x^2 + \phi_z^2) \\ = -p(x) e^{i\omega t} + c.c. \quad \text{on} \quad z = \varepsilon \eta; \end{aligned} \quad (141)$$

$$\phi_z = 0 \quad \text{on} \quad z = -h, \quad (142)$$

where *c.c.* stands for the complex conjugate.

According to the linearized theory, the wave amplitude grows like $\varepsilon \sqrt{t}$ and the dispersive effects decay like $t^{-1/2}$ for large $t \gg 1$. Therefore, nonlinear effects are found to appear when the amplitude is $O(\sqrt{\varepsilon})$ at $T = O(\varepsilon^{-1})$. We now introduce the slow time and slow spatial variables by $T = \varepsilon t$ and $X = \sqrt{\varepsilon} x$, respectively. The finite-amplitude effect is assumed to be in the form of a wave packet of $O(\sqrt{\varepsilon})$ amplitude, modulated by an envelope depending on X and T . Accordingly, we assume the expansions for ϕ and η as

$$\phi = \varepsilon^{-1/2} \phi_0 + \varepsilon^{-1/2} \phi_1 + \phi_2 + \dots, \quad (143)$$

$$\eta = \varepsilon^{-1/2} \eta_1 + \eta_0 + \eta_2 + \dots, \quad (144)$$

where

$$\phi_0 = \Phi_0(X, z, T), \quad \eta_0 = A_0(X, T), \quad (145a, b)$$

$$\begin{aligned} \phi_1 &= \Phi(X, z, T) e^{i\theta} + c.c., \quad \eta_1 \\ &= A(X, T) e^{i\theta} + c.c., \end{aligned} \quad (146a, b)$$

$$\begin{aligned} \phi_2 &= \Phi_2(X, z, T) e^{2i\theta} + c.c., \quad \eta_2 \\ &= A_2(X, T) e^{2i\theta} + c.c., \end{aligned} \quad (147a, b)$$

and $\theta = k_0 x + \omega t$.

In view of nonlinear effects, the primary harmonic produces the mean-flow components ϕ_0 , η_0 and the second harmonics ϕ_2 , η_2 , respectively. On the other hand, the nonlinear interactions associated with the waves of wave numbers k_1 and k_2 , which are found behind the pressure disturbance, can be neglected because they on at higher time scales. The main objective of the nonlinear analysis is to determine an evolution equation for the wave envelope $A(X, T)$. In view of the fact that ϕ satisfies Eqs. (139) and (142), it turns out that

$$\begin{aligned} \Phi_0(X, z, T) &= a_0(X, T) \\ &+ \varepsilon \left[-\frac{1}{2} a_{0XX}(z+h)^2 \right] + \dots, \end{aligned} \quad (148)$$

$$\begin{aligned}\Phi(X, z, T) = & a(X, T) \frac{\cosh k_0(z+h)}{\cosh k_0 h} \\ & + \sqrt{\varepsilon} \left[-ia_X(z+h) \left\{ \frac{\sinh k_0(z+h)}{\cosh k_0 h} \right\} \right] \\ & + \varepsilon \left[-\frac{1}{2}a_{XX}(z+h)^2 \left\{ \frac{\cosh k_0(z+h)}{\cosh k_0 h} \right\} \right] \\ & + \dots, \end{aligned} \quad (149)$$

$$\Phi_2(X, z, T) = a_2(X, T) \frac{\cosh 2k_0(z+h)}{\cosh 2k_0 h} + \dots, \quad (150)$$

We next substitute Eqs. (143) and (144) into the free-surface conditions (140) and (141) to derive the following leading-order equations for the induced mean-flow and the second harmonics:

$$A_{0X} + ha_{0XX} - \left(\frac{2k_0}{\sigma} \right) (AA^*)_X = 0, \quad (151)$$

$$A_0 + a_{0X} + \left(\frac{k_0^2}{\sigma^2} - \sigma^2 \right) (AA^*) = 0, \quad (152)$$

$$\alpha\sigma - \beta k_0 \tanh(2k_0 h) = \frac{k_0^2}{\sigma}, \quad (153)$$

$$\alpha - 2\beta\sigma = \frac{1}{2} \left(3\sigma^2 - \frac{k_0^2}{\sigma^2} \right), \quad (154)$$

where $\alpha_2 = i\beta A^2$, $\Phi_2 = \alpha A^2$, and $\sigma = \omega + k_0$.

The evolution equation for the wave envelope $A(X, T)$ is obtained by using the two equations for the primary harmonic derived from the free-surface boundary conditions (140) and (141). Then, elimination of A gives the following equation for $A(X, T)$:

$$\begin{aligned}A_T - \frac{i\alpha_1}{2} A_{XX} + ik_0 A a_{0X} + \left(\frac{i}{2\sigma} \right) (\sigma^4 - k_0^2) A A_0 \\ + \frac{i}{2} \left[-4k_0^2 \sigma + \frac{\alpha}{\sigma} (\sigma^4 - k_0^2) \right. \\ \left. + \beta (2\sigma^2 k_0 \tanh 2k_0 h) - 4k_0^2 \right] A^2 A^* \\ = \left(\frac{i\sigma}{2} \right) (\varepsilon^{-1/2}) p(X \varepsilon^{-1/2}) \exp(-ik_0 x + i\gamma T), \end{aligned} \quad (155)$$

where

$$\alpha_1 = f''_-(k_0), \quad f_-(k) = k - (k \tanh kh)^{1/2}, \quad (156)$$

and the possibility that the frequency of the oscillating pressure is off the exact critical value ω by $\varepsilon\gamma$, $\gamma = 0(1)$, has been incorporated.

Equations (151), (152), and (153) can be solved by using the condition $a_0(X, T) \rightarrow 0$ as $|X| \rightarrow \infty$ to obtain the mean-flow component as

$$A_0(X, T) = \left[\frac{2k_0}{\sigma(1-h)} - \frac{h}{(1-h)} \left(\sigma^2 - \frac{k_0^2}{\sigma^2} \right) \right] AA^* \quad (157)$$

$$a_{0X} = \left[\left(\sigma^2 - \frac{k_0^2}{\sigma^2} \right) \frac{1}{(1-h)} - \frac{2k_0}{\sigma(1-h)} \right] AA^*. \quad (158)$$

In view of the limit

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{\varepsilon}} p(X/\sqrt{\varepsilon}) \exp(-ik_0 X/\sqrt{\varepsilon}) = 2\pi \bar{p}_0 \delta(X), \quad (159)$$

and the transformation $B = A \exp(-i\gamma T)$, the evolution Eq. (155) reduces to the form

$$\begin{aligned}iB_T - \gamma B + \left(\frac{\alpha_1}{2} \right) B_{XX} - \left(\frac{\alpha_2}{2} \right) B^* B^2 \\ = -\pi \bar{p}_0 \delta(X), \end{aligned} \quad (160)$$

where

$$\begin{aligned}\alpha_2 = & \frac{2k_0}{1-h} \left(\sigma^2 - \frac{k_0^2}{\sigma^2} - \frac{2k_0}{\sigma} \right) \\ & + \frac{(\sigma^4 - k_0^2)}{\sigma(1-h)} \left[\frac{2k_0}{\sigma} - h \left(\sigma^2 - \frac{k_0^2}{\sigma} \right) \right] \\ & - 4k_0^2 \sigma + \left(\frac{\alpha}{\sigma} \right) (\sigma^4 - k_0^2) \\ & + 2\beta (\sigma^2 k_0 \tanh 2k_0 h - 2k_0^2), \end{aligned} \quad (161)$$

where α and β are real constants determined by Eqs. (153) and (154). Clearly, Eq. (160) is the forced nonlinear Schrödinger equation with a

minor modification due to assumed frequency detuning. In fact, the right-hand side of Eq. (160) represents the forcing effect of the applied pressure that includes the Dirac delta function $\delta(X)$ as a factor.

The evolution Eq. (160) has to be solved subject to the initial condition determined by asymptotically matching the nonlinear response to the linearized far-field response. The appropriate initial condition can be written as

$$\begin{aligned} B(X, T) &\sim -\sigma\bar{p}_0 \left(\frac{T}{2\alpha_1}\right)^{1/2} G^\pm(\xi), \quad X \gtrless 0, \quad T \\ &\rightarrow 0, \end{aligned} \quad (162)$$

where

$$\begin{aligned} G^\pm(\xi) &= \int_{C_0^\pm} s^{-2} \exp[i(s\xi - s^2)] ds, \quad (\xi \gtrless 0), \xi \\ &= \sqrt{\frac{x^2}{t}} \left(\frac{2}{\alpha_1}\right)^{1/2}, \quad \alpha_1 > 0. \end{aligned} \quad (163)$$

The initial condition (162) is an even function of X , and since Eq. (160) involves only second spatial derivatives, clearly, B is an even function of X for all values of T . Evidently, Eq. (160), subject to the asymptotic initial condition (162), is equivalent to the equation

$$\begin{aligned} iB_T - \gamma B + \left(\frac{\alpha_1}{2}\right) B_{XX} - \left(\frac{\alpha_2}{2}\right) B^* B^2 &= 0, \\ X > 0, \quad T > 0, \end{aligned} \quad (164)$$

$$B_X = -\left(\frac{\sigma\pi}{\alpha_1}\right) \bar{p}_0(k), \quad X = 0, \quad T > 0, \quad (165)$$

$$\begin{aligned} B &\sim -\sigma\bar{p}_0(k) \left(\frac{T}{2\alpha_1}\right)^{1/2} G^+(\xi), \quad X > 0, \\ T &\rightarrow 0. \end{aligned} \quad (166)$$

The system Eqs. (164), (165), and (166) represent an initial-boundary-value problem that is

similar to that governing the nonlinear problem of an acoustic duct induced by an oscillating piston near a cutoff frequency. So, the existence of a finite-amplitude steady-state solution close to the critical conditions can be investigated from the time-independent solutions of Eqs. (164), (165), and (166). It has been shown by Barnard et al. (1977) that it is possible to determine a steady-state solution that either represents a uniform wave or decays to zero at infinity ($X \rightarrow \infty$). Furthermore, the number and the nature of the possible steady states depend on the sign of $\alpha_1\alpha_2$ and the value of the detuning parameter γ . In particular, for an infinitely deep water ($h \rightarrow \infty$), $\omega = k_0 = \frac{1}{4}$, $\alpha_1 = 2$, $\alpha_2 = -4k_0^2\sigma = -\frac{1}{8}$, so that $\alpha_1\alpha_2 = -\frac{1}{4} < 0$, it can be shown that only one steady-state solution describing a uniform wave is possible for all values of γ . This finding is in excellent agreement with the result of Dagan and Miloh (1982).

On the other hand, Aranha et al. (1982) have made a numerical study of the preceding initial-boundary-value problem and have shown that the steady-state solution found by Dagan and Miloh (1982) is never reached. Indeed, the approach to a steady state is determined by the sign of $\alpha_1\alpha_2$; if $\alpha_1\alpha_2 > 0$, a steady state is attained, while if $\alpha_1\alpha_2 < 0$, the solution is bounded but remains always unsteady. In the present model, α_1 is always positive while α_2 is positive or negative depending on whether the Froude number, $F = (U/\sqrt{gh})$, is less or greater than 0.545. Thus, for water of finite depth, only if $0.545 < F < 1$ ($\omega < 0.191$) a steady state is attained. Of course, this conclusion is based upon the present inviscid weakly nonlinear theory. However, viscous and higher-order nonlinear effects could ultimately lead to a steady-state solution. Finally, it is pertinent to point out that as the Froude number $F \rightarrow 1$ ($h \rightarrow 1$, $\omega \rightarrow 0$, $k_0 \rightarrow 0$), the present theory breaks down, and the long-wave theory of Akylas (1984a) becomes relevant, indicating that, as $F \rightarrow 1$, there is no steady-state solution either. For further details, the reader is referred to Akylas's (1984a) paper.

Recently, Katsis and Akylas (1987) made an interesting investigation of the three-dimensional waves produced by a moving pressure field acting on the free surface of an inviscid fluid of depth h . In terms of the nondimensional variables

$$(x^*, y^*, z^*) = \frac{1}{l}(x, y, z), \quad \eta^* = \frac{\eta}{a}, \quad \phi^* = \frac{c}{agl}\phi, \quad p^* = \frac{p}{ag\rho},$$

the nondimensional form of the three-dimensional equation and the free-surface boundary conditions are, dropping the asterisks,

$$\delta^2(\phi_{xx} + \phi_{yy}) + \phi_{zz} = 0, \quad 0 < z < 1 + \varepsilon\eta, \quad t > 0; \quad (167)$$

$$\eta_t + F\eta_x + \varepsilon(\phi_x\eta_x + \phi_y\eta_y) - \delta^{-2}\phi_z = 0 \quad \text{on} \quad z = 1 + \varepsilon\eta; \quad (168)$$

$$\begin{aligned} \phi_t + F\phi_x + \eta + \frac{1}{2}\varepsilon(\phi_x^2 + \phi_y^2 + \delta^{-2}\phi_z^2) \\ = -p\left(\frac{x}{\delta} \cdot \frac{h}{L}, \frac{y}{\delta} \cdot \frac{h}{D}\right) \quad \text{on} \quad z = 1 + \varepsilon\eta; \end{aligned} \quad (169)$$

$$\phi_z = 0 \quad \text{on} \quad z = 0, \quad (170)$$

where $\phi = \phi(x, y, z)$ is the velocity potential, $\eta = \eta(x, y, t)$ is the free-surface elevation, l is the typical wavelength, a is the wave amplitude, $c = \sqrt{gh}$, L and D represent the typical dimensions of the pressure field in the streamwise and spanwise directions, respectively, $\delta = h/l$ is a measure of dispersion, $\varepsilon = a/h$ is a measure of nonlinearity, and F is the Froude number.

The case of $\varepsilon = 0$ describes the linearized water waves due to the moving disturbance with the dispersion relation

$$\omega^2 = \delta^{-1}\kappa \tanh \delta\kappa \quad (171)$$

where $k = |\kappa|$ is the magnitude of the wave number vector. In the absence of the transient motions, if the wave number vector κ is inclined at an angle α with the streamwise direction, waves persist only if it is stationary in the frame of

moving disturbance (Lighthill 1978, section 3.9), provided

$$\sqrt{\delta}F\kappa \cos \alpha = (\kappa \tanh \delta\kappa)^{1/2}. \quad (172)$$

This represents the wave number in the direction α for any Froude number F and is satisfied for long waves ($\delta \ll 1$) If $F = 1 + O(\delta^2)$ and $\cos \alpha = 1 + O(\delta^2)$. This means that the corresponding wave frequency in the frame of the disturbance is $O(\delta^2)$. However, in the limit as $\delta \rightarrow 0$, nonlinear effects can hardly be neglected. In order to describe the nonlinear wave response near the critical Froude number $F \sim 1$, we write $F = 1 + \lambda\delta^2$ ($\lambda = O(1)$) and define a stretched spanwise coordinate $Y = \delta y$ and a slow time scale $T = \delta^2 t$. Katsis and Akylas (1987) have shown that the nonlinear response satisfies a forced Kadomtsev–Petviashvili (KP) equation:

$$\begin{aligned} \eta_{xT} + \lambda\eta_{xx} - \frac{3}{4}(\eta^2)_{xx} - \frac{1}{6}\eta_{xxxx} - \frac{1}{2}\eta_{yy} \\ = (2\varepsilon)^{-1}p_{xx}. \end{aligned} \quad (173)$$

This is the three-dimensional version of the *forced KdV equation* derived by Akylas (1984a). The steady-state version of Eq. (173) has been derived by Mei (1976) for a slender ship moving with a transcritical velocity under the assumption of existence of a steady-state solution.

It has also been shown by Katsis and Akylas (1987) that, in a channel of finite width $2w$, three-dimensional effects are negligible if $wa/h^2 \ll 1$ and the corresponding two-dimensional response satisfies the forced KdV equation. On the other hand, if $wa/h^2 = O(1)$, numerical analysis of the full KP equation reveals that three-dimensional effects are significant and indicates that a series of straight-crested solitons are found to radiate periodically ahead of the disturbance and three-dimensional waves are found to form behind the disturbance. This prediction is in agreement with experimental findings of Ertekin et al. (1984). In an unbounded channel $wa/h^2 \gg 1$, at critical situation, an unsteady three-dimensional disturbance is found to appear ahead of the pressure field.

The Nonlinear Schrödinger Equation and Evolution of Wave Packets

We consider the evolution of wave packets for gravity waves on the surface of water of uniform, but finite depth ($b \equiv 0$). We retain the shallowness parameter δ in Eqs. (56), (57), (58), and (59) and consider $\varepsilon \rightarrow 0$ for $\sqrt{\delta}$ fixed so that Eqs. (56), (57), (58), and (59) reduce to

$$\phi_{zz} + \delta \phi_{xx} = 0, \quad \text{in } 0 < z < 1 + \varepsilon\eta, \quad t > 0 \quad (174)$$

$$\phi_t + \eta + \frac{\varepsilon}{2}(\phi_x^2) + \frac{\varepsilon}{2\delta}\phi_z^2 = 0, \quad z = 1 + \varepsilon\eta, \quad (175)$$

$$\phi_z - \delta(\eta_t + \varepsilon\phi_x\eta_x) = 0, \quad z = 1 + \varepsilon\eta, \quad (176)$$

$$\phi_z = 0 \quad \text{on } z = 0. \quad (177)$$

In consistent with modulated waves from a Fourier integral representation of a wave described by

$$\phi(x, t) = \int_{-\infty}^{\infty} F(k) \exp[i(kx - \omega t)] dk, \quad (178)$$

where $F(k)$ given by $\omega = \omega(k)$ is also a given dispersion relation. We assume that the main contribution to the wave profile comes from the neighborhood of the carrier wave number $k = k_0$ so that $k - k_0 = \varepsilon\kappa$ and $\omega = \omega(k)$ can be expanded in a Taylor series about $k = k_0$ up to the term ε^2 . It turns out that, $\varepsilon \rightarrow 0$,

$$\phi(x, t) \sim A(x, \tau) \exp[i(k_0x - \omega_0t)], \quad (179)$$

where $A(x, \tau)$ is known, $x = \varepsilon(x - c_g t)$, $\omega_0 = \omega(k_0)$, $\tau = \varepsilon^2 t$, and $c_g = \omega'(k_0)$ is the group velocity. We recognize here that the relevant scales seem to be associated with both ε and ε^2 , and hence, we introduce

$$\xi = x - c_p t, \quad X = \varepsilon(x - c_g t), \quad \tau = \varepsilon^2 t, \quad (180)$$

where c_p is the phase velocity of the wave.

Following Johnson (1997), it turns out the asymptotic solution takes the form

$$\eta_0 = A_0(X, \tau) \exp(ik\xi) + c.c. \quad (181)$$

$$\phi_0 = f_0(X, \tau) + F_0(X, z, \tau) \exp(ik\xi) + c.c. \quad (182)$$

where $c.c.$ stands for the complex conjugate of the terms in $\exp(ik\xi)$. The asymptotic analysis leads (see Debnath (2012), Example 12.9.1) to an equation for $A_0(X, \tau)$ in the form

$$-2ikc_p A_{0\tau} + \alpha A_{0XX} + \beta A_0 |A_0|^2 = 0, \quad (183)$$

where

$$\alpha = c_g^2 - (1 - \delta k \tanh s) \sec^2 h^2 s, \quad s = k\sqrt{\delta}, \quad (184)$$

$$\beta = k^2 c_p^{-2} \left[\frac{1}{2} \left(1 + 9 \coth^2 s - 13 \operatorname{sech}^2 s - \tanh^4 s \right) - 2(2c_p + c_g \operatorname{sech}^2 s)^2 (1 - c_g^2)^{-1} \right]. \quad (185)$$

Equation (183) is well known as the *non-linear Schrödinger (NLS) equation* for the evolution of nonlinear water waves, and it is one of the completely integrable types of equations. It is easy to check that $\alpha > 0$ for all $s = k\sqrt{\delta}$, but β changes its sign from positive to negative as s decreases its value across $s = k\sqrt{\delta} \approx 1.363$. It is important to point out that the nature of the solutions of Eq. (183) depend on the signs of α and β . Indeed, the condition $\beta\alpha^{-1} > 0$ ($\alpha\beta > 0$) ensures the existence of a solitary wave solution of the NLS Eq. (183), and the soliton decays to zero at infinity. There exists some experimental and numerical evidence in support of the solitary wave solution. On the other hand, when

$$\frac{\beta}{\alpha} < 0 \quad (\alpha\beta < 0),$$

no such solitary wave solution exists. In other words, the solution grows exponentially as $\tau \rightarrow \infty$, that is, the solution becomes unstable.

Higher-Order Nonlinear Schrödinger Equations

It is well known from the theory and experiments of Benjamin and Feir (1967) that a finite-amplitude uniform train of surface gravity waves is unstable to modulational perturbations with sufficiently long wavelengths. According to Yuen and Lake (1982), the envelope of a weakly nonlinear wave packet in deep water is governed by the nonlinear Schrödinger equation. This equation seems to provide an accurate description of the evolution of a wave packet of small wave steepness $\varepsilon = ak$ only for a limited time, at most $O(\varepsilon^{-2})$ wave periods. Feir (1967) confirmed experimentally this restriction on the validity of the NLS equation and confirmed that an initially symmetric wave packet of uniform frequency and moderate wave steepness eventually loses its symmetry as it propagates away from the wave maker and splits into two prominent groups of different frequencies. Later on Su (1982a, b) observed similar phenomena in his more comprehensive experiments with initially symmetric wave packets of uniform frequency and various durations. For short pulses his results extend those of Feir and reveal further information on group splitting and frequency downshift in the leading wave group. Both Lake et al. (1977) and Melville (1982) performed the instability experiments of a uniform wave train with the typical value of wave steepness $ka > 0.2$. The side-band disturbances, if they were of equal magnitude initially, were found to grow at equal rate only for a limited time. As nonlinear effects become more and more significant, the lower side band grows faster and attains a greater maximum than the upper side band, whereas the carrier wave drops to a minimum. Before attaining these extrema, local breaking was observed by Melville and probably occurred in Lake et al.'s experiments. Lake et al. (1977) suggested that the unequal growth is possibly responsible for the downward shift of the spectral peak of wind waves with increasing fetch. Janssen (1983a, b) investigated the long time

behavior of the Benjamin–Feir modulational instability. His study based on the NLS equation exhibits the Fermi et al. (1955) *recurrence phenomena* and is in qualitative agreement with experimental work of Lake et al. (1977) and the numerical computation of Yuen and Ferguson (1978a, b). Jassen's analysis also reveals that other effects including dissipation are likely to explain the observed frequency downshift in the experiments of Lake et al. (1977).

For small amplitudes, the NLS equation was derived by several authors (see Debnath 1994) to describe the evolution of a wave train, and this equation seems to be correct to third order in the wave steepness. Subsequently, Longuet-Higgins' (1978a, b) work on the normal mode perturbation analysis of the fully nonlinear water wave problems in deep water confirmed that the NLS equation provides an adequate description for all but the smallest wave steepness. However, the preceding NLS equation is found to compare rather unfavorably with the exact results of Longuet-Higgins for $\varepsilon > 0.10$. Later on, Dysthe (1979) has shown that a significant improvement can be made by extending the perturbation analysis to $O(\varepsilon^4)$. He derived the evolution to fourth order in the amplitude of the wave potential leading to a major improvement in agreement with Longuet-Higgins' analysis. In nondimensional form the complex amplitude A of the first harmonic of the Stokes wave, the average value $\bar{\phi}$ of the velocity potential ϕ over one wave period, and the average surface elevation $\bar{\zeta}$ satisfy Dysthe's equations:

$$\begin{aligned} 2i\left(A_t + \frac{1}{2}A_x\right) + \frac{1}{2}A_{yy} - \frac{1}{4}A_{xx} - A|A|^2 \\ = \frac{i}{8}(A_{xxx} - 6A_{xyy}) + \frac{3i}{2}A(AA_x^* - A^*A_x) \\ - \frac{1}{2}i|A|^2A_x + A(\bar{\phi}_x - i\phi_z), \end{aligned} \quad (186)$$

$$\bar{\phi}_t + 2\bar{\zeta} = 0, \quad (187)$$

$$\bar{\phi}_z - 2\bar{\zeta}_t = (|A|^2)_x. \quad (188)$$

The right-hand side of Eq. (186) is made up of contributions of $O(\varepsilon^4)$. Two linear terms on the right-hand side of Eq. (186) are simply corrections

to the dispersive effects of the NLS equation. In the absence of the fourth-order effects, the whole right-hand side of Eq. (186) becomes zero so that Eq. (186) reduces to the ordinary NLS equation derived by many authors in the 1960s and 1970s. However, the fourth-order NLS equation was first derived by Dysthe (1979) to eliminate the weakness of the ordinary NLS equation. His fourth-order NLS equation gives a significant improvement on the results relating to the stability of finite-amplitude waves. The dominant new effect introduced by adding terms of order ε^4 to the ordinary NLS equation is the mean-flow response to nonuniformities of the radiation stress caused by modulation of a finite-amplitude wave. Moreover, the horizontal component of the mean flow along the direction of wave propagation causes a slowly varying Doppler shift of the wave as represented by the last term in Eq. (186). The Doppler shift seems to have a detuning effect on the modulational instability. Among the new terms in Eq. (186), only the term $A\bar{\phi}_x$ makes significant contribution to the stability of a finite-amplitude wave. Thus, for stability analysis, it is sufficient to use the following simplified Dysthe's equations:

$$2i\left(A_t + \frac{1}{2}A_x\right) + \frac{1}{2}A_{yy} - \frac{1}{4}A_{xx} = A(|A|^2 + \bar{\phi}_x), \quad z = 0 \quad (189)$$

$$\bar{\phi}_z - (|A|)_x = 0 \quad \text{on } z = 0 \quad (190)$$

$$\nabla^2 \bar{\phi} = 0 \quad \text{for } z \leq 0 \quad (191)$$

In deep water so that $kh = O(ka)^{-1} \gg 1$, Dysthe's equations for A and ϕ in dimensional form are

$$\begin{aligned} i\left(A_t + \frac{\omega}{2k}A_x\right) - \frac{\omega}{8k^2}A_{xx} - \frac{1}{2}\omega k^2|A|^2A \\ = \frac{i}{16}\frac{\omega}{k^2}A_{xxx} + \frac{i\omega k}{4}A^2A_x^* - \frac{3i\omega k}{2}|A|^2A_x \\ + kA[\phi_x]_{z=0} = 0, \end{aligned} \quad (192)$$

$$\nabla^2 \phi = 0 \quad -h < z < 0, \quad (193)$$

$$\phi_z = \frac{\omega}{2}(|A|^2)_x \quad \text{on } z = 0, \quad (194)$$

$$\phi_z = 0 \quad \text{on } z = -h, \quad (195)$$

where ϕ is the potential of the induced mean current. The first four terms in Eq. (192) constitute the NLS equation in the fixed frame of reference. All terms on the right-hand side of Eq. (192) are $O(\varepsilon^4)$. Lo and Mei (1985) made a numerical study of the water wave modulation based on Dysthe's fourth-order NLS system (192), (193), (194) and (195). Their analysis shows a reasonable agreement with the recent experimental results. So, the Dysthe's equation represents a useful model for predicting the long-term evolution of the narrow-banded weakly nonlinear waves.

Recently, several authors including Janssen (1983a, b), Stiassnie (1983), and Hogan (1985) derived the fourth-order NLS equation for gravity waves and for capillary-gravity waves. Stiassnie showed that the Dysthe's fourth-order NLS equation is a special case of the more general Zakharov equation that is free from the narrow spectral width assumption. Hogan (1985) extended Stiassnie's analysis to deep water capillary-gravity wave packets. Based on the Zakharov integral equation, Hogan derived the fourth-order NLS equation. In addition to the leading-order dispersive and nonlinear terms in the NLS equation, Hogan's fourth-order NLS equation features certain nonlinear modulation terms and a nonlocal term that describes the coupling of the envelope with the induced mean flow. Hogan's analysis also reveals that there is a band of stable capillary-gravity waves at the fourth order; such a band is known to exist at third order. The effects of the mean flow for pure capillary waves are, in general, of opposite sign to those of pure gravity waves. Hogan showed that the second-order corrections to the first-order stability properties depend on the interaction between the mean flow and the envelope frequency-dispersion term involved in the fourth-order NLS equation. Indeed, Hogan's derivation is based on the Zakharov integral equation under the assumption of a narrow band of waves and including the interactions of capillary-gravity waves. His fourth-order evolution equation for the complex wave amplitude $a(x, t)$ for capillary-gravity waves in deep water is given in the form

$$2i(a_t + c_g a_x) + p a_{xx} + q a_{yy} - \gamma |a|^2 a = -i(r a_{xxx} + s a_{xyy}) + i(v |a|^2 a_x - u a^2 a_x^*) + a \phi_x, \quad (196)$$

where the group velocity c_g is

$$c_g = \left(\frac{\omega_0}{2k_0} \right) \left(\frac{1+3\kappa}{1+\kappa} \right), \quad \kappa = \frac{Sk_0^2}{g},$$

and the other coefficients in Eq. (196) are given by

$$\begin{aligned} p &= \frac{(3\kappa^2 + 6\kappa - 1)}{4(1+\kappa)^2}, & q &= \frac{1}{2} \left(\frac{1+3\kappa}{1+\kappa} \right), & r &= -\frac{(1-\kappa)(1+6\kappa+\kappa^2)}{8(1+\kappa)^3} \\ s &= \frac{(3+2\kappa+3\kappa^2)}{4(1+\kappa)^2}, & \gamma &= \frac{8+\kappa+2\kappa^2}{8(1-2\kappa)(1+\kappa)}, & u &= \frac{(1-\kappa)(8+\kappa+2\kappa^2)}{16(1-2\kappa)(1+\kappa)^2}, \\ v &= \frac{3(4\kappa^4 + 4\kappa^3 - 9\kappa^2 + \kappa - 8)}{8(1+\kappa)^2(1-2\kappa)^2}. \end{aligned}$$

For pure capillary waves, Eq. (196) reduces to the form

$$\begin{aligned} 2i\left(a_t + \frac{3}{2}a_x\right) + \frac{3}{4}a_{xx} + \frac{3}{2}a_{yy} + \frac{1}{8}|a|^2 a \\ = -\frac{i}{8}(a_{xxx} + 6a_{xyy}) + \frac{i}{8}(3|a|^2 a_x - a^2 a_x^*) + a \bar{\phi}_x. \end{aligned} \quad (197)$$

Neglecting the right-hand side of Eq. (197) leads to the third-order envelope equation for capillary-gravity waves in deep water. Hogan showed that the second-order corrections to the first-order stability properties depend essentially on the interaction between the mean flow and the envelope frequency-dispersion term in the governing equation.

From a physical point of view, two kinds of interaction influence the stability of a nonlinear wave train. First, it was shown by Lighthill (1965) that the relative signs of the envelope frequency-dispersion term $p a_{xxx}$ and the nonlinear term $(-\gamma |a|^2 a)$ govern the stability of the solution. There exists a band of stable waves where the product pr is positive. Second, the corrections to the stability characteristics that occur at higher order can be found to arise from an interaction between the mean-flow term $(a \bar{\phi}_x)$ and the frequency-dispersion term $(p a_{xxx})$. It is well known that the frequency-dispersion term for gravity waves is of opposite sign for capillary waves. For gravity

waves the detuned nonlinear term $A(|a|^2 + \bar{\phi}_x)$ balances the dispersion term $-\frac{1}{4}A_{xx}$ as shown by Dysthe (1979). This leads to the fact that a resonant quartet is maintained and energy is transferred to growing side bands. On the other hand, for capillary waves, the nonlinear term is of the form $a(\frac{1}{8}|a|^2 - \bar{\phi}_x)$ and the dispersion term is $-\frac{3}{4}a_{xx}$ so that the detuning arising from the mean flow has the opposite sign. His theoretical predictions are also in good agreement with recent computational results for sufficiently small values of the wave steepness $\varepsilon = ak$.

Based on the fourth-order NLS equation due to Hogan (1985), Akylas et al. (1998) recently derived an asymptotic expression for small-amplitude capillary-gravity solitary waves in deep water that exhibits the algebraic decay like x^{-2} of the tails of these waves. This decay at infinity is solely due to induced mean flow that is not accounted for in the NLS equation. Their numerical computations based on full nonlinear water wave equations also confirm the algebraic decay of the solitary wave tails in deep water.

The Davey-Stewartson (DS) Equations in Water of Finite Depth

The ordinary NLS equation is used to describe the situation where the wave propagates only in one direction and for which the wave profile evolves

only in the same direction. Such a wave is generated by an initial profile (at $t = 0$) in the form

$$A(\varepsilon x) \exp(ikx) + c.c.$$

We consider Davey and Stewartson's (1974) analysis for the development of a wave generated by the initial profile (at $t = 0$) in the form

$$A(\varepsilon x, \varepsilon y) \exp(ikx) + c.c.$$

According to Davey and Stewartson, the slow (or weak) dependence occurs in both the x - and y -directions, but the rapid oscillation is only in the x -direction and the wave packet propagates in the x -direction with a slowly varying nature in both x - and y -directions. The group velocity is assumed to be still associated with the propagation in the x -direction. The governing equations are, in the nondimensional form,

$$\begin{aligned} \delta(\phi_{xx} + \phi_{yy}) + \phi_{zz} &= 0, \quad 0 < z < 1 + \varepsilon\eta, \\ t &> 0, \end{aligned} \quad (198)$$

$$\eta_t + \varepsilon(\phi_x \eta_x + \phi_y \eta_y) - \delta^{-1} \phi_z = 0 \quad \text{on } z = 1 + \varepsilon\eta, \quad (199)$$

$$\phi_t + \eta + \frac{1}{2} \varepsilon (\nabla \phi)^2 + \frac{\varepsilon}{2\delta} \phi_z^2 = 0 \quad \text{on } z = 1 + \varepsilon\eta, \quad (200)$$

$$\phi_z = 0 \quad \text{on } z = 0. \quad (201)$$

It is convenient to introduce new variables

$$\begin{aligned} \xi &= x - c_p t, \quad X = \varepsilon(x - c_g t), \quad Y = \varepsilon y, \\ \tau &= \varepsilon^2 t \end{aligned} \quad (202)$$

so that Eqs. (198), (199), (200), and (201) can be readily be transformed into a system:

$$\phi_{zz} + \delta[\phi_{\xi\xi} + 2\varepsilon\phi_{\xi X} + \varepsilon(\phi_{XX} + \phi_{YY})] = 0, \quad (203)$$

$$\begin{aligned} \phi_z &= \delta \left[(\varepsilon^2 \eta_t - \varepsilon c_g \eta_X - c_p \eta_\xi) \right. \\ &\quad \left. + \varepsilon(\phi_\xi + \varepsilon\phi_X)(\eta_\xi + \varepsilon\eta_X) + \varepsilon^3 \phi_Y \eta_Y \right] \text{ on } \\ z &= 1 + \varepsilon\eta, \end{aligned} \quad (204)$$

$$\begin{aligned} \varepsilon^2 \phi_\tau - \varepsilon c_g \phi_X - c_p \phi_\xi + \eta + \frac{1}{2} \varepsilon (\phi_\xi + \varepsilon\phi_X)^2 \\ + \frac{\varepsilon^2}{2} \phi_Y^2 + \frac{\varepsilon}{2\delta} \phi_z^2 &= 0, \\ z &= 1 + \varepsilon\eta \end{aligned} \quad (205)$$

$$\phi_z = 0 \quad \text{on } z = 0. \quad (206)$$

Retaining terms $O(\varepsilon^2)$ generates the only contribution from the dependence in Y from the term ϕ_{yy} in the Laplace Eq. (203). The other terms involving derivatives in Y produce new nonlinear interactions that arise first at $O(\varepsilon^3)$.

We seek a solution of the form

$$\begin{aligned} \phi &= f_0(X, Y, \tau) \\ &+ \sum_{n=0}^{\infty} \varepsilon^n \left\{ \sum_{m=0}^{n+1} F_{nm}(X, Y, z, \tau) e^m + c.c. \right\} \end{aligned} \quad (207)$$

$$\eta = \sum_{n=0}^{\infty} \varepsilon^n \left\{ \sum_{m=0}^{n+1} A_{nm}(X, Y, \tau) e^m + c.c. \right\}, \quad (208)$$

where $e = \exp(ik\xi)$ and $A_{00} = 0$ so that the first approximation to the surface gravity wave is purely harmonic.

To the order $O(\varepsilon^2)$, the problem at $\varepsilon^2 e^0$ gives equation for f_0

$$\begin{aligned} (1 - c_g^2) f_{0XX} + f_{0YY} \\ = -c_p^{-2} (2c_p + c_g \sec h^2 s) (|A_0|^2)_X, \end{aligned} \quad (209)$$

$$s = k\sqrt{\delta}.$$

Given $A_0 \equiv A_{01}$, the surface boundary conditions for the $\varepsilon^2 e^1$ leads to

$$\begin{aligned}
& -2ikc_p A_{0\tau} + \alpha A_{0XX} - c_p c_g A_{0YY} + \frac{1}{2} k^2 c_p^{-2} \\
& \left(1 + 9\coth^2 s - 13\operatorname{sech}^2 s - 2\tanh^4 s\right) A_0 |A_0|^2 \\
& + k^2 (2c_p + c_g \operatorname{sech}^2 s) A_0 f_{0X} = 0.
\end{aligned} \tag{210}$$

Thus the coupled Eqs. (209) and (210) are known as the *Davey–Stewartson* (DS) equations for the modulation of harmonic waves. In the absence of Y dependence, it turns out that Eq. (209) takes the form

$$\left(1 - c_g^2\right) f_{0X} = -c_p^{-2} (2c_p + c_g \operatorname{sech}^2 s) |A_0|^2. \tag{211}$$

This equation provides the leading contribution to the mean drift generated by the nonlinear interaction of the wave motion, usually called the *Stokes drift*. Similarly, Eq. (210) reduces to the ordinary nonlinear Schrödinger equation

$$-2ikc_p A_{0\tau} + \alpha A_{0XX} + \beta A_0 |A_0|^2 = 0 \tag{212}$$

which is identical to Eq. (183).

Finally, the Davey–Stewartson equations can also be written in the compact form

$$\begin{aligned}
& \left(1 - c_g^2 f_{0XX} + f_{0YY} = -\frac{\gamma}{c_p^2} \left(|A_0|^2\right)_X\right), \tag{213} \\
& -2ikc_p A_{0\tau} + \alpha A_{0XX} - C_p c_g A_{0YY} \\
& + \left\{ \beta + \frac{\gamma^2 k^2}{c_p^2 (1 - c_g^2)} \right\} A_0 |A_0|^2 + k^2 \gamma A_0 f_{0X} \\
& = 0,
\end{aligned} \tag{214}$$

where α and β are given by Eqs. (184) and (185), and

$$\gamma = 2c_p + c_g \operatorname{sech}^2 s, \tag{215}$$

where $\gamma > 0$ and $c_p c_g > 0$.

These evolution equations can be further approximated for long waves ($\delta \rightarrow 0$) and short waves ($\delta \rightarrow \infty$), respectively. However, their validity suffers from criticism because other terms become important. The long-wave approximation ($\delta \rightarrow 0$) leads to the one-dimensional propagation of long waves so that the relevant equation is the KdV equation. Thus KdV and NLS equations are two fundamental equations for weakly nonlinear waves. The former equation represents long waves which can be derived in the limit as $\delta \rightarrow 0$ and $\varepsilon \rightarrow 0$ with $\delta = O(\varepsilon)$. However, the use of a suitable rescaled variable allows us to obtain the KdV equation for arbitrary $\sqrt{\delta}$. On the other hand, the NLS equation requires scaled variables which are defined with respect to ε only with $\delta = O(1)$ that is retained as a parameter. At least for a class of nonlinear waves, there are two representations:

- (i) $\eta(x, t, \varepsilon, \delta)$ with $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$ gives the KdV equation.
- (ii) $\eta(x, t, \varepsilon, \delta)$ with $\varepsilon \rightarrow 0$ for fixed δ leads to the NLS equation.

The same match also occurs between the Davey–Stewartson Eqs. (209) and (210) and the two-dimensional KdV equation (see Freeman and Davey 1975). In fact, the long-wave limit ($\delta \rightarrow 0$) of the DS equation matches the short wave limit ($\delta \rightarrow \infty$) of the (1 + 2)-dimensional KdV equation:

$$\left(2\eta_{0\tau} + 3\eta_0 \eta_{0\xi} + \frac{1}{3} \eta_{0\xi\xi\xi}\right)_\xi + \eta_{0YY} = 0. \tag{216}$$

This equation is called the *Kadomtsev–Petviashvili* (KP) equation (Kadomtsev and Petviashvili 1970). This is another completely integrable evolution equation. This equation also admits an exact analytical solution describing any number of waves that cross obliquely and interact nonlinearly. For the case of three such waves, spatial solutions exists corresponding to a resonance phenomenon.

It has been an interesting exercise to present several conservation laws for the nonlinear Schrödinger equation. However, the derivation of the conservation laws for the Davey–Stewartson

equations is not easy because these equations are coupled and depend on the three variables: τ , X , and y . In order to derive a few conservation laws, it is convenient to rewrite the DS Eqs. (213) and (214) by replacing τ by t , X by x , Y by y , f_0 by f , and A_0 by A so that they read as

$$af_{xx} + f_{yy} - b(|A|^2)_x = 0, \quad (217)$$

$$-icA_t + dA_{xx} - eA_{yy} + hA|A|^2 + Af_x = 0, \quad (218)$$

Equation (217) is already in the conservation form

$$\frac{\partial}{\partial x}(af_x - b|A|^2) + \frac{\partial}{\partial y}(f_y) = 0, \quad (219)$$

which gives

$$\frac{\partial}{\partial x} \left[\int_{-\infty}^{+\infty} (af_x - b|A|^2) dy \right] + [f_y]_{-\infty}^{\infty} = 0. \quad (220)$$

Under the decay conditions as $|x| \rightarrow \infty$, this equation takes the form

$$a \frac{\partial}{\partial x} \left(\int_{-\infty}^{\infty} f dy \right) = b \int_{-\infty}^{\infty} |A|^2 dy = g(t), \quad (221)$$

where $g(t)$ is an arbitrary function of t . If for some appropriate value of x , the left-hand side of Eq. (221) vanishes, then $g(t) = 0$ for all t . Consequently, Eq. (221) gives

$$a \left(\int_{-\infty}^{\infty} f dy \right) = b \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A|^2 dx dy = \text{const.} \quad (222)$$

This leads to another conservation law:

$$\int_{-\infty}^{\infty} f dy = \text{const.} \quad (223)$$

We next take the complex conjugate of Eq. (218) to obtain

$$ic\bar{A}_t + d\bar{A}_{xx} - e\bar{A}_{yy} + h\bar{A}|A|^2 + \bar{A}f_x = 0. \quad (224)$$

We next multiply Eq. (224) by A and Eq. (218) by $\bar{A}$ and then subtract the latter result from the former result to find

$$\begin{aligned} ic \frac{\partial}{\partial x} (A\bar{A}) + d \frac{\partial}{\partial x} (A\bar{A}_x - \bar{A}A_x) \\ - e \frac{\partial}{\partial y} (\bar{A}A_y - A\bar{A}_y) = 0. \end{aligned} \quad (225)$$

This leads to the following results in the conservation form:

$$\begin{aligned} ic \frac{\partial}{\partial t} \left[\int_{-\infty}^{\infty} |A|^2 dx \right] + e \frac{\partial}{\partial y} \left[\int_{-\infty}^{\infty} (\bar{A}A_y - A\bar{A}_y) dx \right] \\ = 0, \end{aligned} \quad (226)$$

$$\begin{aligned} ic \frac{\partial}{\partial t} \left[\int_{-\infty}^{\infty} |A|^2 dy \right] + d \frac{\partial}{\partial x} \left[\int_{-\infty}^{\infty} (A\bar{A}_x - \bar{A}A_x) dy \right] \\ = 0, \end{aligned} \quad (227)$$

provided decay conditions are satisfied as $x \rightarrow \infty$ for fixed y and as $y \rightarrow \infty$ for fixed x . This means that wave at infinity is not parallel to either the x - or y -directions. Moreover, if decay conditions hold for $x \rightarrow \infty$ to and $y \rightarrow \infty$ to, that is, the solution decays sufficiently rapidly as $(x^2 + y^2) \rightarrow \infty$, it turns out that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A|^2 dx dy = \text{const.} \quad (228)$$

This represents a conserved quantity that arises for a certain class of solutions. This is usually referred to as the law of the *conservation of mass*.

Finally, we close this section by representing one of the most direct applications of the NLS and DS equations. This application deals with the stability of the Stokes waves. We recall the NLS Eq. (212) and rewrite it in the form, by replacing X by ζ ,

$$-2ikc_p A_{0\tau} + \alpha A_{0\zeta\zeta} + \beta A_0 |A_0|^2 = 0, \quad (229)$$

where α and β are given by Eqs. (184) and (185).

We seek a nonlinear plane wave solution of Eq. (229) in the form

$$A_0(\zeta, \tau) = A \exp [i(m\zeta - \sigma\tau)], \quad (230)$$

where the amplitude A is a complex constant and the wave number m is real. This solution exists provided σ satisfies the dispersion relation

$$2kc_p\sigma = \beta|A|^2 - \alpha m^2, \quad (231)$$

where c_p , α and β are functions of the wave number $k(>0)$ of primary wave described by Eq. (12.9.14) in Debnath's book (2012) on page 664 and $m(>0)$ is the wave number of the modulation. Thus, the primary wave solution becomes

$$\eta_0 = A \exp \{i(kx - \omega t + m\zeta - \sigma\tau)\} + c.c., \quad (232)$$

where $\tau = \varepsilon^2 t$ and σ is given by Eq. (231) with $m = 0$.

The solution Eq. (232) represents the Stokes wave of constant amplitude A and wave number k with the dispersion relation:

$$2kc_p\sigma = \beta|A|^2. \quad (233)$$

Thus, the Stokes wave assumes the form

$$\eta_0(x, t) = A \exp [i\{kx - (\omega + \varepsilon^2\sigma)t\}]. \quad (234)$$

We next employ the NLS equation to examine the stability of the Stokes wave. The NLS equation describes the modulation of the amplitude of the harmonic wave represented by $\exp(ikx)$. We seek a solution which is a small perturbation of the nonlinear plane wave solution Eq. (234) so that

$$A_0 = A(1 + \mu a) \exp [i(-\sigma\tau + \mu\theta)], \quad (235)$$

where $a = a(\zeta, \tau)$ and $\theta = \theta(\zeta, \tau)$ are real perturbation functions and μ is the parameter with the dispersion relation Eq. (233).

Substituting Eq. (235) into the original NLS Eq. (229) gives, dropping the exponential term is Eq. (235),

$$\begin{aligned} & -2ikc_p[\mu a_\tau + i(1 + \mu a)(\mu\theta_\tau - \sigma)] \\ & + \alpha[\mu a_{\zeta\zeta} + 2i\mu^2 a_\zeta\theta_\zeta + \mu(1 + \mu a)(i\theta_{\zeta\zeta} - \mu\theta_\zeta^2)] \\ & + \beta(1 + \mu a)^3|A|^2 = 0. \end{aligned} \quad (236)$$

The leading term $O(1)$ as $\mu \rightarrow 0$ cancels due to Eq. (233), and hence, the main perturbation terms $O(\mu)$ are given by

$$\begin{aligned} & -2ikc_p[a_\tau + i(\theta_\tau - \sigma)] + \alpha(a_{\zeta\zeta} + i\theta_{\zeta\zeta}) \\ & + 3\beta|A|^2 a = 0. \end{aligned} \quad (237)$$

In view of the fact that a and θ are real functions, we use Eq. (233) to eliminate σ from Eq. (237) so that

$$2kc_p\theta_\tau + \alpha a_{\zeta\zeta} + 2\beta|A|^2 a = 0, \quad (238)$$

$$-2kc_p a_\tau + \alpha\theta_{\zeta\zeta} = 0. \quad (239)$$

This is a pair of linear partial differential equations with constant coefficients. We seek a solution of the form

$$\begin{pmatrix} a \\ \theta \end{pmatrix} = \begin{pmatrix} a_0 \\ \theta_0 \end{pmatrix} \exp [i(\kappa\zeta - \Omega\tau)] + c.c., \quad (240)$$

where a_0 , θ_0 , $\kappa(>0)$, and Ω are constant. This solution exists, provided the dispersion relation for Ω

$$(2kc_p\Omega)^2 = \alpha^2\kappa^2 \left[\kappa^2 - 2\left(\frac{\beta}{\alpha}\right)|A|^2 \right] \quad (241)$$

is satisfied. Or, equivalently,

$$(2kc_p\Omega) = (\alpha\kappa) \left[\kappa^2 - 2 \left(\frac{\beta}{\alpha} \right) |A|^2 \right]^{\frac{1}{2}}. \quad (242)$$

Some new and striking conclusions can be drawn from Eq. (242). If $\beta\alpha^{-1} > 0$ ($\alpha\beta > 0$), Ω is imaginary for some wave number κ , provided

$$0 < \kappa < \sqrt{2}|A| \sqrt{\frac{\beta}{\alpha}}. \quad (243)$$

In this case, the amplitude modulation grows exponentially as $\tau \rightarrow \infty$. This means that the Stokes wave is definitely unstable for a range of wave numbers κ given by Eq. (243). On the other hand, if $\beta\alpha^{-1} < 0$ ($\alpha\beta < 0$), Ω is real for all values of the wave number κ . This implies that the amplitude modulation persists, but does not grow. In other words, the Stokes wave is neutrally stable for all values of κ .

Another most simple and direct interpretation of the instability phenomena of the Stokes wave can be given by writing the leading-order fundamental wave mode at $t = 0$ as

$$\eta_0 = A(1 + \mu a) \exp[i(kx + \mu\theta)] + c.c., \quad (244)$$

which is, by using Eq. (240) with $\zeta = \varepsilon(x - c_g t)$ at $t = 0$,

$$= A \exp[i(kx + \mu\theta)] + A\mu a_0 \exp[i(k + \varepsilon\kappa)x + i\mu\theta] + c.c. \quad (245)$$

as $\mu \rightarrow 0$ for fixed x and ε .

Evidently, the perturbation to the fundamental mode has the term μa_0 which has a wavelength $k + \varepsilon\kappa$ that is close to k . This means that a perturbation with a wave number close to that of the fundamental mode will produce an unstable solution whenever $\beta\alpha^{-1} > 0$. Thus, it is impossible to generate stable waves with a precisely fixed wave number both in nature and in the laboratory. In other words, a wave with a small deviation from a fixed wave number k would always occur and lead to the instability phenomena. This kind of instability is associated with a small change in

the wave number of the fundamental mode and is now well known as the Benjamin and Feir (1967) *side-band instability*. It is often observed in nature that a set of plane waves gradually breaks down along the wave fronts into a series of wave groups.

The Camassa–Holm (CH) and Degasperis–Procesi (DP) Nonlinear Model Equations

From a physical point of view, it is evident that nonlinearity produces the steepening effects which are counterbalanced by the smoothing effects of dispersion. These effects play a major role in wave peaking and breaking and other physical features of wave phenomena including a variety of weakly singular patterns. In order to understand the major role of these effects, several strongly nonlinear and dispersive models have been developed without a full resolution of the problems, despite over 150 years of progress. Recently, Camassa and Holm (1993) and Camassa et al. (1994) first developed a new, strongly nonlinear, completely integrable model by using asymptotic expansions of Euler equations for an inviscid incompressible fluid flow in the shallow water regime. Their model is governed by the nonlinear dispersive equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (246)$$

$$x \in \mathbb{R} \quad t > 0,$$

where $u(x, t)$ is the free-surface elevation over a flat rigid bottom. In fact, Fokas and Fuchssteiner (1981) derived Eq. (246) much earlier as an abstract *bi*-Hamiltonian partial differential equation with infinitely many conservation laws (also see McKean 2003).

On the other hand, Johnson (2002) gave an alternative derivation of the CH model equation. In fact, Camassa and Holm discovered that their equation is formally integrable in the sense that there exists a Lax pair. Further, the CH model equation has solitary wave solutions which retain

their shape and velocity after nonlinear interaction with waves of the same kind, and they are solitons. The most fundamental feature of the CH equation is that it has not only solutions that are global in time but also possesses wave-breaking (singular) solutions. In other words, the CH equation has smooth solutions that develop singularities in finite time (or solutions blow up in finite time). It can be shown that the regular peakon solutions of the CH equation are given by

$$u(x, t) = c \exp(-|x - ct|), \quad (247)$$

where c is the velocity of the wave. These are stable solutions (see Constantin and Strauss 2000) and are not classical solutions because they have peak at their crests and are called *weak solutions*. These solutions are called *peakons*, as shown in Fig. 6. Indeed, the CH solitary waves lead to breaking, the solution remains bounded, but its slope becomes unbounded in finite time (see Constantin and Escher 1998a, b, c). It has recently been shown that the CH equation is locally well-posed in the Sobolev space $H^s(\mathbb{R})$ for $s > \frac{3}{2}$, with solutions depending continuously on initial data, and has global conservative solutions in $H^1(\mathbb{R})$.

The peakons have to be understood as weak solutions, as it is suitable to rewrite the CH Eq. (246) in the nonlocal conservation law form:

$$(u_t + uu_x) + \partial_x(1 - \partial_x^2)^{-1}\left(u^2 + \frac{1}{2}u_x^2\right) = 0. \quad (248)$$

Several authors including Dullin et al. (2003, 2004) dealt with the traveling wave solutions of generalized version of the CH equation and found its solutions in an implicit form. Qiao and Zhang (2006) also discussed all possible explicit solutions of the CH Eq. (246) with boundary condition $u \rightarrow A = \text{const.}$ as $|x| \rightarrow \infty$. When $A = 0$, they found the regular peakon solutions, and when $A \neq 0$, both

new peaked solutions and a new kind of smooth solutions can be expressed in terms of trigonometric and hyperbolic functions. Using the traveling wave solution as $u(x, t) = \eta(\xi)$, $\xi = x - ct$, with c being the wave velocity, and substituting it in Eq. (246) yields

$$(\eta - c)(\eta - \eta'')' + 2\eta'(\eta - \eta'') = 0, \quad (249)$$

where $\eta' = \frac{d\eta}{d\xi}$.

The CH Eq. (246) has solution of the form

$$u(x, t) = \eta(\xi) = c \exp(-|x - ct - \xi_0|),$$

where $\xi_0 = (x_0 - ct_0)$, $\eta(\xi_0) = c$, $\eta(\pm\infty) = 0$, $\eta'(\xi_0-) = c$, $\eta'(\xi_0+) = -c$, $\eta'(\xi_0-)$ and $\eta'(\xi_0+)$ denote the left-hand derivative and the right-hand derivative of η at ξ_0 , respectively.

It can be verified that the CH Eq. (246) has the weak traveling wave solution

$$u(x, t) = c \exp(-|x - ct - \xi_0|) - \sinh(|x - ct - \xi_0|), \quad (250)$$

where d and ξ_0 are arbitrary real constants.

In particular, when $A = 0$, $d = 0$, and $\xi_0 = 0$, Eq. (250) reduces to the regular peakon solution Eq. (247) that was originally found by Camassa et al. (1994).

When $A \neq 0$, both new peaked solitons and a new kind of smooth solitons which can be expressed in terms of trigonometric and hyperbolic functions emerge.

On the other hand, Degasperis and Procesi (1999) discovered a new nonlinear dispersive equation, known as the *DP equation*, in the form

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R}, \quad t > 0. \quad (251)$$

Both CH and DP equations represent models for the propagation of nonlinear shallow water waves which capture the essential features of



Nonlinear Water Waves and Nonlinear Evolution Equations with Applications, Fig. 6 A peakon or singular solution

wave-breaking phenomena. The solution of the DP equation is similar to the CH equation. In particular, their solutions are singular, leading to a wave breaking, that is, they are bounded, but their slopes become unbounded in finite time. Escher et al. (2006) examined weak global solutions and the blow-up feature of the DP equation. The asymptotic analysis and numerical simulations confirmed that the DP equation admits smooth solitons and cusp solitons.

Vakhnenko and Parkes (2004) discussed periodic and solitary wave solutions of the DP equation, and Lenells (2005) investigated the traveling solitary wave solutions of the DP equation. On the other hand, Zhang and Qiao (2007) studied cusp soliton and smooth soliton solutions of the DP Eq. (251) with the nonhomogeneous boundary condition $u(x, t) \rightarrow A \neq 0$ as $|x| \rightarrow \infty$. They also found by direct calculation that the DP Eq. (251) admits a new stationary cusp soliton solution in the form

$$u(x, t) = \{1 - \exp(-2|x|)\}^{\frac{1}{2}} \in W_{loc}^{1,1}, \quad (252)$$

for any $x \neq 0$.

If $u \in H^1$, the DP Eq. (251) can equivalently be expressed into the following nonlocal conservation law form:

$$L(u) = u_t + uu_x + \partial_x(1 - \partial_x^2)^{-1} \left(\frac{3}{2} u^2 \right) = 0. \quad (253)$$

However, if $u \notin H^1$, Eq. (253) is no longer equivalent to the DP Eq. (251). The cusp soliton solution does not satisfy Eq. (253) because $L(u) = \text{sgn} x \exp(-|x|)$. This means that it is impossible to find the cusp soliton solution of the DP Eq. (251) through its nonlocal conservation form (253).

The DP equation has not only peakon solitons Eq. (247) but also a shock-peakon solution in the form

$$u(x, t) = \frac{1}{(t+k)} \text{sgn}(x) e^{-|x|}. \quad (254)$$

This represents a significantly different solution from that of the CH equation.

The shock-peakon solutions (see Lundmark 2007) can be observed by substituting $(x, t) \rightarrow$

$(\varepsilon x, \varepsilon t)$ to Eq. (251) and letting $\varepsilon \rightarrow 0$ so that it yields the *inviscid derivative Burgers equation* as

$$(u_t + uu_x)_{xx} = 0, \quad (255)$$

which produces shock wave.

On the other hand, the celebrated KdV equation is an integrable model for nonlinear shallow water waves and the dispersive effect incorporated into the KdV model prevents breaking. But they do not shed light on the *breaking process* or what happens *after breaking* which are the most fundamental features of water waves for which there appears to be no satisfactory mathematical, physical, or computational theory for a long period of time.

In recent years, Holm and Stanley (2003) investigated the following family of $(1+1)$ -dimensional nonlinear viscous equation for the fluid velocity $u(x, t)$ in the form

$$m_t + um_x + bu_x m = vm_{xx}, \quad (256)$$

where $u = g * m$ is the *convolution* defined by

$$u(x) = \int_{\mathbb{R}} g(x - \xi) m(\xi) d\xi, \quad (257)$$

which related velocity u to momentum density m with the kernel $g(x)$ over $\mathbb{R}$. The kernel $g(x)$ is chosen to be the Green's function for the Helmholtz operator $(1 - \partial_x^2)$ on the line, that is, $g(x) = \frac{1}{2} \exp(-|x|)$. This means that $m = (u - u_{xx})$. The family of Eq. (256) is characterized by the kernel $g(x)$ and the real nondimensional parameter b which is the ratio of the stretching term $(bu_x m)$ to the convective term (um_x) in Eq. (256). The parameter b is also the number of covariant dimensions associated with m , and $v(>0)$ is the viscosity coefficient associated with m . It is to be noted that the kernel $g(x)$ determines the traveling wave shape and the length scale for Eq. (256), whereas the constant b represents a balance (or bifurcation) parameter for the nonlinear solution. The quadratic terms in Eq. (256) describe the balance in fluid convection between nonlinear transport and amplification due to b -dimensional stretching term. In a recent work on soliton dynamics, it is shown that Eq. (256) for $v = 0$ and $b \neq -1$ is included

in the family of shallow water equations at quadratic order accuracy that are asymptotically equivalent under Kodama transformations (see Dullin et al. 2003).

In the absence of viscosity ($\nu = 0$), Eq. (256) reduces to $(1 + 1)$ -dimensional b -family equations in the form

$$m_t + um_x + bmu_x = 0, \quad u = g * m. \quad (258)$$

Using the method of asymptotic integrability, Degasperis and Procesi (1999) showed that the Eq. (258) cannot be completely integrable unless $b = 2$ or $b = 3$. When $b = 2$, Eq. (258) becomes the CH Eq. (246). When $b = 3$, Eq. (258) reduces to DP Eq. (251). It is shown by Dullin et al. (2003) that the DP equation can be derived from the shallow water elevation equation by an appropriate Kodama transformation. Lundmark and Szmigielski (2003) used the universe scattering method for finding n -peakon solutions of the DP equation, and Holm and Stanley (2003) used numerical method to study stability of solitons and peakons. Recently, Gui et al. (2008) investigated the local well-posedness for the peakon b -family of Eq. (258) that includes both the CH and the DP equations as special cases and found the precise blow-up feature of strong solutions of Eq. (258) with certain initial conditions.

Among the singular entities, the peakon, a soliton with a finite discontinuity in gradients at its crests, is perhaps the weakest nonanalytic solution ever observed. The peakon solutions have been known for some time (see Fuchssteiner 1981). Camassa and Holm (1993) proved the integrability of the equation

$$u_t - u_{xxt} = bu_x + 3uu_x - \left(uu_{xx} + \frac{1}{2}u_x^2 \right)_x. \quad (259)$$

Even if $b = 0$ in Eq. (259), it admits peakon solutions $u(x, t) = c \exp(-|x + ct|)$ which are obtained as a solution of the equation

$$(c - u) \left(u_\xi^2 - u^2 \right) = 0, \quad \xi = x + ct. \quad (260)$$

If $b \neq 0$, Eq. (259) yields soliton solutions which are analytic functions.

Another nonlinear model due to Camassa and Holm (1993) is described by the equation

$$(u \pm u_{xx})_t = bu_x + \frac{1}{2} \left[(u^2 \pm u_x^2)(u \pm u_{xx}) \right]_x. \quad (261)$$

This equation also admits peakon solutions if $b > 0$. When $b < 0$, ordinary solitons emerge, but if $b = 0$, no solitons are possible. For a detailed discussion of this Eq. (261) and its peakon solutions, the reader is referred to Rosenau (1997).

It is important to note that the singular solutions of the Camassa–Holm equation look very familiar to Stokes' wave of extreme height, but they only appear in the rest frame of reference. This equation is not Galilean invariant.

Another generalized version of the *Camassa–Holm equation* is

$$u_t - u_{xxt} + 3uu_x + 2\kappa u_x = 2u_x u_{xx} + uu_{xxx}, \quad (262)$$

where κ is a real constant. It has many conservation laws and it has peakon solution when $\kappa = 0$; and its solitary wave solution is stable for $\kappa > 0$. The local well-posedness, global existence, blow-up phenomena, and the well-posedness of global weak solutions of Eq. (262) have been discussed by many authors including Constantin and Escher (1998a, b, c) and Constantin and McKean (1999). Bressan and Constantin (2007) obtained the sharpest results for the Eq. (262) for the global existence and blow-up solutions. Parker (2004, 2005a, b) and Ohta et al. (2008) constructed soliton solutions of Eq. (262).

For $\kappa \neq 0$, the soliton solution of Eq. (262) for the initial value problem is represented graphically by Camassa and Holm (1993). Indeed, the solution for the Gaussian initial condition breaks up into a series of ordered solitons as time progresses. Thus, the soliton train eventually wraps around the periodic domain so that the leading solitons overtake the slower emergent solitons from behind after interaction that causes the phase shifts. Hence, the CH Eq. (262) represents a model of unidirectional propagation of shallow water waves over a flat bottom. It also arises as a model equation for the axially symmetric waves

in a hyperelastic rod. The analysis of the CH equation has provided a challenge and a source of inspiration for many new developments in nonlinear water waves, integrable systems, asymptotics, geometry, and Lie groups.

Camassa and Holm (1993) discovered the peakon solitary traveling wave solution of Eq. (262) for nonlinear shallow water waves in the form

$$u(x, t) = c \exp \left[- \left(\frac{|x - ct|}{\alpha} \right) \right], \quad (263)$$

where the fluid velocity $u(x, t)$ is a function of position x in $\mathbb{R}$ and time t . The peakon solitary wave moves with a velocity equal to its maximum height, at which it has a sharp peak with jump in derivative. So, peakons can be obtained after solving the initial value problem for a partial differential equation derived by an asymptotic expansion of the Euler equations using small parameters of shallow water wave dynamics. In fact, peakons are nonanalytic solitons that are the superposition of N -soliton solutions as

$$u(x, t) = \frac{1}{2} \sum_{n=1}^N p_n(t) \exp [-|x - q_n(t)|/\alpha], \quad (264)$$

where the discrete sets $\{p_n(t)\}$ and $\{q_n(t)\}$ of peakon parameters satisfy the canonical Hamiltonian equations. They also arise in the absence of linear dispersion ($\kappa = 0$) in Eq. (262). Each term in Eq. (264) is a soliton with a sharp peak at its maximum. Expressed using its momentum $m = (1 - \alpha^2 \partial_x^2)x$, the peakon velocity solution Eq. (264) of dispersionless CH equation becomes a sum of delta functions, supported on a set of points traveling on the real axis. In the other words, the peakon velocity Eq. (264) implies

$$m(x, t) = \alpha \sum_{n=1}^N p_n(t) \delta(x - q_n(t)), \quad (265)$$

due to the fact that $(1 - \alpha^2 \partial_x^2) \exp \left(-\frac{|x|}{\alpha} \right) = 2\alpha \delta(x)$. These solutions satisfy Eq. (262) for the zero dispersion parameter ($\kappa = 0$).

Substituting the peakon solution Eqs. (264) and (265) into the dispersionless CH equation

$$m_t + um_x + 2mu_x = 0, \quad m = u - \alpha^2 u_{xx}, \quad (266)$$

gives the celebrated Hamilton's canonical equations for the discrete set of peakon parameters $\{p_n(t)\}$ and $\{q_n(t)\}$ as

$$\dot{q}_n(t) = \frac{\partial H_N}{\partial p_n} \quad \text{and} \quad \dot{p}_n(t) = -\frac{\partial H_N}{\partial q_n}, \quad (267)$$

where the Hamiltonian H_N is given by

$$H_N = \frac{1}{4} \sum_{n,m=1}^N p_n p_m \exp \left(-\frac{|q_n - q_m|}{\alpha} \right), \quad (268)$$

$n, m = 1, 2, \dots, N$.

In their recent work of weakly nonlinear analysis combined with the variational principle for Euler equations, Holm and Stanley (2003) made a two-dimensional generalization of the dispersionless CH Eq. (266). This generalization is known as the *Euler–Poincaré (EP) equation* for the Lagrangian consisting of the kinetic energy

$$K = \frac{1}{2} \int \left[|\mathbf{u}|^2 + \alpha^2 (\operatorname{div} \mathbf{u})^2 \right] dx dy, \quad (269)$$

where $\mathbf{u} = (u, v)$ is a two-dimensional fluid velocity vector. They have shown that evolution generated by the kinetic energy in Hamilton's principle results in geodesic motion with respect to the velocity norm $\|\mathbf{u}\|$ which is provided by the Lagrangian of the kinetic energy. The EP equation produced by any choice of kinetic energy norm without imposing incompressibility is called “*EPDiff*” for the *Euler–Poincaré (EP) equation for geodesic motion on the diffeomorphisms*. Thus, the EPDiff equation given by Holm and Stanley (2003) is

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{m} + \nabla \mathbf{u}^T \cdot \mathbf{m} + \mathbf{m} (\operatorname{div} \mathbf{u}) = 0, \quad (270)$$

where $\mathbf{m} = \frac{\delta K}{\delta \mathbf{u}}$ is the momentum density and $K = \frac{1}{2} \|\mathbf{u}\|^2$ is the kinetic energy which defines a norm in the fluid velocity $\|\mathbf{u}\|$. Clearly, this equation has no contribution from either the pressure or

the potential energy. However, this equation conserves the velocity norm $\|\mathbf{u}\|$ given by kinetic energy. In fact, its evolution describes geodesic motion on the diffeomorphisms with respect to this norm. There is an alternative way of expressing the EPDiff Eq. (270) in either two or three dimensions as

$$\frac{\partial \mathbf{m}}{\partial t} - \mathbf{u} \times \text{curl} \mathbf{m} + \nabla(\mathbf{u} \cdot \mathbf{m}) + \mathbf{m}(\text{div} \mathbf{u}) = 0. \quad (271)$$

Remarkably, all three differential operators – gradients, divergence, and curl – are present in this form of EPDiff equation. For the kinetic energy Lagrangian K given by Eq. (269) which is a norm for an *irrotational flow* ($\text{curl} \mathbf{u} = 0$), Eq. (270) is the EPDiff equation with the momentum $\mathbf{m} = \delta K / \delta \mathbf{u} = \mathbf{u} - \alpha^2 \nabla(\text{div} \mathbf{u})$.

It is important to point out that the EPDiff equation has many other interpretation beyond applications in fluid flows. However, the stability problem for EPDiff singular momentum solutions has not yet been solved, even through the stability of the peakon in on one dimension was proved by Constantin and Strauss (2000).

On the other hand, Degasperis and Procesi (1999) derived another generalized form of the shallow water model equation

$$u_t - u_{xxt} + 2\kappa u_x + 4uu_x + 3u_x u_{xx} + uu_{xxx}, \quad (272)$$

Like Eq. (262), Degasperis et al. (2002) proved that Eq. (272) is also completely integrable and has the Lax pair formulation. It admits an infinite number of conservation laws and its solitary waves interact like solitons (see Matsuno 2005a, b).

Recently, Constantin and Lannes (2009) derived a generalized version of Eqs. (262) and (263) in the form

$$\begin{aligned} u_t - u_x + \frac{3}{2} \varepsilon uu_x + \delta (\alpha u_{xxx} + \beta u_{xxx} + \beta u_{xxt}) \\ = \varepsilon \delta (\alpha u_x u_{xx} + \beta uu_{xxx}), \end{aligned} \quad (273)$$

where ε and δ are defined by Eq. (55) and α, β, a , and b satisfy certain conditions. They also proved the large time well-posedness on a time scale $O(|\varepsilon|^{-1})$ provided the initial value u_0 belongs to H^s for $s > \frac{5}{2}$. They also discussed the wave-breaking phenomena. Using suitable transformations, Eq. (273) can be transformed into the form

$$u_t - u_{xxt} + 2\kappa u_x + \alpha uu_x = \alpha u_x u_{xx} + \beta uu_{xxx}, \quad (274)$$

where κ, α, a , and b are constants. Obviously, Eq. (274) is a generalization of both Eqs. (262) and (272). The major difference between Eq. (274) for the case $a \neq 2b$ and the CH Eq. (262) is that Eq. (274) does not have the conservation law

$$I = \int_{\mathbb{R}} (u^2 + u_x^2) dx \quad (275)$$

which plays a major role in the investigation of the CH Eq. (262). Recently, Lai and Wu (2011) investigated local existence and uniqueness of solution for the generalized Eq. (274). They also proved the local well-posedness of solution of Eq. (274) in the Sobolev space $H^s(\mathbb{R})$ for $s > \frac{3}{2}$.

We close this discussion by simply stating two other strongly nonlinear models: the nonlinear shallow water model and the Harry Dym equation (see Kruskal 1975; Leo et al. 1983)

$$u_t = \left(u^{\frac{1}{2}}\right)_{xxx}, \quad (276)$$

which arises as a generalization of the class of isospectral flows of the Schrödinger operator. They lead to essentially new scattering problems, which have not yet been fully explored. There is another version of the Harry Dym equation

$$u_t = \frac{1}{2} (D^3 - 4D) u^{-\frac{1}{2}}, D \equiv \frac{\partial}{\partial x}, \quad (277)$$

with an extra transport term.

A recent literature search also suggests that there is a class of hierarchies containing the KdV hierarchy that are obtained from the isospectral problem:

$$D^2 + k^2 u(x) - q(x). \quad (278)$$

These include the KdV flows, the negative KdV flows, the Harry Dym equation, and the Camassa–Holm equation (see Qiao 2003).

Periodic and Solitary Waves with Constant Vorticity

Almost all studies of steady and unsteady water waves are based on the assumption that flows are irrotational (that is, $\text{curl} \mathbf{u} = \mathbf{0}$). In general, this is a good assumption. However, vorticity is generated at the free surface by wind stress or current or by friction at the rigid bottom boundary. In fact, a very thin vorticity (or shear) layer is produced at the free or at the bottom boundary. In recent years, considerable attention has been given to study water waves by assuming for simplicity that vorticity is constant in the fluid flows. Simmen and Saffman (1985) first investigated numerical solution for periodic waves in deep water with constant vorticity. Their study showed that the waves have either a limiting configuration with a 120° angle at the crest or a trapped bubble at their troughs.

Following Simmen and Saffman (1985), we consider a two-dimensional periodic waves in an inviscid incompressible fluid of infinite depth. The flow is assumed to be rotational and described by a constant vorticity, Ω . We take a frame of reference with the x -axis along the mean water level and in which the flow is steady with gravity is in the negative y -axis. We assume that the steady flow is symmetric with respect to the y -axis. The flow can be described in terms of a stream function $\psi(x, y)$ satisfying

$$\nabla^2 \psi = \Omega \quad (279)$$

in the flow domain.

We reduce the problem to one governed by the Laplace equation by subtracting a particular solution of Eq. (279). Thus, if we write

$$\psi = \Psi + \frac{\Omega}{2} y^2 - cy, \quad (280)$$

then

$$\nabla^2 \Psi = 0 \quad (281)$$

with the requirement that $\Psi \rightarrow 0$ as $y \rightarrow -\infty$. This defines the quantity c in Eq. (280) uniquely and c is referred to as the *wave velocity*. In terms of the dimensionless variables by choosing λ as the unit length and c as unit velocity, the dimensionless Eq. (280) becomes

$$\psi = \Psi + \frac{\omega}{2} y^2 - y, \quad (282)$$

where ω is the dimensionless vorticity defined by

$$\omega = \left(\frac{\Omega \lambda}{c} \right). \quad (283)$$

The function $\omega(z) = u - iv = \Psi_y + i\Psi_x$ is an analytic function of $z = x + iy$, and the fluid velocity vector is $(u + \omega y - 1, v)$. The function $\omega(z)$ vanishes at infinity.

Vanden-Broeck's (1996a, b) numerical analysis first confirmed the results of Simmen and Saffman (1985). He then found that there are solution branches that bifurcate from the uniform shear flow $(u, v) = (0, 0)$. As we move along the solution branches, the wave ultimately attain limiting configuration with a 120° angle at the crest or a trapped bubble at their troughs. Further more, new families of solution can also be constructed.

Several authors including Teles da Silva and Peregrine (1988), Pullin and Grimshaw (1988), and Vanden-Broeck (1994, 1995) obtained numerical solutions for solitary waves with constant vorticity. They formulated the problem in terms of stream function ψ such that

$$\nabla^2 \psi = -\omega, \quad (284)$$

where the flow is rotational and characterized by a constant vorticity Ω so that $\omega = \left(\frac{\Omega h}{c}\right)$ is the non-dimensional vorticity, h is the depth of the fluid, and c is the wave velocity. These authors reduced the problem to the Laplace equation $\nabla^2 \Psi = 0$ by subtracting a particular solution of Eq. (284), that is, $\psi = \Psi - \left\{\frac{\omega}{2}y^2 - (1 + \omega)y\right\}$. Thus, $\omega(z) = u - iv = \Psi_y + i\Psi_x$ is an analytic function of $z = x + iy$, where the fluid velocity vector is $\{(u - \omega(y - 1) + 1, v)\}$. They require the kinematic condition that $v = 0$ on the bottom by reflecting the flow in the bottom. The function $\omega(z)$ vanishes at infinity.

This problem is then numerically investigated using the flow variables nondimensional by choosing h as the unit length and c as the unit velocity. The flow is then described by the above parameter ω and two other parameters

$$G = \left(\frac{gh}{c^2}\right) \quad \text{and} \quad a = \frac{\eta}{h}, \quad (285)$$

where η is the elevation of the wave crests above the level of the free surface. Clearly, G is the nondimensional gravitational parameter, and a is the amplitude parameter. When $\omega = 0$, the flow becomes irrotational. The above authors numerically examined solutions for various values of the parameters ω , G , and a . Their solitary wave solutions are solution branches that bifurcate from the trivial solution ($u = v = 0$ and $a = 0$) at the critical values $G = 1 + \omega$ which is obtained by Benjamin (1962). He studied asymptotic solutions for solitary waves of small amplitude. His analysis showed that for each value of ω , there is one-parameter family of solutions that bifurcates from the uniform shear flow at the critical value $G = 1 + \omega$. He also found an asymptotic solution for small values of the parameter a and obtained the following relation G , ω and a :

$$G = 1 + \omega - a \left(1 + \omega + \frac{\omega^2}{3}\right). \quad (286)$$

Vanden-Broeck (1994, 1995) and others generalized Benjamin's results for waves of finite amplitude. They obtained some solution branches which are not associated with Benjamin's

solutions and have shown that their numerical values agree with Eq. (286) as $a \rightarrow 0$. As they progress along the solution branches, there exists a critical value $\omega_c \approx -0.32$ of ω so that different limiting configurations occur for $\omega > \omega_c$ and $\omega < \omega_c$. For $\omega < \omega_c$, they found that the solution branches which bifurcate from a uniform shear flow at the critical value $G = 1 + \omega$ do not have limiting configuration with 120° angle at crests. In the limit as $G \rightarrow 0$, the solution branches eventually tend to closed regions of constant vorticity in contact with the bottom boundary if G is allowed to become negative. If $\omega > \omega_c$, the solution branches that bifurcate from a uniform shear flow at the critical values $G = 1 + \omega$ have limiting configuration with 120° angle at their crests (see Teles da Silva and Peregrine 1988 and Pullin and Grimshaw 1988).

As a concluding remark, it is necessary to point out that the boundary integral equation method is used by Vanden-Broeck (1996a, b) to compute periodic waves with constant vorticity and to find the solution in the limit as $G \rightarrow 0$. This computational results reveal that there are solution branches that tend to configurations with a closed region of fluid in the rigid body rotation as $G \rightarrow 0$. Further, there are solitary waves with circular closed regions at their crests. Since the solitary waves can be considered as the limit of periodic waves as $\left(\frac{\lambda}{h}\right) \rightarrow \infty$, Vanden-Broeck's (1996a, b) numerical analysis strongly suggest that configurations with circular regions at the wave crests are a general feature of waves in water of finite depth. For more detailed information on periodic and solitary waves with constant vorticity, the reader is referred to a recent book by Vanden-Broeck (2010).

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Inverse Scattering Transform and the Theory of Solitons

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Article Outline

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Glossary

AKNS method A method introduced by Ablowitz, Kaup, Newell, and Segur in 1973 that identifies the nonlinear partial differential equation (NPDE) associated with a given first-order system of linear ordinary differential equations (LODEs) so that the initial value problem (IVP) for that NPDE can be solved by the inverse scattering transform (IST) method.

Direct scattering problem The problem of determining the scattering data corresponding to a given potential in a differential equation.

Integrability A NPDE is said to be integrable if its IVP can be solved via an IST.

Inverse scattering problem The problem of determining the potential that corresponds to a given set of scattering data in a differential equation.

Inverse scattering transform A method introduced in 1967 by Gardner, Greene, Kruskal,

and Miura that yields a solution to the IVP for a NPDE with the help of the solutions to the direct and inverse scattering problems for an associated LODE.

Lax method A method introduced by Lax in 1968 that determines the integrable NPDE associated with a given LODE so that the IVP for that NPDE can be solved with the help of an IST.

Scattering data The scattering data associated with a LODE usually consists of a reflection coefficient which is a function of the spectral parameter λ , a finite number of constants λ_j that correspond to the poles of the transmission coefficient in the upper half complex plane, and the bound-state norming constants whose number for each bound-state pole λ_j is the same as the order of that pole. It is desirable that the potential in the LODE is uniquely determined by the corresponding scattering data and vice versa.

Soliton The part of a solution to an integrable NPDE due to a pole of the transmission coefficient in the upper half complex plane. The term soliton was introduced by Zabusky and Kruskal in 1965 to denote a solitary wave pulse with a particle-like behavior in the solution to the Korteweg-de Vries (KdV) equation.

Time evolution of the scattering data The evolution of the scattering data from its initial value $S(\lambda, 0)$ at $t = 0$ to its value $S(\lambda, t)$ at a later time t .

Definition of the Subject

A general theory to solve NPDEs does not seem to exist. However, there are certain NPDEs, usually first order in time, for which the corresponding IVPs can be solved by the IST method. Such NPDEs are sometimes referred to as integrable evolution equations. Some exact solutions to such equations may be available in terms of elementary functions, and such solutions are important to understand nonlinearity better and they

may also be useful in testing accuracy of numerical methods to solve such NPDEs.

Certain special solutions to some of such NPDEs exhibit particle-like behaviors. A single-soliton solution is usually a localized disturbance that retains its shape but only changes its location in time. A multi-soliton solution consists of several solitons that interact nonlinearly when they are close to each other but come out of such interactions unchanged in shape except for a phase shift.

Integrable NPDEs have important physical applications. For example, the KdV equation is used to describe (Korteweg and de Vries 1895; Zabusky and Kruskal 1965) surface water waves in long, narrow, shallow canals; it also arises (Zabusky and Kruskal 1965) in the description of hydromagnetic waves in a cold plasma, and ion-acoustic waves in anharmonic crystals. The nonlinear Schrödinger (NLS) equation arises in modeling (Zakharov and Shabat 1972) electro-magnetic waves in optical fibers as well as surface waves in deep waters. The sine-Gordon equation is helpful (Ablowitz and Clarkson 1991) in analyzing the magnetic field in a Josephson junction (gap between two superconductors).

Introduction

The first observation of a soliton was made in 1834 by the Scottish engineer John Scott Russell at the Union Canal between Edinburgh and Glasgow. Russell reported (Russell 1845) his observation to the British Association of the Advancement of Science in September 1844, but he did not seem to be successful in convincing the scientific community. For example, his contemporary George Airy, the influential mathematician of the time, did not believe in the existence of solitary water waves (Ablowitz and Clarkson 1991).

The Dutch mathematician Korteweg and his doctoral student de Vries published (Korteweg and de Vries 1895) a paper in 1895 based on de Vries' Ph.D. dissertation, in which surface waves in shallow, narrow canals were modeled by what is now known as the KdV equation. The importance of this paper was not understood

until 1965 even though it contained as a special solution what is now known as the one-soliton solution.

Enrico Fermi in his summer visits to the Los Alamos National Laboratory, together with J. Pasta and S. Ulam, used the computer named Maniac I to computationally analyze a one-dimensional dynamical system of 64 particles in which adjacent particles were joined by springs where the forces also included some nonlinear terms. Their main goal was to determine the rate of approach to the equipartition of energy among different modes of the system. Contrary to their expectations there was little tendency towards the equipartition of energy but instead the almost ongoing recurrence to the initial state, which was puzzling. After Fermi died in November 1954, Pasta and Ulam completed their last few computational examples and finished writing a preprint (Fermi et al. 1955), which was never published as a journal article. This preprint appears in Fermi's Collected Papers (Fermi 1965) and is also available on the internet (<http://www.osti.gov/accomplishments/pdf/A80037041/A80037041.pdf>).

In 1965 Zabusky and Kruskal explained (Zabusky and Kruskal 1965) the Fermi-Pasta-Ulam puzzle in terms of solitary wave solutions to the KdV equation. In their numerical analysis they observed "solitary-wave pulses", named such pulses "solitons" because of their particle-like behavior, and noted that such pulses interact with each other nonlinearly but come out of interactions unaffected in size or shape except for some phase shifts. Such unusual interactions among solitons generated a lot of excitement, but at that time no one knew how to solve the IVP for the KdV equation, except numerically. In 1967 Gardner, Greene, Kruskal, and Miura presented (Gardner et al. 1967) a method, now known as the IST, to solve that IVP, assuming that the initial profile $u(x, 0)$ decays to zero sufficiently rapidly as $x \rightarrow \pm \infty$. They showed that the integrable NPDE, i. e. the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0, \quad (1)$$

is associated with a LODE, i. e. the 1-D Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + u(x, t)\psi = k^2\psi, \quad (2)$$

and that the solution $u(x, t)$ to (1) can be recovered from the initial profile $u(x, 0)$ as explained in the diagram given in section “[Inverse Scattering Transform](#).” They also explained that soliton solutions to the KdV equation correspond to a zero reflection coefficient in the associated scattering data. Note that the subscripts x and t in (1) and throughout denote the partial derivatives with respect to those variables.

In 1972 Zakharov and Shabat showed (Zakharov and Shabat 1972) that the IST method is applicable also to the IVP for the NLS equation

$$iu_t + u_{xx} + 2u|u|^2 = 0, \quad (3)$$

where i denotes the imaginary number $\sqrt{-1}$. They proved that the associated LODE is the first-order linear system

$$\begin{cases} \frac{d\xi}{dx} = -i\lambda\xi + u(x, t)\eta, \\ \frac{d\eta}{dx} = i\lambda\eta - \overline{u(x, t)}\xi, \end{cases} \quad (4)$$

where λ is the spectral parameter and an overline denotes complex conjugation. The system in (4) is now known as the Zakharov–Shabat system.

Soon afterwards, again in 1972 Wadati showed in a one-page publication (Wadati 1972) that the IVP for the modified Korteweg-de Vries (mKdV) equation

$$u_t + 6u^2u_x + u_{xxx} = 0, \quad (5)$$

can be solved with the help of the inverse scattering problem for the linear system

$$\begin{cases} \frac{d\xi}{dx} = -i\lambda\xi + u(x, t)\eta, \\ \frac{d\eta}{dx} = i\lambda\eta - u(x, t)\xi. \end{cases} \quad (6)$$

Next, in 1973 Ablowitz, Kaup, Newell, and Segur showed (Ablowitz et al. 1973, 1974) that the IVP for the sine-Gordon equation

$$u_{xt} = \sin u,$$

can be solved in the same way by exploiting the inverse scattering problem associated with the linear system

$$\begin{cases} \frac{d\xi}{dx} = -i\lambda\xi - \frac{1}{2}u_x(x, t)\eta, \\ \frac{d\eta}{dx} = i\lambda\eta + \frac{1}{2}u_x(x, t)\xi. \end{cases}$$

Since then, many other NPDEs have been discovered to be solvable by the IST method.

Our review is organized as follows. In the next section we explain the idea behind the IST. Given a LODE known to be associated with an integrable NPDE, there are two primary methods enabling us to determine the corresponding NPDE. We review those two methods, the Lax method and the AKNS method, in section “[The Lax Method](#)” and in section “[The AKNS Method](#),” respectively. In section “[Direct Scattering](#)” we introduce the scattering data associated with a LODE containing a spectral parameter and a potential, and we illustrate it for the Schrödinger equation and for the Zakharov–Shabat system. In section “[Time Evolution of the Scattering Data](#)” we explain the time evolution of the scattering data and indicate how the scattering data sets evolve for those two LODEs. In section “[Inverse Scattering Problem](#)” we summarize the Marchenko method to solve the inverse scattering problem for the Schrödinger equation and that for the Zakharov–Shabat system, and we outline how the solutions to the IVPs for the KdV equation and the NLS equation are obtained with the help of the IST. In section “[Solitons](#)” we present soliton solutions to the KdV and NLS eqs. A brief conclusion is provided in section “[Future Directions](#).”

Inverse Scattering Transform

Certain NPDEs are classified as integrable in the sense that their corresponding IVPs can be solved with the help of an IST. The idea behind the IST method is as follows: Each integrable NPDE is associated with a LODE (or a system of LODEs)

containing a parameter λ (usually known as the spectral parameter), and the solution $u(x, t)$ to the NPDE appears as a coefficient (usually known as the potential) in the corresponding LODE. In the NPDE the quantities x and t appear as independent variables (usually known as the spatial and temporal coordinates, respectively), and in the LODE x is an independent variable and λ and t appear as parameters. It is usually the case that $u(x, t)$ vanishes at each fixed t as x becomes infinite so that a scattering scenario can be created for the related LODE, in which the potential $u(x, t)$ can uniquely be associated with some scattering data $S(\lambda, t)$. The problem of determining $S(\lambda, t)$ for all λ values from $u(x, t)$ given for all x values is known as the direct scattering problem for the LODE. On the other hand, the problem of determining $u(x, t)$ from $S(\lambda, t)$ is known as the inverse scattering problem for that LODE.

The IST method for an integrable NPDE can be explained with the help of Diagram 1.

In order to solve the IVP for the NPDE, i. e. in order to determine $u(x, t)$ from $u(x, 0)$, one needs to perform the following three steps:

- (i) Solve the corresponding direct scattering problem for the associated LODE at $t = 0$, i.e. determine the initial scattering data $S(\lambda, 0)$ from the initial potential $u(x, 0)$.
- (ii) Time evolve the scattering data from its initial value $S(\lambda, 0)$ to its value $S(\lambda, t)$ at time t . Such an evolution is usually a simple one and is particular to each integrable NPDE.
- (iii) Solve the corresponding inverse scattering problem for the associated LODE at fixed t , i.e. determine the potential $u(x, t)$ from the scattering data $S(\lambda, t)$.

It is amazing that the resulting $u(x, t)$ satisfies the integrable NPDE and that the limiting value of $u(x, t)$ as $t \rightarrow 0$ agrees with the initial profile $u(x, 0)$.

The Lax Method

In 1968 Peter Lax introduced (Lax 1968) a method yielding an integrable NPDE corresponding to a given LODE. The basic idea behind the Lax method is the following. Given a linear differential operator $\mathcal{L}$ appearing in the spectral problem $\mathcal{L}\psi = \lambda\psi$, find an operator $\mathcal{A}$ (the operators $\mathcal{A}$ and $\mathcal{L}$ are said to form a Lax pair) such that:

- (i) The spectral parameter λ does not change in time, i.e. $\lambda_t = 0$.
- (ii) The quantity $\psi_t - \mathcal{A}\psi$ remains a solution to the same linear problem $\mathcal{L}\psi = \lambda\psi$.
- (iii) The quantity $\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}$ is a multiplication operator, i. e. it is not a differential operator.

From condition (ii) we get

$$\mathcal{L}(\psi_t - \mathcal{A}\psi) = \lambda(\psi_t - \mathcal{A}\psi), \quad (7)$$

and with the help of $\mathcal{L}\psi = \lambda\psi$ and $\lambda_t = 0$, from (7) we obtain

$$\begin{aligned} \mathcal{L}\psi_t - \mathcal{L}\mathcal{A}\psi &= \lambda\psi_t - \mathcal{A}(\lambda\psi) = \partial_t(\lambda\psi) - \mathcal{A}\mathcal{L}\psi \\ &= \partial_t(\mathcal{L}\psi) - \mathcal{A}\mathcal{L}\psi = \mathcal{L}_t\psi + \mathcal{L}\psi_t - \mathcal{A}\mathcal{L}\psi, \end{aligned} \quad (8)$$

where ∂_t denotes the partial differential operator with respect to t . After canceling the term $\mathcal{L}\psi_t$ on the left and right hand sides of (8), we get

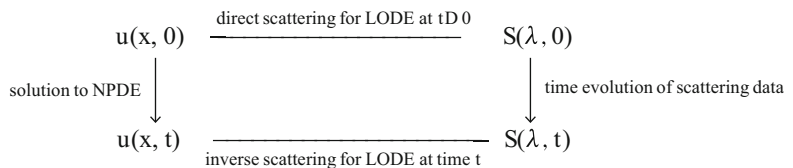
$$(\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L})\psi = 0,$$

which, because of (iii), yields

$$\mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L} = 0. \quad (9)$$

Note that (9) is an evolution equation containing a first-order time derivative, and it is

Inverse Scattering Transform and the Theory of Solitons, Diagram 1 The method of inverse scattering transform



the desired integrable NPDE. The equation in (9) is often called a compatibility condition.

Having outlined the Lax method, let us now illustrate it to derive the KdV equation in (1) from the Schrödinger equation in (2). For this purpose, we write the Schrödinger equation as $\mathcal{L}\psi = \lambda\psi$ with $\lambda := k^2$ and

$$\mathcal{L} := -\partial_x^2 + u(x, t), \quad (10)$$

where the notation $:=$ is used to indicate a definition so that the quantity on the left should be understood as the quantity on the right hand side. Given the linear differential operator $\mathcal{L}$ defined as in (10), let us try to determine the associated operator $\mathcal{A}$ by assuming that it has the form

$$\mathcal{A} = \alpha_3 \partial_x^3 + \alpha_2 \partial_x^2 + \alpha_1 \partial_x + \alpha_0, \quad (11)$$

where the coefficients α_j with $j = 0, 1, 2, 3$ may depend on x and t , but not on the spectral parameter λ . Note that $\mathcal{L}_t = u_t$. Using (10) and (11) in (9), we obtain

$$\begin{aligned} & (\) \partial_x^5 + (\) \partial_x^4 + (\) \partial_x^3 + (\) \partial_x^2 + (\) \partial_x \\ & + (\) \\ & = 0, \end{aligned} \quad (12)$$

where, because of (iii), each coefficient denoted by $(\)$ must vanish. The coefficient of ∂_x^5 vanishes automatically. Setting the coefficients of ∂_x^j to zero for $j = 4, 3, 2, 1$, we obtain

$$\begin{aligned} \alpha_3 &= c_1, \quad \alpha_2 = c_2, \quad \alpha_1 = c_3 - \frac{3}{2}c_1 u, \\ \alpha_0 &= c_4 - \frac{3}{4}c_1 u_x - c_2 u, \end{aligned}$$

with c_1, c_2, c_3 , and c_4 denoting arbitrary constants. Choosing $c_1 = -4$ and $c_3 = 0$ in the last coefficient in (12) and setting that coefficient to zero, we get the KdV equation in (1). Moreover, by letting $c_2 = c_4 = 0$, we obtain the operator $\mathcal{A}$ as

$$\mathcal{A} = -4\partial_x^3 + 6u\partial_x + 3u_x. \quad (13)$$

For the Zakharov–Shabat system in (4), we proceed in a similar way. Let us write it as

$\mathcal{L}\psi = \lambda\psi$, where the linear differential operator $\mathcal{L}$ is defined via

$$\mathcal{L} := i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x - i \begin{bmatrix} 0 & u(x, t) \\ \overline{u(x, t)} & 0 \end{bmatrix}.$$

Then, the operator $\mathcal{A}$ is obtained as

$$\begin{aligned} \mathcal{A} &= 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x^2 - 2i \begin{bmatrix} 0 & u \\ \overline{u} & 0 \end{bmatrix} \partial_x \\ &\quad - i \begin{bmatrix} -|u|^2 & u_x \\ \overline{u}_x & |u|^2 \end{bmatrix}, \end{aligned} \quad (14)$$

and the compatibility condition (9) gives us the NLS equation in (3).

For the first-order system (6), by writing it as $\mathcal{L}\psi = \lambda\psi$, where the linear operator $\mathcal{L}$ is defined by

$$\mathcal{L} := i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x - i \begin{bmatrix} 0 & u(x, t) \\ u(x, t) & 0 \end{bmatrix},$$

we obtain the corresponding operator $\mathcal{A}$ as

$$\begin{aligned} \mathcal{A} &= -4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \partial_x^3 - 6 \begin{bmatrix} u^2 & -u_x \\ u_x & u^2 \end{bmatrix} \partial_x \\ &\quad - \begin{bmatrix} 6uu_x & -3u_{xx} \\ 3u_{xx} & 6uu_x \end{bmatrix}, \end{aligned}$$

and the compatibility condition (9) yields the mKdV equation in (5).

The AKNS Method

In 1973 Ablowitz, Kaup, Newell, and Segur introduced (Ablowitz et al. 1973, 1974) another method to determine an integrable NPDE corresponding to a LODE. This method is now known as the AKNS method, and the basic idea behind it is the following. Given a linear operator $\mathcal{X}$ associated with the first-order system $v_x = \mathcal{X}v$, we are interested in finding an operator $\mathcal{T}$ (the operators $\mathcal{T}$ and $\mathcal{X}$ are said to form an AKNS pair) such that:

- (i) The spectral parameter λ does not change in time, i. e. $\lambda_t = 0$.

- (ii) The quantity $v_t - \mathcal{T}v$ is also a solution to $v_x = \mathcal{X}v$, i. e. we have $(v_t - \mathcal{T}v)_x = \mathcal{X}(v_t - \mathcal{T}v)$.
- (iii) The quantity $\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X}$ is a (matrix) multiplication operator, i. e. it is not a differential operator.

From condition (ii) we get

$$\begin{aligned} v_{tx} - \mathcal{T}_x v - \mathcal{T}v_x &= \mathcal{X}v_t - \mathcal{X}\mathcal{T}v \\ &= (\mathcal{X}v)_t - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v \\ &= (v_x)_t - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v \\ &= v_{xt} - \mathcal{X}_t v - \mathcal{X}\mathcal{T}v. \end{aligned} \quad (15)$$

Using $v_{tx} = v_{xt}$ and replacing $\mathcal{T}v_x$ by $\mathcal{T}\mathcal{X}v$ on the left side and equating the left and right hand sides in (15), we obtain

$$(\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X})v = 0,$$

which in turn, because of (iii), implies

$$\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\mathcal{T} - \mathcal{T}\mathcal{X} = 0. \quad (16)$$

We can view (16) as an integrable NPDE solvable with the help of the solutions to the direct and inverse scattering problems for the linear system $v_x = \mathcal{X}v$. Like (9), the compatibility condition (16) yields a nonlinear evolution equation containing a first-order time derivative. Note that $\mathcal{X}$ contains the spectral parameter λ , and hence $\mathcal{T}$ also depends on λ as well. This is in contrast with the Lax method in the sense that the operator $\mathcal{A}$ does not contain λ .

Let us illustrate the AKNS method by deriving the KdV equation in (1) from the Schrödinger equation in (2). For this purpose we write the Schrödinger equation, by replacing the spectral parameter k^2 with λ , as a first-order linear system $v_x = \mathcal{X}v$, where we have defined

$$v := \begin{bmatrix} \psi_x \\ \psi \end{bmatrix}, \quad \mathcal{X} := \begin{bmatrix} 0 & u(x, t) - \lambda \\ 1 & 0 \end{bmatrix}.$$

Let us look for $\mathcal{T}$ in the form

$$\mathcal{T} = \begin{bmatrix} \alpha & \beta \\ \rho & \sigma \end{bmatrix},$$

where the entries α , β , ρ , and σ may depend on x , t , and λ . The compatibility condition (16) yields

$$\begin{aligned} \begin{bmatrix} -\alpha_x - \beta + \rho(u - \lambda) & u_t - \beta_x + \sigma(u - \lambda) - \alpha(u - \lambda) \\ -\rho_x + \alpha - \sigma & -\sigma_x + \beta - \rho(u - \lambda) \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (17)$$

The (1, 1), (2, 1), and (2, 2)-entries in the matrix equation in (17) imply

$$\begin{aligned} \beta &= -\alpha_x + (u - \lambda)\rho, \quad \sigma = \alpha - \rho_x, \quad \sigma_x \\ &= -\alpha_{xx}. \end{aligned} \quad (18)$$

Then from the (1, 2)-entry in (17) we obtain

$$u_t + \frac{1}{2}\rho_{xxx} - u_x\rho - 2\rho_x(u - \lambda) = 0. \quad (19)$$

Assuming a linear dependence of ρ on the spectral parameter and hence letting $\rho = \lambda\zeta + \mu$ in (19), we get

$$\begin{aligned} 2\zeta_x\lambda^2 + \left(\frac{1}{2}\zeta_{xxx} - 2\zeta_x u + 2\mu_x - u_x\zeta\right)\lambda \\ + \left(u_t + \frac{1}{2}\mu_{xxx} - 2\mu_x u - u_x\mu\right) \\ = 0. \end{aligned}$$

Equating the coefficients of each power of λ to zero, we have

$$\begin{aligned} \zeta &= c_1, \quad \mu = \frac{1}{2}c_1 u + c_2, \\ u_t - \frac{3}{2}c_1 u u_x - c_2 u_x + \frac{1}{4}c_1 u_{xxx} &= 0, \end{aligned} \quad (20)$$

with c_1 and c_2 denoting arbitrary constants. Choosing $c_1 = 4$ and $c_2 = 0$, from (20) we obtain the KdV equation given in (1). Moreover, with the help of (18) we get

$$\begin{aligned}\alpha &= u_x + c_3, \\ \beta &= -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx}, \\ \rho &= 4\lambda + 2u, \quad \sigma = c_3 - u_x,\end{aligned}$$

where c_3 is an arbitrary constant. Choosing $c_3 = 0$, we find

$$\mathcal{T} = \begin{bmatrix} u_x & -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda + 2u & -u_x \end{bmatrix}.$$

As for the Zakharov–Shabat system in (4), writing it as $v_x = \mathcal{X}v$, where we have defined

$$\mathcal{X} := \begin{bmatrix} \frac{-i\lambda}{-u(x, t)} & u(x, t) \\ -u(x, t) & i\lambda \end{bmatrix},$$

we obtain the matrix operator $\mathcal{T}$ as

$$\mathcal{T} = \begin{bmatrix} -2i\lambda^2 + i|u|^2 & 2\lambda u + iu_x \\ -2\lambda\bar{u} + i\bar{u}_x & 2i\lambda^2 - i|u|^2 \end{bmatrix},$$

and the compatibility condition (16) yields the NLS equation in (3).

As for the first-order linear system (6), by writing it as $v_x = \mathcal{X}v$, where

$$\mathcal{X} := \begin{bmatrix} -i\lambda & u(x, t) \\ -u(x, t) & i\lambda \end{bmatrix},$$

we obtain the matrix operator $\mathcal{T}$ as

$$\mathcal{T} = \begin{bmatrix} -4i\lambda^3 + 2i\lambda u^2 & 4\lambda^2 u + 2i\lambda u_x - u_{xx} - 2u^3 \\ -4\lambda^2 u + 2i\lambda u_x + u_{xx} + 2u^3 & 4i\lambda^3 - 2i\lambda u^2 \end{bmatrix}$$

and the compatibility condition (16) yields the mKdV equation in (5).

As for the first-order system $v_x = \mathcal{X}v$, where

$$\mathcal{X} := \begin{bmatrix} -i\lambda & -\frac{1}{2}u_x(x, t) \\ \frac{1}{2}u_x(x, t) & i\lambda \end{bmatrix},$$

we obtain the matrix operator $\mathcal{T}$ as

$$\mathcal{T} = \frac{i}{4\lambda} \begin{bmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{bmatrix}.$$

Then, the compatibility condition (16) gives us the sine-Gordon equation

$$u_{xt} = \sin u.$$

Direct Scattering Problem

The direct scattering problem consists of determining the scattering data when the potential is known. This problem is usually solved by obtaining certain specific solutions, known as the Jost solutions, to the relevant LODE. The appropriate scattering data can be constructed with the help of spatial asymptotics of the Jost solutions at infinity or from certain Wronskian relations among the Jost solutions. In this section we review the scattering data corresponding to the Schrödinger equation in (2) and to the Zakharov–Shabat system in (4). The scattering data sets for other LODEs can similarly be obtained.

Consider (2) at fixed t by assuming that the potential $u(x, t)$ belongs to the Faddeev class, i. e. $u(x, t)$ is real valued and $\int_{-\infty}^{\infty} dx(1+|x|) |u(x, t)|$ is finite. The Schrödinger equation has two types of solutions; namely, scattering solutions and bound-state solutions. The scattering solutions are those that consist of linear combinations of e^{ikx} and e^{-ikx} as $x \rightarrow \pm \infty$, and they occur for $k \in \mathbf{R} \setminus \{0\}$, i. e. for real nonzero values of k . Two linearly independent scattering solutions f_l and f_r , known as the Jost solution from the left and from the right, respectively, are those solutions to (2) satisfying the respective asymptotic conditions

$$\begin{aligned}f_l(k, x, t) &= e^{ikx} + o(1), \\ f_l'(k, x, t) &= ike^{ikx} + o(1), \quad x \rightarrow +\infty,\end{aligned} \tag{21}$$

$$\begin{aligned}f_r(k, x, t) &= e^{-ikx} + o(1), \\ f_r'(k, x, t) &= -ike^{-ikx} + o(1), \quad x \rightarrow -\infty,\end{aligned}$$

where the notation $o(1)$ indicates the quantities that vanish. Writing their remaining spatial asymptotics in the form

$$f_1(k, x, t) = \frac{e^{ikx}}{T(k, t)} + \frac{L(k, t)e^{-ikx}}{T(k, t)} + o(1), \quad x \rightarrow -\infty, \quad (22)$$

$$f_r(k, x, t) = \frac{e^{-ikx}}{T(k, t)} + \frac{R(k, t)e^{ikx}}{T(k, t)} + o(1), \quad x \rightarrow +\infty,$$

we obtain the scattering coefficients; namely, the transmission coefficient T and the reflection coefficients L and R , from the left and right, respectively.

Let $\mathbf{C}^+$ denote the upper half complex plane. A bound-state solution to (2) is a solution that belongs to $L^2(\mathbf{R})$ in the x variable. Note that $L^2(\mathbf{R})$ denotes the set of complex-valued functions whose absolute squares are integrable on the real line $\mathbf{R}$. When $u(x, t)$ is in the Faddeev class, it is known (Aktosun and Klaus 2001; Chadan and Sabatier 1989; Deift and Trubowitz 1979; Faddeev 1967; Levitan 1987; Marchenko 1986; Melin 1985; Newton 1983) that the number of bound states is finite, the multiplicity of each bound state is one, and the bound-state solutions can occur only at certain k -values on the imaginary axis in $\mathbf{C}^+$. Let us use N to denote the number of bound states, and suppose that the bound states occur at $k = i\kappa_j$ with the ordering $0 < \kappa_1 < \dots < \kappa_N$. Each bound state corresponds to a pole of T in $\mathbf{C}^+$. Any bound-state solution at $k = i\kappa_j$ is a constant multiple of $f_1(i\kappa_j, x, t)$. The left and right bound-state norming constants $c_{lj}(t)$ and $c_{rj}(t)$, respectively, can be defined as

$$c_{lj}(t) := \left[\int_{-\infty}^{\infty} dx f_1(i\kappa_j, x, t)^2 \right]^{-1/2},$$

$$c_{rj}(t) := \left[\int_{-\infty}^{\infty} dx f_r(i\kappa_j, x, t)^2 \right]^{-1/2},$$

and they are related to each other through the residues of T via

$$\text{Res}(T, i\kappa_j) = i c_{lj}(t)^2 \gamma_j(t) = i \frac{c_{rj}(t)^2}{\gamma_j(t)}, \quad (23)$$

where the $\gamma_j(t)$ are the dependency constants defined as

$$\gamma_j(t) := \frac{f_1(i\kappa_j, x, t)}{f_r(i\kappa_j, x, t)}. \quad (24)$$

The sign of $\gamma_j(t)$ is the same as that of $(-1)^{N-j}$, and hence $c_{rj}(t) = (-1)^{N-j} \gamma_j(t) c_{lj}(t)$.

The scattering matrix associated with (2) consists of the transmission coefficient T and the two reflection coefficients R and L , and it can be constructed from $\{\kappa_j\}_{j=1}^N$ and one of the reflection coefficients. For example, if we start with the right reflection coefficient $R(k, t)$ for $k \in \mathbf{R}$, we get

$$T(k, t) = \left(\prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j} \right) \times \exp \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds \frac{\log \left(1 - |R(s, t)|^2 \right)}{s - k - i0^+} \right),$$

$$k \in \mathbf{C}^+ \cup \mathbf{R},$$

where the quantity $i0^+$ indicates that the value for $k \in \mathbf{R}$ must be obtained as a limit from $\mathbf{C}^+$. Then, the left reflection coefficient $L(k, t)$ can be constructed via

$$L(k, t) = - \frac{\overline{R(k, t)} T(k, t)}{\overline{T(k, t)}}, \quad k \in \mathbf{R}.$$

We will see in the next section that $T(k, t) = T(k, 0)$, $|R(k, t)| = |R(k, 0)|$, and $|L(k, t)| = |L(k, 0)|$.

For a detailed study of the direct scattering problem for the 1-D Schrödinger equation, we refer the reader to (Aktosun and Klaus 2001; Chadan and Sabatier 1989; Deift and Trubowitz 1979; Faddeev 1967; Levitan 1987; Marchenko 1986; Melin 1985; Newton 1983). It is important to remember that $u(x, t)$ for $x \in \mathbf{R}$ at each fixed t is uniquely determined (Aktosun and Klaus 2001; Chadan and Sabatier 1989; Deift and Trubowitz 1979; Faddeev 1967; Levitan 1987; Marchenko

1986; Melin 1985) by the scattering data $\{R, \{\kappa_j\}, \{c_j(t)\}\}$ or one of its equivalents. Letting $c_j(t) := c_j(t)^2$, we will work with one such data set, namely $\{R, \{\kappa_j\}, \{c_j(t)\}\}$, in section “Time Evolution of the Scattering Data” and section “Inverse Scattering Problem”.

Having described the scattering data associated with the Schrödinger equation, let us briefly describe the scattering data associated with the Zakharov–Shabat system in (4). Assuming that $u(x, t)$ for each t is integrable in x on $\mathbf{R}$, the two Jost solutions $\psi(\lambda, x, t)$ and $\phi(\lambda, x, t)$, from the left and from the right, respectively, are those unique solutions to (4) satisfying the respective asymptotic conditions

$$\begin{aligned} \psi(\lambda, x, t) &= \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad x \rightarrow +\infty, \\ \phi(\lambda, x, t) &= \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad x \rightarrow -\infty. \end{aligned} \quad (25)$$

The transmission coefficient T , the left reflection coefficient L , and the right reflection coefficient R are obtained via the asymptotics

$$\begin{aligned} \psi(\lambda, x, t) &= \begin{bmatrix} \frac{L(\lambda, t)e^{-i\lambda x}}{T(\lambda, t)} \\ \frac{e^{i\lambda x}}{T(\lambda, t)} \end{bmatrix} + o(1), \quad x \rightarrow -\infty, \\ \phi(\lambda, x, t) &= \begin{bmatrix} \frac{e^{-i\lambda x}}{T(\lambda, t)} \\ \frac{R(\lambda, t)e^{i\lambda x}}{T(\lambda, t)} \end{bmatrix} + o(1), \quad x \rightarrow +\infty. \end{aligned} \quad (26)$$

The bound-state solutions to (4) occur at those λ values corresponding to the poles of T in $\mathbf{C}^+$. Let us use $\{\lambda_j\}_{j=1}^N$ to denote the set of such poles. It should be noted that such poles are not necessarily located on the positive imaginary axis. Furthermore, unlike the Schrödinger equation, the multiplicities of such poles may be greater than one. Let us assume that the pole λ_j has multiplicity n_j . Corresponding to the pole λ_j , one associates (Aktosun et al. 2007; Olmedilla 1987) n_j bound-

state norming constants $c_{js}(t)$ for $s = 0, 1, \dots, n_j - 1$. We assume that, for each fixed t , the potential $u(x, t)$ in the Zakharov–Shabat system is uniquely determined by the scattering data $\{R, \{\lambda_j\}, \{c_{js}(t)\}\}$ and vice versa.

Time Evolution of the Scattering Data

As the initial profile $u(x, 0)$ evolves to $u(x, t)$ while satisfying the NPDE, the corresponding initial scattering data $S(\lambda, 0)$ evolves to $S(\lambda, t)$. Since the scattering data can be obtained from the Jost solutions to the associated LODE, in order to determine the time evolution of the scattering data, we can analyze the time evolution of the Jost solutions with the help of the Lax method or the AKNS method.

Let us illustrate how to determine the time evolution of the scattering data in the Schrödinger equation with the help of the Lax method. As indicated in section “The Lax Method”, the spectral parameter k and hence also the values κ_j related to the bound states remain unchanged in time. Let us obtain the time evolution of $f_l(k, x, t)$, the Jost solution from the left. From condition (ii) in section “The Lax Method”, we see that the quantity $\partial_t f_l - \mathcal{A}f_l$ remains a solution to (2) and hence we can write it as a linear combination of the two linearly independent Jost solutions f_l and f_r as

$$\begin{aligned} \partial_t f_l(k, x, t) &= (-4\partial_x^3 + 6u\partial_x + 3u_x)f_l(k, x, t) \\ &= p(k, t)f_l(k, x, t) + q(k, t)f_r(k, x, t), \end{aligned} \quad (27)$$

where the coefficients $p(k, t)$ and $q(k, t)$ are yet to be determined and $\mathcal{A}$ is the operator in (13). For each fixed t , assuming $u(x, t) = o(1)$ and $u_x(x, t) = o(1)$ as $x \rightarrow +\infty$ and using (21) and (22) in (27) as $x \rightarrow +\infty$, we get

$$\begin{aligned} \partial_t e^{ikx} + 4\partial_x^3 e^{ikx} &= p(k, t)e^{ikx} \\ &+ q(k, t) \left[\frac{1}{T(k, t)} e^{-ikx} + \frac{R(k, t)}{T(k, t)} e^{ikx} \right] + o(1). \end{aligned} \quad (28)$$

Comparing the coefficients of e^{ikx} and e^{-ikx} on the two sides of (28), we obtain.

$$q(k, t) = 0, \quad p(k, t) = -4ik^3.$$

Thus, $f_l(k, x, t)$ evolves in time by obeying the linear third-order PDE

$$\partial_t f_l - \mathcal{A}f_l = -4ik^3 f_l. \quad (29)$$

Proceeding in a similar manner, we find that $f_r(k, x, t)$ evolves in time according to

$$\partial_t f_r - \mathcal{A}f_r = 4ik^3 f_r. \quad (30)$$

Notice that the time evolution of each Jost solution is fairly complicated. We will see, however, that the time evolution of the scattering data is very simple. Letting $x \rightarrow -\infty$ in (29), using (22) and $u(x, t) = o(1)$ and $u_x(x, t) = o(1)$ as $x \rightarrow -\infty$, and comparing the coefficients of e^{ikx} and e^{-ikx} on both sides, we obtain

$$\partial_t T(k, t) = 0, \quad \partial_t L(k, t) = -8ik^3 L(k, t),$$

yielding

$$T(k, t) = T(k, 0), \quad L(k, t) = L(k, 0)e^{-8ik^3 t}.$$

In a similar way, from (30) as $x \rightarrow +\infty$, we get

$$R(k, t) = R(k, 0)e^{8ik^3 t}. \quad (31)$$

Thus, the transmission coefficient remains unchanged and only the phases of the reflection coefficients change as time progresses.

Let us also evaluate the time evolution of the dependency constants $\gamma_j(t)$ defined in (24). Evaluating (29) at $k = i\kappa_j$ and replacing $f_l(i\kappa_j, x, t)$ by $\gamma_j(t)f_r(i\kappa_j, x, t)$, we get

$$\begin{aligned} & f_r(i\kappa_j, x, t) \partial_t \gamma_j(t) + \gamma_j(t) \partial_t f_r(i\kappa_j, x, t) \\ & - \gamma_j(t) \mathcal{A}f_r(i\kappa_j, x, t) \\ & = -4\kappa_j^3 \gamma_j(t) f_r(i\kappa_j, x, t). \end{aligned} \quad (32)$$

On the other hand, evaluating (30) at $k = i\kappa_j$, we obtain

$$\begin{aligned} & \gamma_j(t) \partial_t f_r(i\kappa_j, x, t) - \gamma_j(t) \mathcal{A}f_r(i\kappa_j, x, t) \\ & = 4\kappa_j^3 \gamma_j(t) f_r(i\kappa_j, x, t). \end{aligned} \quad (33)$$

Comparing (32) and (33) we see that $\partial_t \gamma_j(t) = -8\kappa_j^3 \gamma_j(t)$, or equivalently

$$\gamma_j(t) = \gamma_j(0)e^{-8\kappa_j^3 t}. \quad (34)$$

Then, with the help of (23) and (34), we determine the time evolutions of the norming constants as

$$c_{lj}(t) = c_{lj}(0)e^{4\kappa_j^3 t}, \quad c_{rj}(t) = c_{rj}(0)e^{-4\kappa_j^3 t}.$$

The norming constants $c_j(t)$ appearing in the Marchenko kernel (38) are related to $c_{lj}(t)$ as $c_j(t) := c_{lj}(t)^2$, and hence their time evolution is described as

$$c_j(t) = c_j(0)e^{8\kappa_j^3 t}. \quad (35)$$

As for the NLS equation and other integrable NPDEs, the time evolution of the related scattering data sets can be obtained in a similar way. For the former, in terms of the operator $\mathcal{A}$ in (14), the Jost solutions $\psi(\lambda, x, t)$ and $\phi(\lambda, x, t)$ appearing in (25) evolve according to the respective linear PDEs

$$\psi_t - \mathcal{A}\psi = -2i\lambda^2 \psi, \quad \phi_t - \mathcal{A}\phi = 2i\lambda^2 \phi.$$

The scattering coefficients appearing in (26) evolve according to

$$\begin{aligned} T(\lambda, t) &= T(\lambda, 0), \\ R(\lambda, t) &= R(\lambda, 0)e^{4i\lambda^2 t}, \\ L(\lambda, t) &= L(\lambda, 0)e^{-4i\lambda^2 t}. \end{aligned} \quad (36)$$

Associated with the bound-state pole λ_j of T , we have the bound-state norming constants $c_{js}(t)$ appearing in the Marchenko kernel $\mathcal{Q}(y, t)$ given in (41). Their time evolution is governed (Aktosun et al. 2007) by

$$\begin{aligned}
& \begin{bmatrix} c_{j(n_j-1)}(t) & c_{j(n_j-2)}(t) & \dots & c_{j0}(t) \end{bmatrix} \\
& = \begin{bmatrix} c_{j(n_j-1)}(0) & c_{j(n_j-2)}(0) & \dots & c_{j0}(0) \end{bmatrix} e^{-4iA_j^2 t},
\end{aligned} \tag{37}$$

where the $n_j \times n_j$ matrix A_j appearing in the exponent is defined as.

$$A_j := \begin{bmatrix} -i\lambda_j & -1 & 0 & \dots & 0 & 0 \\ 0 & -i\lambda_j & -1 & \dots & 0 & 0 \\ 0 & 0 & -i\lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -i\lambda_j & -1 \\ 0 & 0 & 0 & \dots & 0 & -i\lambda_j \end{bmatrix}.$$

Inverse Scattering Problem

In section “[Direct Scattering Problem](#)” we have seen how the initial scattering data $S(\lambda, 0)$ can be constructed from the initial profile $u(x, 0)$ of the potential by solving the direct scattering problem for the relevant LODE. Then, in section “[Time Evolution of the Scattering Data](#)” we have seen how to obtain the time-evolved scattering data $S(\lambda, t)$ from the initial scattering data $S(\lambda, 0)$. As the final step in the IST, in this section we outline how to obtain $u(x, t)$ from $S(\lambda, t)$ by solving the relevant inverse scattering problem. Such an inverse scattering problem may be solved by the Marchenko method (Aktosun and Klaus 2001; Chadan and Sabatier 1989; Deift and Trubowitz 1979; Faddeev 1967; Levitan 1987; Marchenko 1986; Melin 1985; Newton 1983). Unfortunately, in the literature many researchers refer to this method as the Gel’fand–Levitan method or the Gel’fand–Levitan–Marchenko method, both of which are misnomers. The Gel’fand–Levitan method (Aktosun and Klaus 2001; Chadan and Sabatier 1989; Levitan 1987; Marchenko 1986; Newton 1983) is a different method to solve the inverse scattering problem, and the corresponding Gel’fand–Levitan integral equation involves an integration on the finite interval $(0, x)$ and its kernel is related to the Fourier transform of the

spectral measure associated with the LODE. On the other hand, the Marchenko integral equation involves an integration on the semi-infinite interval $(x, +\infty)$, and its kernel is related to the Fourier transform of the scattering data.

In this section we first outline the recovery of the solution $u(x, t)$ to the KdV equation from the corresponding time-evolved scattering data $\{R, \{\kappa_j\}, \{c_j(t)\}\}$ appearing in (31) and (35). Later, we will also outline the recovery of the solution $u(x, t)$ to the NLS equation from the corresponding time-evolved scattering data $\{R, \{\lambda_j\}, \{c_{js}(t)\}\}$ appearing in (36) and (37).

The solution $u(x, t)$ to the KdV equation in (1) can be obtained from the time-evolved scattering data by using the Marchenko method as follows:

- (a) From the scattering data $\{R(k, t), \{\kappa_j\}, \{c_j(t)\}\}$ appearing in (31) and (35), form the Marchenko kernel Ω defined via

$$\begin{aligned}
\Omega(y, t) &:= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk R(k, t) e^{iky} \\
&+ \sum_{j=1}^N c_j(t) e^{-\kappa_j y}.
\end{aligned} \tag{38}$$

- (b) Solve the corresponding Marchenko integral equation

$$\begin{aligned}
& K(x, y, t) + \Omega(x + y, t) \\
& + \int_x^{\infty} dz K(x, z, t) \Omega(z + y, t) = 0, \tag{39} \\
& x < y < +\infty,
\end{aligned}$$

and obtain its solution $K(x, y, t)$.

- (c) Recover $u(x, t)$ by using

$$u(x, t) = -2 \frac{\partial K(x, x, t)}{\partial x}. \tag{40}$$

The solution $u(x, t)$ to the NLS equation in (3) can be obtained from the time-evolved scattering data by using the Marchenko method as follows:

- (i) From the scattering data $\{R(\lambda, t), \{\lambda_j\}, \{c_{js}(t)\}\}$ appearing in (36) and (37), form the Marchenko kernel Ω as

$$\begin{aligned} \Omega(y, t) := & \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda R(\lambda, t) e^{i\lambda y} + \sum_{j=1}^N \\ & \times \sum_{s=0}^{n_j-1} c_{js}(t) \frac{y^s}{s!} e^{i\lambda_j y}. \end{aligned} \quad (41)$$

- (ii) Solve the Marchenko integral equation

$$\begin{aligned} K(x, y, t) - \overline{\Omega(x+y, t)} + \int_x^{\infty} dz \\ \times \int_x^{\infty} ds K(x, s, t) \Omega(s+z, t) \overline{\Omega(z+y, t)} = 0, \\ x < y < +\infty, \end{aligned}$$

and obtain its solution $K(x, y, t)$.

- (iii) Recover $u(x, t)$ from the solution $K(x, y, t)$ to the Marchenko equation via

$$u(x, t) = -2K(x, x, t).$$

- (iv) Having determined $K(x, y, t)$, one can alternatively get $|u(x, t)|^2$ from

$$|u(x, t)|^2 = 2 \frac{\partial G(x, x, t)}{\partial x},$$

where we have defined

$$G(x, y, t) := - \int_x^{\infty} dz K(x, z, t) \overline{\Omega(z+y, t)}.$$

Solitons

A soliton solution to an integrable NPDE is a solution $u(x, t)$ for which the reflection coefficient in the corresponding scattering data is zero. In other words, a soliton solution $u(x, t)$ to an integrable NPDE is nothing but a reflectionless

potential in the associated LODE. When the reflection coefficient is zero, the kernel of the relevant Marchenko integral equation becomes separable. An integral equation Error! Reference source not found. a separable kernel can be solved explicitly by transforming that linear equation into a system of linear algebraic equations. In that case, we get exact solutions to the integrable NPDE, which are known as soliton solutions.

For the KdV equation the N -soliton solution is obtained by using $R(k, t) = 0$ in (38). In that case, letting

$$\begin{aligned} X(x) &:= \begin{bmatrix} e^{-\kappa_1 x} & e^{-\kappa_2 x} & \dots & e^{-\kappa_N x} \end{bmatrix}, \\ Y(y, t) &:= \begin{bmatrix} c_1(t) e^{-\kappa_1 y} \\ c_2(t) e^{-\kappa_2 y} \\ \vdots \\ c_N(t) e^{-\kappa_N y} \end{bmatrix}, \end{aligned}$$

we get $\Omega(x+y, t) = X(x) Y(y, t)$. As a result of this separability the Marchenko integral equation can be solved algebraically and the solution has the form $K(x, y, t) = H(x, t) Y(y, t)$, where $H(x, t)$ is a row vector with N entries that are functions of x and t . A substitution in (39) yields

$$K(x, y, t) = -X(x) \Gamma(x, t)^{-1} Y(y, t), \quad (42)$$

where the $N \times N$ matrix $\Gamma(x, t)$ is given by

$$\Gamma(x, t) := I + \int_x^{\infty} dz Y(z, t) X(z), \quad (43)$$

with I denoting the or $N \times N$ identity matrix. Equivalently, the (j, l) -entry of Γ is given by

$$\Gamma_{jl} = \delta_{jl} + \frac{c_j(0) e^{-2\kappa_j x + 8\kappa_j^3 t}}{\kappa_j + \kappa_l},$$

with δ_{jl} denoting the Kronecker delta. Using (42) in (40) we obtain

$$\begin{aligned} u(x, t) &= 2 \frac{\partial}{\partial x} [X(x) \Gamma(x, t)^{-1} Y(x, t)] \\ &= 2 \text{tr} \frac{\partial}{\partial x} [Y(x, t) X(x) \Gamma(x, t)^{-1}], \end{aligned}$$

where tr denotes the matrix trace (the sum of diagonal entries in a square matrix). From (43) we see that $-Y(x, t)X(x)$ is equal to the x -derivative of $\Gamma(x, t)$ and hence the N -soliton solution can also be written as

$$\begin{aligned} u(x, t) &= -2\text{tr} \frac{\partial}{\partial x} \left[\frac{\partial \Gamma(x, t)}{\partial x} \Gamma(x, t)^{-1} \right] \\ &= -2 \frac{\partial}{\partial x} \left[\frac{\frac{\partial}{\partial x} \det \Gamma(x, t)}{\det \Gamma(x, t)} \right], \end{aligned} \quad (44)$$

where $\det$ denotes the matrix determinant. When $N = 1$, we can express the one-soliton solution $u(x, t)$ to the KdV equation in the equivalent form

$$u(x, t) = -2\kappa_1^2 \text{sech}^2(\kappa_1 x - 4\kappa_1^3 t + \theta),$$

with $\theta := \log \sqrt{2\kappa_1/c_1(0)}$.

Let us mention that, using matrix exponentials, we can express (Aktosun and van der Mee 2006) the N -soliton solution appearing in (44) in various other equivalent forms such as

$$u(x, t) = -4C e^{-Ax+8A^3t} \Gamma(x, t)^{-1} A \Gamma(x, t)^{-1} e^{-Ax} B,$$

where

$$\begin{aligned} A &:= \text{diag}\{\kappa_1, \kappa_2, \dots, \kappa_N\}, \\ B^\dagger &:= \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}, \\ C &:= \begin{bmatrix} c_1(0) & c_2(0) & \dots & c_N(0) \end{bmatrix}. \end{aligned} \quad (45)$$

Note that a dagger is used for the matrix adjoint (transpose and complex conjugate), and B has N entries. In this notation we can express (43) as

$$\Gamma(x, t) = I + \int_x^\infty dz e^{-zA} B C e^{-zA} e^{8tA^3}.$$

As for the NLS equation, the well-known N -soliton solution (with simple bound-state poles) is obtained by choosing $R(\lambda, t) = 0$ and $n_j = 1$ in (41). Proceeding as in the KdV case, we obtain

the N -soliton solution in terms of the triplet A, B, C with

$$A := \text{diag}\{-i\lambda_1, -i\lambda_2, \dots, -i\lambda_N\}, \quad (46)$$

where the complex constants λ_j are the distinct poles of the transmission coefficient in $\mathbb{C}^+$, B and C are as in (45) except for the fact that the constants $c_j(0)$ are now allowed to be nonzero complex numbers. In terms of the matrices $P(x, t)$, M , and Q defined as

$$\begin{aligned} P(x, t) &:= \text{diag}\left\{e^{2i\lambda_1 x + 4i\lambda_1^2 t}, e^{2i\lambda_2 x + 4i\lambda_2^2 t}, \dots, \right. \\ &\quad \left. e^{2i\lambda_N x + 4i\lambda_N^2 t}\right\}, \\ M_{jl} &:= \frac{i}{\lambda_j - \bar{\lambda}_l}, Q_{jl} := \frac{-i\bar{c}_j c_l}{\bar{\lambda}_j - \lambda_l}. \end{aligned}$$

we construct the N -soliton solution $u(x, t)$ to the NLS equation as

$$\begin{aligned} u(x, t) &= -2B^\dagger \left[I + P(x, t)^\dagger Q P(x, t) M \right]^{-1} \\ &\quad P(x, t)^\dagger C^\dagger, \end{aligned} \quad (47)$$

or equivalently as

$$u(x, t) = -2B^\dagger e^{-A^\dagger x} \Gamma(x, t)^{-1} e^{-A^\dagger x + 4i(A^\dagger)^2 t} C^\dagger, \quad (48)$$

where we have defined

$$\begin{aligned} \Gamma(x, t) &:= I \\ &\quad + \left[\int_x^\infty ds \left(C e^{-As - 4iA^2 t} \right)^\dagger \left(C e^{-As - 4iA^2 t} \right) \right] \\ &\quad \times \left[\int_x^\infty dz \left(e^{-Az} B \right) \left(e^{-Az} B \right)^\dagger \right]. \end{aligned} \quad (49)$$

Using (45) and (46) in (49), we get the (j, l) -entry of $\Gamma(x, t)$ as

$$\Gamma_{jl} = \delta_{jl} - \sum_{m=1}^N \frac{\bar{c}_j c_l e^{i(2\lambda_m - \bar{\lambda}_j - \bar{\lambda}_l)x + 4i(\lambda_m^2 - \bar{\lambda}_j^2)t}}{(\lambda_m - \bar{\lambda}_j)(\lambda_m - \bar{\lambda}_l)}.$$

Note that the absolute square of $u(x, t)$ is given by

$$\begin{aligned} |u(x, t)|^2 &= \operatorname{tr} \left[\frac{\partial}{\partial x} \left(\Gamma(x, t)^{-1} \frac{\partial \Gamma(x, t)}{\partial x} \right) \right] \\ &= \frac{\partial}{\partial x} \left[\frac{\frac{\partial}{\partial x} \det \Gamma(x, t)}{\det \Gamma(x, t)} \right]. \end{aligned}$$

For the NLS equation, when $N = 1$, from (47) or (48) we obtain the single-soliton solution

$$u(x, t) = \frac{-8\bar{c}_1 (\operatorname{Im}[\lambda_1])^2 e^{-2i\bar{\lambda}_1 x - 4i(\bar{\lambda}_1)^2 t}}{4(\operatorname{Im}[\lambda_1])^2 + |c_1|^2 e^{-4x(\operatorname{Im}[\lambda_1]) - 8t(\operatorname{Im}[\lambda_1]^2)}},$$

where Im denotes the imaginary part.

Future Directions

There are many issues related to the IST and solitons that cannot be discussed in such a short review. We will briefly mention only a few.

Can we characterize integrable NPDEs? In other words, can we find a set of necessary and sufficient conditions that guarantee that an IVP for a NPDE is solvable via an IST? Integrable NPDEs seem to have some common characteristic features (Ablowitz and Clarkson 1991) such as possessing Lax pairs, AKNS pairs, soliton solutions, infinite number of conserved quantities, a Hamiltonian formalism, the Painlevé property, and the Bäcklund transformation. Yet, there does not seem to be a satisfactory solution to their characterization problem.

Another interesting question is the determination of the LODE associated with an IST. In other words, given an integrable NPDE, can we determine the corresponding LODE? There does not yet seem to be a completely satisfactory answer to this question.

When the initial scattering coefficients are rational functions of the spectral parameter, representing the time-evolved scattering data in terms of matrix exponentials results in the separability of the kernel of the Marchenko integral equation. In that case, one obtains

explicit formulas (Aktosun et al. 2007; Aktosun and van der Mee 2006) for exact solutions to some integrable NPDEs and such solutions are constructed in terms of a triplet of constant matrices A , B , C whose sizes are $p \times p$, $p \times 1$, and $1 \times p$, respectively, for any positive integer p . Some special cases of such solutions have been mentioned in section “Solitons”, and it would be interesting to determine if such exact solutions can be constructed also when p becomes infinite.

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Different Analytical Methods for Solving the Korteweg-de Vries Equation (KdV)

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Article Outline

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Glossary

Analytical method for solving an equation The
method for obtaining the exact solutions of an
equation.

Solitons The special solitary waves which retain
their original shapes and speeds after collision
and exhibit only a small overall phase shift.

e ± 1

t the time

(x, y) Cartesian coordinates of a point

∂_x^{-1} an indefinite integrate operator $\int dx$.

Definition of the Subject

This paper presents the analytical methods for
obtaining the exact solutions of the Korteweg–de
Vries Equation (KdV equation).

The KdV equation and its exact solutions can
describe and explain many physical problems. In
addition, it is a typical, relatively simple and classi-
cal equation among the many nonlinear equations in
physics. Much of the literature of nonlinear equation
theory customarily uses solving soliton solutions of
the KdV equation as an example to introduce the
nonlinear theory, method and character of soliton
solutions. During the last five decades, the construc-
tion of exact solution for a wide class of nonlinear
equations has been an exciting and extremely active
area of research. This includes the most famous
nonlinear example of the KdV equation.

In the family of the KdV equations, a well
known KdV equation is expressed in its simplest
form as (Korteweg and de Vries 1895; Khaters et al.
1997; Helal 2001; Brugarino and Sciacca 2008):

$$u_t(x, t) + \alpha u(x, t)u_x(x, t) + \beta u_{xxx}(x, t) = 0. \quad (1)$$

where the coefficient of the nonlinear term α and
the coefficient of the dispersive term β are inde-
pendent of x and t .

The KdV equation was derived in 1885 by the
two scientists, Korteweg and de Vries (1895), to
describe long wave propagation on shallow water.
Its importance comes from its many applications in
weakly nonlinear and weakly dispersive physical
systems and its interesting mathematical properties.
It can be considered as a paradigm in nonlinear
science. The equation is so powerful it has been

used to simulate the red spots on Jupiter. Moreover, it describes non-linear waves in rotating fluids (Leibovitch 1970), the giant ocean waves known as Tsunami, and other aspects of solitary waves in plasma. The giant internal waves in the interior of the ocean arising from temperature differences which may destroy marine vessels can be also described by the powerful KdV equation. It possesses infinitely many generalized symmetries and infinitely many integrals of motion (Miura et al. 1968), and can be viewed as a completely integrable Hamiltonian system and solved by the inverse scattering transform. It also provides resources for studying integrability of nonlinear differential (and difference) equations, serving as a typical model of integrable equations. More remarkable is that various physically important solutions to the KdV equation can be presented explicitly in a simple way, among which are solitons, rational solutions, positons and negatons.

The KdV equation has been widely investigated and developed continually in recent decades. Several generalizations of the KdV equation have found applications in many areas, including quantum field theory, plasma physics, solid-state physics, liquid-gas bubble mixtures, and anharmonic crystals. For instance, the classical generalized Korteweg–de Vries equation (gKdV) has the following form (Helal and Mehanna 2007; Hamdia et al. 2005; Sarma 2008)

$$u_t(x, t) + \alpha u^p(x, t)u_x(x, t) + \beta u_{xxx}(x, t) = 0. \quad (2)$$

where the coefficient of the nonlinear term α and the coefficient of the dispersive term β are independent of x and t .

The more generalized form of the KdV equation is defined by parameters (l, p) as follows (Karpman 1998; Hassan 2004; Khare and Cooper 1993; Dodan 2005; Shen and Xu 2007)

$$\begin{aligned} u_t(x, t) = & u^{l-2}(x, t)u_x(x, t) + \alpha \left[2u^l(x, t)u_{xxx}(x, t) \right. \\ & + 4pu^{p-1}(x, t)u_x(x, t)u_{xx}(x, t) \\ & \left. + p(p-1)u^{p-2}(x, t)u_x^3(x, t) \right]. \end{aligned} \quad (3)$$

Propagation of weakly nonlinear long waves in an inhomogeneous waveguide is governed by a

variable-coefficient KdV equation of the form (Clarke et al. 2002).

$$u_t(x, t) + 6u(x, t)u_x(x, t) + B(t)u_{xxx}(x, t) = 0. \quad (4)$$

where $u(x, t)$ is the wave amplitude, t the propagation coordinate, x the temporal variable and $B(t)$ is the local dispersion coefficient.

The applicability of the variable-coefficient KdV equation (4) arises in many areas of physics as, for example, the description of the propagation of gravity-capillary and interfacial-capillary waves, internal waves and Rossby waves (Clarke et al. 2002). In order to study the propagation of weakly nonlinear, weakly dispersive waves in the inhomogeneous media, Eq. (4) is rewritten as follows (Houria 2007)

$$u_t(x, t) + 6A(t)u(x, t)u_x(x, t) + B(t)u_{xxx}(x, t) = 0. \quad (5)$$

which includes a variable nonlinearity coefficient $A(t)$.

In studying long waves, tides, solitary waves, and related phenomena, one is led to an equation of the form (Kaya and Aassila 2002)

$$u_t(x, t) + (m+1)(m+2)u^m(x, t)u_x(x, t) + u_{xxx}(x, t) = f(x, t). \quad (6)$$

where $f(x, t)$ is a given function and $m = 1, 2, \dots$, with $u(x, t), u_x(x, t), u_{xx}(x, t) \rightarrow 0$ as $|x| \rightarrow +\infty$. This equation is referred to as a generalized Korteweg–de Vries equation in Clarke et al. (2002).

The extended Korteweg–de Vries (eKdV) equation (Shek and Chow 2008)

$$u_t(x, t) + \alpha u(x, t)u_x(x, t) + \beta u^2(x, t)u_x(x, t) + \delta u_{xxx}(x, t) = 0. \quad (7)$$

incorporates both quadratic and cubic nonlinearities, and serves as a useful model in the propagation of long waves in physical oceanography.

A system of two coupled Korteweg–de Vries equations in Brugarino and Sciacca (2008) is given by the following equation:

$$\begin{aligned} u_{it}(x, t) + \sum_{h,k} A_i^{hk} u_h(x, t) u_{kx}(x, t) \\ + \sum_k A_i^k u_{kxxx}(x, t) \\ = 0, \quad i, k, h = 1, 2. \end{aligned} \quad (8)$$

The (2 + 1)-dimensional KdV equation is presented by the following equation (Zhang 2007):

$$\begin{cases} u_t(x, y, t) - 3v(x, y, t)u_x(x, y, t) \\ - 3v_x(x, y, t)u(x, y, t) + u_{xxx}(x, y, t) = 0, \\ u_x(x, y, t) - v_y(x, y, t) = 0. \end{cases} \quad (9)$$

Another (2 + 1)-dimensional KdV equation in Song and Zhang (2007) is written the following form

$$\begin{aligned} u_t(x, y, t) - 4u(x, y, t)u_y(x, y, t) \\ - 4u_x(x, y, t)\partial_x^{-1}u_y(x, y, t) \\ - u_{xy}(x, y, t) = 0. \end{aligned} \quad (10)$$

where ∂_x^{-1} is an indefinite integrate operator $\int dx$, possessing some interesting coherent structures.

In addition, the complex coupled KdV equation in Wu et al. (1999) is written the following form

$$\begin{cases} u_t(x, t) = \frac{1}{2}[u_{xxx}(x, t) - 6u(x, t)u_x(x, t)] \\ + 3\left(|v(x, t)|^2\right)_x, \quad v_t(x, t) = -v_{xxx}(x, t) \\ + 3u(x, t)v_x(x, t). \end{cases} \quad (11)$$

Introduction

It is well known that nonlinear complex physical phenomena are related to nonlinear partial differential equations (NLPDEs), which are involved in many fields from physics to biology, chemistry,

mechanics, etc. The investigation of exact solutions of NLPDEs as mathematical models of phenomena can help one to better understand a variety of phenomena. In the past several decades, many analytical methods for obtaining exact solutions of NLPDEs have been presented, such as the inverse scattering method, Hirota's bilinear method (Hirota 1971), the Bäcklund transformation (Wadati et al. 1975), the Painlevé expansion (Weiss et al. 1983), the tanh function method (Fan and Zhang 1998), the sine-cosine method (Wazwaz 2005), the homogenous balance method (Wang 1996), the homotopy perturbation method (He 2005), the variational method (He 2004), asymptotic methods (He 2006a), non-perturbative methods (He 2006b), the exp-function method (He and Wu 2006), the Adomian Pade approximation (Abassy et al. 2004), the Jacobi elliptic function expansion method (Liu et al. 2001), the F-expansion method (Zhou et al. 2003), the Weierstrass semi-rational expansion method (Chen and Yan 2006), the unified rational expansion method (Chen and Wang 2006), the algebraic method (Zhang and Xia 2006; Fan and Dai 2003; Yan 2004), the auxiliary equation method (Xu and Li 2005), and so on.

In recent years, based on the ideas of unification methods, algorithm realization and mechanization for solving NLPDEs, we have improved and presented some analytical methods for obtaining exact solutions of NLPDEs, such as the generalized hyperbolic function – the Bäcklund transformation method (Ren and Zhang 2006a), the method for constructing higher order and higher dimension Bäcklund transformation (Ren 2007a), the generalized hyperbolic function – the Riccati method (Ren 2007a), the generalized F-expansion method (Ren and Zhang 2006b; Ren et al. 2006a), the extended generalized algebraic method (Ren et al. 2007, 2008a), the Exp-Bäcklund method (Ren et al. 2008b; Ren 2007b), the Exp-N soliton-like method (Ren et al. 2008c; Ren 2007c), the extended sine-cosine method (Wang et al. 2005), and so on. The present article is motivated by the desire to introduce and make use of our works published in Ren and Zhang (2006a, b), Ren (2007a, b, c), and Ren et al. (2006a, 2007, 2008a, b, c) to construct more general exact solutions, which contain not

only the results obtained by using the methods (Korteweg and de Vries 1895; Khaters et al. 1997; Helal 2001; Brugarino and Sciacca 2008; Leibovitch 1970; Miura et al. 1968; Helal and Mehanna 2007; Hamdia et al. 2005; Sarma 2008; Karpman 1998; Hassan 2004; Khare and Cooper 1993; Dodan 2005; Shen and Xu 2007; Clarke et al. 2002; Houria 2007; Kaya and Aassila 2002; Shek and Chow 2008; Zhang 2007; Song and Zhang 2007; Hirota 1971; Wadati et al. 1975; Weiss et al. 1983; Fan and Zhang 1998; Wazwaz 2005; Wang 1996; He 2004, 2005, 2006a, b; He and Wu 2006; Abassy et al. 2004; Liu et al. 2001; Zhou et al. 2003; Chen and Yan 2006; Chen and Wang 2006; Zhang and Xia 2006; Xu and Li 2005; Wu et al. 1999; Fan and Dai 2003; Yan 2004; Ablowitz and Clarkson 1991; Dodd et al. 1984; Drazin and Johnson 1989; Fan 2004; Guo 1996; Liu and Liu 2000; Miura 1976; Miurs 1978), but also a series of new and more general exact solutions. For illustration, we apply our analytical methods above to the different types of the KdV equations, and successfully obtain many new and more general exact solutions.

The remainder of this entry is organized as follows. In section “[The Generalized Hyperbolic Function–Bäcklund Transformation Method and Its Application in the \(2+1\)-Dimensional KdV Equation](#)”, in order to construct the unification solutions of the hyperbolic function solution and exponential function solution of NLPDEs, we first find a new theory of generalized hyperbolic functions which includes two new definitions of generalized hyperbolic functions and generalized hyperbolic function transformations and their properties. Then we present a new higher order and higher dimension Bäcklund transformation method and a new generalized hyperbolic function – Bäcklund transformation method. The validity of the methods are tested by their application in the $(2 + 1)$ -dimensional KdV equation (9). In section “[The Generalized F-Expansion Method and Its Application in Another \(2 + 1\)-Dimensional KdV Equation](#)”, we present a new and more general formal transformation and a new generalized F-expansion method and apply them to another $(2 + 1)$ -dimensional KdV equation (10). In section “[The Generalized Algebra Method and Its Application in](#)

[\(1 + 1\)-Dimensional Generalized Variable – Coefficient KdV Equation](#)”, in order to develop the algebraic method (Fan and Dai 2003; Yan 2004) for constructing the traveling wave solutions of NLPDEs, we first present two new and general transformations, two new theorems and their proofs, by using Maple, and a new mechanization method to find the exact solutions of a first-order nonlinear ordinary differential equations with any degree. The validity of the method is tested by its application in the first-order nonlinear ordinary differential equation with six degrees. Next, we present a new and more general transformation and a new generalized algebra method and apply them to the $(1 + 1)$ -dimensional generalized variable – coefficient KdV equation (5). In section “[A New Exp-N Solitary-Like Method and Its Application in the \(1 + 1\)-Dimensional Generalized KdV Equation](#)”, in order to develop the Exp-function method (He and Wu 2006), we present two new generic transformations, a new Exp-N solitary-like method, and its algorithm. In addition, we apply our methods to construct new exact solutions of the $(1 + 1)$ -dimensional classical generalized KdV (gKdV) equation (2); In section “[The Exp-Bäcklund Transformation Method and Its Application in \(1 + 1\)-Dimensional KdV Equation](#)”, a new Exp-Bäcklund transformation method and its algorithm is presented to find more exact solutions of NLPDEs. We choose the $(1 + 1)$ -dimensional KdV equation (1) to illustrate the effectiveness and convenience of our algorithm. As a result, we obtain more general exact solutions including non-traveling wave solutions and traveling wave solutions. In addition, the long-term behavior of the non-traveling wave solutions and traveling wave solutions are illustrated by some Figures. In section “[Future Directions](#)”, future directions for Solving NLPDEs are given.

The Generalized Hyperbolic Function–Bäcklund Transformation Method and Its Application in the $(2 + 1)$ -Dimensional KdV Equation

In this section, in order to construct the unification solutions of the hyperbolic function solution and the

exponential function solution of NLPDEs, we establish a new theory of generalized hyperbolic functions, which includes two new definitions of generalized hyperbolic functions and generalized hyperbolic function transformations and their properties that we first presented in Ren and Zhang (2006a) and Ren (2007a). Then we present a new higher order and higher dimension Bäcklund transformation method and a new generalized hyperbolic function-Bäcklund transformation method that we first presented in Ren and Zhang (2006a) and Ren (2007a). With the aid of symbolic computation, we choose the $(2 + 1)$ -dimensional Korteweg-de Vries equations to illustrate the validity and advantages of the methods. As a result, many new and more general exact non-traveling waves are obtained.

The Definition and Properties of Generalized Hyperbolic Functions

In Ren and Zhang (2006a), we first defined the following new functions which named generalized hyperbolic functions and studied the properties of these functions for constructing new exact solutions of NLPDEs.

Definition 1 Suppose that ξ is an independent variable, p , q and k are constants.

The **generalized hyperbolic sine function** is

$$\sinh_{pqk}(\xi) = \frac{pe^{k\xi} - qe^{-k\xi}}{2} \quad (12)$$

generalized hyperbolic cosine function is

$$\cosh_{pqk}(\xi) = \frac{pe^{k\xi} + qe^{-k\xi}}{2} \quad (13)$$

generalized hyperbolic tangent function is

$$\tanh_{pqk}(\xi) = \frac{pe^{k\xi} - qe^{-k\xi}}{pe^{k\xi} + qe^{-k\xi}} \quad (14)$$

generalized hyperbolic cotangent function is

$$\coth_{pqk}(\xi) = \frac{pe^{k\xi} + qe^{-k\xi}}{pe^{k\xi} - qe^{-k\xi}} \quad (15)$$

generalized hyperbolic secant function is

$$\operatorname{sech}_{pqk}(\xi) = \frac{2}{pe^{k\xi} + qe^{-k\xi}} \quad (16)$$

generalized hyperbolic cosecant function is

$$\operatorname{csch}_{pqk}(\xi) = \frac{2}{pe^{k\xi} - qe^{-k\xi}} \quad (17)$$

the above six kinds of functions are said **generalized hyperbolic functions**.

Thus we can prove the following theory of generalized hyperbolic functions on the basis of Definition 1.

Theorem 1 The generalized hyperbolic functions satisfy the following relations:

$$\cosh_{pqk}^2(\xi) - \sinh_{pqk}^2(\xi) = pq, \quad (18)$$

$$1 - \tanh_{pqk}^2(\xi) = pq \cdot \operatorname{sech}_{pqk}^2(\xi), \quad (19)$$

$$1 - \coth_{pqk}^2(\xi) = -pq \cdot \operatorname{csch}_{pqk}^2(\xi), \quad (20)$$

$$\operatorname{sech}_{pqk}(\xi) = \frac{1}{\cosh_{pqk}(\xi)}, \quad (21)$$

$$\operatorname{csch}_{pqk}(\xi) = \frac{1}{\sinh_{pqk}(\xi)}, \quad (22)$$

$$\tanh_{pqk}(\xi) = \frac{\sinh_{pqk}(\xi)}{\cosh_{pqk}(\xi)}, \quad (23)$$

$$\coth_{pqk}(\xi) = \frac{\cosh_{pqk}(\xi)}{\sinh_{pqk}(\xi)}, \quad (24)$$

$$\tanh_{pqk}(\xi) = \frac{1}{\coth_{pqk}(\xi)}, \quad (25)$$

$$\sinh_{pqk}(-\xi) = -\sinh_{pqk}(\xi), \quad (26)$$

$$\cosh_{pqk}(-\xi) = \cosh_{pqk}(\xi), \quad (27)$$

$$\tanh_{pqk}(-\xi) = -\tanh_{pqk}(\xi), \quad (28)$$

$$\coth_{pqk}(-\xi) = -\coth_{pqk}(\xi), \quad (29)$$

$$\operatorname{sech}_{pqk}(-\xi) = \operatorname{sech}_{pqk}(\xi), \quad (30)$$

$$\operatorname{csch}_{pqk}(-\xi) = -\operatorname{csch}_{pqk}(\xi), \quad (31) \quad \textbf{Proof} \text{ By Eqs. (18) and (21), we obtain}$$

$$\begin{aligned} \sinh_{11k}(\xi - \eta) &= \frac{1}{pq} \left[\sinh_{pqk}(\xi) \cdot \cosh_{pqk}(\eta) \right. \\ &\quad \left. - \cosh_{pqk}(\xi) \cdot \sinh_{pqk}(\eta) \right], \end{aligned} \quad (32)$$

$$\begin{aligned} \sinh_{p^2q^2k}(\xi + \eta) &= \sinh_{pqk}(\xi) \cdot \cosh_{pqk}(\eta) \\ &+ \cosh_{pqk}(\xi) \cdot \sinh_{pqk}(\eta), \end{aligned} \quad (33)$$

$$\begin{aligned} \cosh_{p^2q^2k}(\xi + \eta) &= \cosh_{pqk}(\xi) \cdot \cosh_{pqk}(\eta) \\ &+ \sinh_{pqk}(\xi) \cdot \sinh_{pqk}(\eta), \end{aligned} \quad (34)$$

$$\begin{aligned} \cosh_{11k}(\xi - \eta) &= \frac{1}{pq} \left[\cosh_{pqk}(\xi) \cdot \cosh_{pqk}(\eta) \right. \\ &\quad \left. - \sinh_{pqk}(\xi) \cdot \sinh_{pqk}(\eta) \right], \end{aligned} \quad (35)$$

$$\sinh_{q^2p^2k}(2\xi) = 2 \sinh_{pqk}(\xi) \cdot \cosh_{pqk}(\xi), \quad (36)$$

$$\cosh_{p^2q^2k}(2\xi) = \sinh_{pqk}^2(\xi) + \cosh_{pqk}^2(\xi). \quad (37)$$

$$\begin{aligned} &\frac{\tanh_{pqk}(\xi) + \tanh_{pqk}(\eta)}{1 - \tanh_{pqk}(\xi) \tanh_{pqk}(\eta)} \\ &= \frac{\sinh_{p^2q^2k}(\xi + \eta)}{p \cdot q \cdot \cosh_{11k}(\xi - \eta)}, \end{aligned} \quad (38)$$

$$\begin{aligned} &\frac{\tanh_{pqk}(\xi) - \tanh_{pqk}(\eta)}{1 + \tanh_{pqk}(\xi) \tanh_{pqk}(\eta)} \\ &= \frac{p \cdot q \cdot \sinh_{11k}(\xi - \eta)}{\cosh_{p^2q^2k}(\xi + \eta)}. \end{aligned} \quad (39)$$

For simplicity, only a sample of these are proved here.

$$\begin{aligned} 1 - \tanh_{pqk}^2(\xi) &= 1 - \frac{\sinh_{pqk}^2(\xi)}{\cosh_{pqk}^2(\xi)} \\ &= \frac{\cosh_{pqk}^2(\xi) - \sinh_{pqk}^2(\xi)}{\cosh_{pqk}^2(\xi)} \\ &= \frac{pq}{\cosh_{pqk}^2(\xi)} = pq \cdot \operatorname{sech}_{pqk}^2(\xi). \end{aligned}$$

Similarly, we can prove other conclusions in Theorem 1.

Theorem 2 The derivative formulae of generalized hyperbolic functions are as follows:

$$\frac{d(\sinh_{pqk}(\xi))}{d\xi} = k \cdot \cosh_{pqk}(\xi), \quad (40)$$

$$\frac{d(\cosh_{pqk}(\xi))}{d\xi} = k \cdot \sinh_{pqk}(\xi), \quad (41)$$

$$\frac{d(\tanh_{pqk}(\xi))}{d\xi} = kpq \cdot \operatorname{sech}_{pqk}^2(\xi), \quad (42)$$

$$\frac{d(\coth_{pqk}(\xi))}{d\xi} = -kpq \cdot \operatorname{csch}_{pqk}^2(\xi), \quad (43)$$

$$\begin{aligned} \frac{d(\operatorname{sech}_{pqk}(\xi))}{d\xi} &= -k \cdot \operatorname{sech}_{pqk}(\xi) \\ &\quad \cdot \tanh_{pqk}(\xi), \end{aligned} \quad (44)$$

$$\begin{aligned} \frac{d(\operatorname{csch}_{pqk}(\xi))}{d\xi} &= -k \cdot \operatorname{csch}_{pqk}(\xi) \\ &\quad \cdot \coth_{pqk}(\xi). \end{aligned} \quad (45)$$

Proof According to Eqs. (40) and (41), we can get

$$\begin{aligned} \frac{d(\tanh_{pqk}(\xi))}{d\xi} &= \left(\frac{\sinh_{pqk}(\xi)}{\cosh_{pqk}(\xi)} \right)' \\ &= \frac{(\sinh_{pqk}(\xi))' \cosh_{pqk}(\xi) - \sinh_{pqk}(\xi) (\cosh_{pqk}(\xi))'}{\cosh_{pqk}^2(\xi)} \\ &= \frac{k \cdot \cosh_{pqk}(\xi) \cdot \cosh_{pqk}(\xi) - \sinh_{pqk}(\xi) \cdot (k \cdot \sinh_{pqk}(\xi))}{\cosh_{pqk}^2(\xi)} \\ &= kpq \cdot \operatorname{sech}_{pqk}^2(\xi). \end{aligned}$$

Similarly, we can prove other differential coefficient formulae in Theorem 2.

Remark 1 We see that when $p = 1, q = 1, k = 1$ in Eqs. (12), (13), (14), (15), (16), and (17), new generalized hyperbolic functions $\sinh_{pqk}(\xi)$, $\cosh_{pqk}(\xi)$, $\tanh_{pqk}(\xi)$, $\coth_{pqk}(\xi)$, $\operatorname{sech}_{pqk}(\xi)$ and $\operatorname{csch}_{pqk}(\xi)$ degenerate as hyperbolic functions $\sinh(\xi)$, $\cosh(\xi)$, $\tanh(\xi)$, $\coth(\xi)$, $\operatorname{sech}(\xi)$ and $\operatorname{csch}(\xi)$, respectively. In addition, when $p = 0$ or $q = 0$ in Eqs. (12), (13), (14), (15), (16), and (17), $\sinh_{pqk}(\xi)$, $\cosh_{pqk}(\xi)$, $\operatorname{sech}_{pqk}(\xi)$, $\operatorname{csch}_{pqk}(\xi)$, $\tanh_{pqk}(\xi)$ and $\coth_{pqk}(\xi)$ degenerate as exponential functions $\frac{1}{2}pe^{k\xi}$, $\pm \frac{1}{2}qe^{-k\xi}$, $2pe^{-k\xi}$, $\pm 2qe^{k\xi}$ and ± 1 , respectively.

A New Higher Order and Higher Dimension Bäcklund Transformation Method to Construct an Auto-Bäcklund Transformation of the (2 + 1)-Dimensional KdV Equation

In this section, we obtain an auto-Bäcklund transformation of the following (2 + 1)-dimensional KdV equation by making use of the method for constructing higher order and higher dimension Bäcklund transformations which we presented in Ren and Zhang (2006a) and Ren (2007a) via symbolic computation.

The (2 + 1)-dimensional KdV equations (Zhang 2007) take the form

$$\begin{cases} u_t(x, y, t) - 3v(x, y, t)u_x(x, y, t) \\ - 3v_x(x, y, t)u(x, y, t) + u_{xx}(x, y, t) = 0, \\ u_x(x, y, t) - v_y(x, y, t) = 0. \end{cases} \quad (46)$$

The auto-Bäcklund transformations of Eq. (46) have the following forms:

$$\begin{cases} u(x, y, t) = \sum_{j_1=0}^{m_1} u_{j_1}(x, y, t) f^{j_1-m_1}(x, y, t), \\ v(x, y, t) = \sum_{j_2=0}^{m_2} v_{j_2}(x, y, t) f^{j_2-m_2}(x, y, t), \end{cases} \quad (47)$$

where $f(x, y, t), u_{j_1}(x, y, t), j_1 = 0, 1, 2, \dots, m_1 - 1$ and $v_{j_2}(x, y, t), j_2 = 0, 1, 2, \dots, m_2 - 1$ are all differential functions to be determined later, and $u_{m_1}(x, y, t), v_{m_2}(x, y, t)$ are the trivial seed solutions of Eq. (46).

By balancing the highest order linear term and nonlinear terms in Eq. (46), we get $m_1 = 2$ and $m_2 = 2$, and Eq. (47) has the following formal

$$\begin{cases} u(x, y, t) = u_0(x, y, t)f^{-2}(x, y, t) \\ + u_1(x, y, t)f^{-1}(x, y, t) + u_2(x, y, t), \\ v(x, y, t) = v_0(x, y, t)f^{-2}(x, y, t) \\ + v_1(x, y, t)f^{-1}(x, y, t) + v_2(x, y, t), \end{cases} \quad (48)$$

We take the trivial seed solutions as

$$u_2 = u_2(x, y, t), v_2 = v_2(x, y, t). \quad (49)$$

With the aid of Maple symbolic computation software, substituting Eqs. (48) and (49) into Eq. (46), and collecting all terms with $f^{-i}(x, y, t)$, $i = 0, 1, 2, \dots$, we obtain

$$\begin{aligned} & (-24u_0f_x^3 + 12u_0v_0f_x)f^{-5} \\ & + (12u_1f_x^3 + 36f_x^2f_yf_{x^2} + 18v_1f_x^2f_y + 24f_x^3f_{xy})f^{-4} \\ & + \dots = 0, \end{aligned} \quad (50)$$

$$\begin{aligned} & (-2u_0f_x + 2v_0f_y)f^{-3} \\ & + (u_{0x} - u_1f_x + v_1f_y - v_{0y})f^{-2} \\ & + (u_{1x} - v_{1y})f^{-1} + u_{2x} - v_{2y} = 0, \end{aligned} \quad (51)$$

Setting the coefficient of f^{-5} in Eq. (50) and f^{-3} in Eq. (51) to be zero, we obtain a differential equation

$$-24u_0f_x^3 + 12u_0v_0f_x = 0, \quad (52)$$

$$-2u_0f_x + 2v_0f_y = 0, \quad (53)$$

which has the solution

$$u_0 = 2f_xf_y, \quad v_0 = 2f_x^2. \quad (54)$$

Setting the coefficients of f^{-4} in Eq. (50) and f^{-2} in Eq. (51) to be zero, we obtain a differential equation

$$12u_1f_x^3 + 36f_x^2f_yf_{x^2} + 18v_1f_x^2f_y + 24f_x^3f_{xy} = 0, \quad (55)$$

$$u_{0x} - u_1f_x + v_1f_y - v_{0y} = 0, \quad (56)$$

from which we can get the following expression:

$$u_1 = -2f_{yx}, \quad v_1 = -2f_{xx}. \quad (57)$$

By Eqs. (48), (49), (54), and (57), we obtain an auto-Bäcklund transformation of Eq. (46)

$$\begin{aligned} u &= -2\partial_{yx} \ln(f(x, y, t)) + u_2(x, y, t), \\ v &= -2\partial_{xx} \ln(f(x, y, t)) + v_2(x, y, t), \end{aligned} \quad (58)$$

where $u_2(x, y, t)$, $v_2(x, y, t)$ are the trivial seed solutions of Eq. (46), $f = f(x, y, t)$ satisfies the Eqs. (52), (53), (55), (56) and the following equations

$$\begin{aligned} &-u_0v_{1x} - u_1v_{0x} + 2u_1v_1f_x + 2u_2v_0f_x - 2/3u_0f_t \\ &-2/3u_0f_{x^3} + 2u_1f_xf_{x^2} - 2u_0f_{x^2} \\ &+ 2u_1f_x^2 - 2u_0x^2f_x - v_0u_{1x} - v_1u_{0x} \\ &+ 2v_2u_0f_x = 0, \\ &u_{0x^3} - u_1f_{x^3} - 3u_0v_{2x} - 3u_1v_{1x} - 3u_2v_{0x} \\ &+ 3u_2v_1f_x - 3u_1f_{x^2} - u_1f_t - 3v_0u_{2x} - 3v_1u_{1x} \\ &- 3v_2u_{0x} + 3v_2u_1f_x + u_{0t} - 3u_{1x^2}f_x = 0. \end{aligned} \quad (59)$$

The Generalized Hyperbolic Function–Bäcklund Transformation Method and Its Application in the (2 + 1)-Dimensional KdV Equation

We first give the definition of a generalized hyperbolic function transformation (Ren 2007a) as follows:

Definition 2 If a transformation includes the generalized hyperbolic functions, then the transformation is defined to be a generalized hyperbolic function transformation.

In this section, we will use the generalized hyperbolic function–Bäcklund transformation

method to seek new exact solutions of Eq. (46). We take $f(x, y, t)$ in Eqs. (48), (49), (54), (57), and (59) as being of a new form which is the following generalized hyperbolic function transformation

$$\begin{aligned} f(x, y, t) &= H(t) + \sum_{i=1}^n K_i(t) \\ &\quad \cdot F_i(\xi_i(x, y, t)) \\ &\quad \cdot G_i(\eta_i(x, y, t)), \end{aligned} \quad (60)$$

where $F_i(\xi_i)$ and $G_i(\eta_i)$ may take any two generalized hyperbolic functions among $\sinh_{pqk}(\xi)$, $\cosh_{pqk}(\xi)$, $\tanh_{pqk}(\xi)$, $\coth_{pqk}(\xi)$, $\operatorname{sech}_{pqk}(\xi)$ and $\operatorname{csch}_{pqk}(\xi)$. And then Eq. (60) has 21 types of general forms. For example.

$$\begin{aligned} f(x, y, t) &= H(t) + \sum_{i=1}^n K_i(t) \\ &\quad \cdot \operatorname{sech}_{p_{1i}q_{1i}k_{1i}}(\xi_i(x, y, t)) \\ &\quad \cdot \tanh_{p_{2i}q_{2i}k_{2i}}(\eta_i(x, y, t)), \end{aligned} \quad (61)$$

$$\begin{aligned} f(x, y, t) &= H(t) + \sum_{i=1}^n K_i(t) \\ &\quad \cdot \operatorname{csch}_{p_iq_ik_i}^2(\xi_i(x, y, t)), \end{aligned} \quad (62)$$

We see that when $p = 1$, $q = 1$, $k = 1$ in (12)–(12), new generalized hyperbolic function $\sinh_{pqk}(\xi)$, $\cosh_{pqk}(\xi)$, $\tanh_{pqk}(\xi)$, $\coth_{pqk}(\xi)$, $\operatorname{sech}_{pqk}(\xi)$ and $\operatorname{csch}_{pqk}(\xi)$ degenerate as hyperbolic functions $\sinh(\xi)$, $\cosh(\xi)$, $\tanh(\xi)$, $\coth(\xi)$, $\operatorname{sech}(\xi)$ and $\operatorname{csch}(\xi)$ respectively. Therefore (3.49) includes 21 types of hyperbolic function forms. For example

$$\begin{aligned} f(x, y, t) &= H(t) + \sum_{i=1}^n K_i(t) \\ &\quad \cdot \sinh(\xi_i(x, y, t)) \\ &\quad \cdot \operatorname{sech}(\eta_i(x, y, t)), \end{aligned} \quad (63)$$

and thirty-six types of the omnibus forms of generalized hyperbolic functions and hyperbolic functions. For example

$$f(x, y, t) = H(t) + \sum_{i=1}^n K_i(t) \cdot \operatorname{sech}(\xi_i(x, y, t)) \cdot \operatorname{csch}_{p_{2i}q_{2i}k_{2i}}(\eta_i(x, y, t)), \quad (64)$$

In addition, when $p = 0$ or $q = 0$ in Eqs. (12), (13), (14), (15), (16), and (17), $\sinh_{pqk}(\xi)$, $\cosh_{pqk}(\xi)$, $\operatorname{sech}_{pqk}(\xi)$, $\operatorname{csch}_{pqk}(\xi)$, $\tanh_{pqk}(\xi)$ and $\coth_{pqk}(\xi)$ degenerate as exponential functions $\frac{1}{2}pe^{k\xi}$, $\pm \frac{1}{2}qe^{-k\xi}$, $2pe^{-k\xi}$, $\pm 2qe^{k\xi}$ and ± 1 respectively.

For example, we take $F_i(\xi_i(x, y, t)) = \tanh_{p_{1i}q_{1i}k_{1i}}(\xi_i)$, $G_i(\eta_i(x, y, t)) = \sinh_{p_{2i}q_{2i}k_{2i}}(\eta_i)$. Then Eq. (60) has the new form

$$f(x, y, t) = H(t) + \sum_{i=1}^n K_i(t) \tanh_{p_1q_1k_1}(\xi_i) \operatorname{sech}_{p_2q_2k_2}(\eta_i), \quad (65)$$

where $\xi_i = \alpha_i(t)x + \beta_i(t)y + r_i(t)$, $\eta_i = \mu_i(t)x + v_i(t)y + l_i(t)$, ($i = 1, 2, \dots, n$), and $H(t)$, $K_i(t)$, $\alpha_i(t)$, $\beta_i(t)$, $r_i(t)$, $\mu_i(t)$, $v_i(t)$ and $l_i(t)$ are the functions of t , p_{1i} , q_{1i} , k_{1i} , p_{2i} , q_{2i} , k_{2i} are the constants, and $\sum_{i=1}^n K_i^2(t) \neq 0$ ($i = 0, 1, 2, \dots, n$).

We take the initial solution of Eq. (46) as $u_2(x, y, t) = u_2(t)$, and $v_2(x, y, t) = v_2(t)$ in Eq. (58) for convenience. So based on the Bäcklund transformation (58), the generalized hyperbolic function transformation (65), and

$$\frac{d^2(\tanh_{pqk}(\xi))}{d\xi^2} = -2k^2pq \cdot \operatorname{sech}_{pqk}^2(\xi) \cdot \tanh_{pqk}(\xi), \quad (66)$$

$$\frac{d^2(\operatorname{sech}_{pqk}(\xi))}{d\xi^2} = k^2 \cdot \operatorname{sech}_{pqk}(\xi) \cdot \left(2 \tanh_{pqk}^2(\xi) - 1\right), \quad (67)$$

we can get the following new multi-soliton-like solution which we call a generalized hyperbolic function solution of Eq. (46):

$$\begin{aligned} u = & \frac{2 \sum_{i=1}^n p_1 q_1 k_1 k_2 K_i(t) v_i(t) \alpha_i(t) \tanh_{p_2 q_2 k_2}(\eta_i) \operatorname{sech}_{p_1 q_1 k_1}^2(\xi_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)}{H(t) + \sum_{i=1}^n K_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)} \\ & + \frac{2 \sum_{i=1}^n k_2^2 K_i(t) \mu_i(t) v_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i) \left(1 - 2 \tanh_{p_2 q_2 k_2}^2(\eta_i)\right)}{H(t) + \sum_{i=1}^n K_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)} \\ & + \frac{2 \sum_{i=1}^n p_1 q_1 k_1 k_2 K_i(t) \beta_i(t) \mu_i(t) \operatorname{sech}_{p_1 q_1 k_1}^2(\xi_i) \tanh_{p_2 q_2 k_2}(\eta_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)}{H(t) + \sum_{i=1}^n K_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)} \\ & - \frac{2 \sum_{i=1}^n K_i(t) \left(p_1 q_1 k_1 \beta_i(t) \operatorname{sech}_{p_1 q_1 k_1}^2(\xi_i) - k_2 v_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \tanh_{p_2 q_2 k_2}(\eta_i)\right) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)}{H(t) + \sum_{i=1}^n K_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)} \\ & - \frac{\sum_{i=1}^n K_i(t) \left(p_1 q_1 k_1 \alpha_i(t) \operatorname{sech}_{p_1 q_1 k_1}^2(\xi_i) - k_2 \mu_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \tanh_{p_2 q_2 k_2}(\eta_i)\right) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)}{H(t) + \sum_{i=1}^n K_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)} + u_2, \end{aligned} \quad (68)$$

$$\begin{aligned}
& \sum_{i=1}^n 4p_1 q_1 k_1 \alpha_i(t) K_i(t) \operatorname{sech}_{p_1 q_1 k_1}^2(\xi_i) \\
& \operatorname{sech}_{p_2 q_2 k_2}(\eta_i) (k_1 \alpha_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \\
& + k_2 \mu_i(t) \tanh_{p_2 q_2 k_2}(\eta_i)) \\
v = & \frac{H(t) + \sum_{i=1}^n K_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)}{2 \sum_{i=1}^n K_i(t) k_2^2 (\mu_i(t))^2 \tanh_{p_1 q_1 k_1}(\xi_i)} \\
& + \frac{\operatorname{sech}_{p_2 q_2 k_2}(\eta_i) \left(1 - 2 \tanh_{p_2 q_2 k_2}^2(\eta_i)\right)}{H(t) + \sum_{i=1}^n K_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i)} \\
& \left(\sum_{i=1}^n K_i(t) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i) (p_1 q_1 k_1 \alpha_i(t) \operatorname{sech}_{p_1 q_1 k_1}^2(\xi_i) - k_2 \mu_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \tanh_{p_2 q_2 k_2}(\eta_i)) \right) 2 \\
& \left(H(t) + \sum_{i=1}^n K_i(t) \tanh_{p_1 q_1 k_1}(\xi_i) \operatorname{sech}_{p_2 q_2 k_2}(\eta_i) \right)^2 \\
& + v_2,
\end{aligned} \tag{69}$$

where $\xi_i = \alpha_i(t)x + \beta_i(t)y + r_i(t)$, $\eta_i = \mu_i(t)x + v_i(t)y + l_i(t)$.

For example, when we take $n = 2$, $H, K_i, p_i, q_i, k_i, \alpha_i, \beta_i$ and r_i , $i = 1, 2$ are all constants, then Eq. (65) becomes

$$\begin{aligned}
f(x, y, t) = & H + K_1 \tanh_{p_1 q_1 k_1}(\alpha_1 x + \beta_1 y + r_1) \\
& \cdot \operatorname{sech}_{p_1 q_1 k_1}(\alpha_1 x + \beta_1 y + r_1) \\
& + K_2 \tanh_{p_2 q_2 k_2}(\alpha_2 x + \beta_2 y + r_2) \\
& \operatorname{sech}_{p_2 q_2 k_2}(\alpha_2 x + \beta_2 y + r_2),
\end{aligned} \tag{70}$$

With the help of symbolic computation, substituting Eq. (70) and $u_2(x, y, t) = u_2(t)$, $v_2(x, y, t) = v_2(t)$ into Eqs. (52), (53), (55), (56), and (59), we can get $H, K_i, p_i, q_i, k_i, \alpha_i, \beta_i$ and r_i , $i = 1, 2$ respectively as follows:

Case 1

$$\begin{aligned}
u_2 = & C_1, \\
v_2 = & \frac{C_1 (\beta_2^2 \alpha_1^2 + \beta_1^2 \alpha_2^2 + \beta_1 \alpha_2 \beta_2 \alpha_1)}{3 \beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1)}, \\
p_1 = & p_1, \quad p_2 = p_2, \quad q_1 = 0, \quad q_2 = 0, \\
k_2 = & \varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1} (2 \beta_1 \alpha_2 + \beta_2 \alpha_1)}{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) \alpha_2} \text{ or} \\
k_2 = & \varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1} (2 \beta_1 \alpha_2 + \beta_2 \alpha_1)}{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) \alpha_2}, \\
k_1 = & \varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1} (\beta_1 \alpha_2 + 2 \beta_2 \alpha_1)}{(\beta_1 \beta_2^2 \alpha_1 + \beta_1^2 \beta_2 \alpha_2) \alpha_1}, \\
H = & H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 = & \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 = & K_1, \quad K_2 = K_2.
\end{aligned} \tag{71}$$

Case 2

$$\begin{aligned}
p_2 = & p_2, u_2 = C_1, \\
v_2 = & \frac{C_1 (\beta_2^2 \alpha_1^2 + \beta_1^2 \alpha_2^2 + \beta_1 \alpha_2 \beta_2 \alpha_1)}{3 \beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1)}, \\
p_1 = & p_1, \quad q_1 = 0, \\
k_1 = & \varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1} (\beta_1 \alpha_2 + 2 \beta_2 \alpha_1)}{(\beta_1 \beta_2^2 \alpha_1 + \beta_1^2 \beta_2 \alpha_2) \alpha_1}, \\
q_2 = & q_2, k_2 = 0, H = H, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \\
\beta_1 = & \beta_1, \beta_2 = \beta_2, \gamma_1 = \gamma_1, \gamma_2 = \gamma_2, \\
K_1 = & K_1, K_2 = K_2.
\end{aligned} \tag{72}$$

Case 3

$$\begin{aligned}
p_2 = & 0, u_2 = C_1, \\
v_2 = & \frac{C_1 (\beta_2^2 \alpha_1^2 + \beta_1^2 \alpha_2^2 + \beta_1 \alpha_2 \beta_2 \alpha_1)}{3 \beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1)}, \\
p_1 = & p_1, q_1 = 0, K_1 = K_1, K_2 = K_2, \\
k_1 = & \varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1} (\beta_1 \alpha_2 + 2 \beta_2 \alpha_1)}{(\beta_1 \beta_2^2 \alpha_1 + \beta_1^2 \beta_2 \alpha_2) \alpha_1}, \\
q_2 = & q_2, k_2 = 0, H = H, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \\
\beta_1 = & \beta_1, \beta_2 = \beta_2, \gamma_1 = \gamma_1, \gamma_2 = \gamma_2.
\end{aligned} \tag{73}$$

Case 4

$$\begin{aligned}
 k_2 &= \varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1 (2\beta_1 \alpha_2 + \beta_2 \alpha_1)}}{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) \alpha_2} \text{ or} \\
 k_2 &= -\varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1 (2\beta_1 \alpha_2 + \beta_2 \alpha_1)}}{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) \alpha_2}, \\
 p_2 &= 0, \quad u_2 = C_1, \\
 v_2 &= \frac{C_1 (\beta_2^2 \alpha_1^2 + \beta_1^2 \alpha_2^2 + \beta_1 \alpha_2 \beta_2 \alpha_1)}{3\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1)}, \\
 p_1 &= p_1, \quad q_1 = 0, \\
 k_1 &= \varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1 (\beta_1 \alpha_2 + 2\beta_2 \alpha_1)}}{(\beta_1 \beta_2^2 \alpha_1 + \beta_1^2 \beta_2 \alpha_2) \alpha_1}, \\
 q_2 &= q_2, H = H, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \\
 \beta_1 &= \beta_1, \beta_2 = \beta_2, \gamma_1 = \gamma_1, \gamma_2 = \gamma_2, \\
 K_1 &= K_1, K_2 = K_2.
 \end{aligned}$$

(74)

Case 5

$$\begin{aligned}
 k_2 &= \varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1 (2\beta_1 \alpha_2 + \beta_2 \alpha_1)}}{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) \alpha_2} \text{ or} \\
 k_2 &= -\varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1 (2\beta_1 \alpha_2 + \beta_2 \alpha_1)}}{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) \alpha_2}, \\
 v_2 &= \frac{C_1 (\beta_2^2 \alpha_1^2 + \beta_1^2 \alpha_2^2 + \beta_1 \alpha_2 \beta_2 \alpha_1)}{3\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1)}, \\
 k_1 &= -\varepsilon \frac{\sqrt{\beta_1 \beta_2 (\beta_1 \alpha_2 + \beta_2 \alpha_1) C_1 (\beta_1 \alpha_2 + 2\beta_2 \alpha_1)}}{(\beta_1 \beta_2^2 \alpha_1 + \beta_1^2 \beta_2 \alpha_2) \alpha_1}, \\
 H &= H, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \\
 \beta_1 &= \beta_1, \beta_2 = \beta_2, \gamma_1 = \gamma_1, \gamma_2 = \gamma_2, \\
 q_2 &= 0, \quad p_1 = 0, \quad q_1 = q_1, \\
 u_2 &= C_1, \quad p_2 = p_2, \\
 K_1 &= K_1, \quad K_2 = K_2.
 \end{aligned}$$

(75)

Case 6

$$\begin{aligned}
 p_1 &= 0, \quad k_2 = k_2, \quad k_1 = k_1, \quad v_2 = v_2, \\
 u_2 &= C_1, \quad p_2 = p_2, \quad q_1 = 0, \quad q_2 = q_2, \\
 H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
 \beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
 K_1 &= K_1, \quad K_2 = K_2.
 \end{aligned}$$

(76)

Case 7

$$\begin{aligned}
 p_2 &= 0, \quad q_2 = 0, \quad p_1 = 0, \quad q_1 = q_1, \quad v_2 = C_1, \\
 k_2 &= \varepsilon \frac{\sqrt{3C_1}}{\alpha_2} \text{ or } k_2 = -\varepsilon \frac{\sqrt{3C_1}}{\alpha_2}, \\
 k_1 &= \varepsilon \frac{\sqrt{3C_1}}{\alpha_1}, \\
 u_2 &= 0, \quad H = H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
 \beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
 K_1 &= K_1, \quad K_2 = K_2.
 \end{aligned}$$

(77)

Case 8

$$\begin{aligned}
 v_2 &= C_2, \quad p_2 = 0, \quad k_1 = \varepsilon \frac{\sqrt{3\beta_1 (C_2 \beta_1 + C_1 \alpha_1)}}{\beta_1 \alpha_1} \text{ or} \\
 k_1 &= -\varepsilon \frac{\sqrt{3\beta_1 (C_2 \beta_1 + C_1 \alpha_1)}}{\beta_1 \alpha_1}, \quad q_1 = q_1, \quad u_2 = C_1, \\
 k_2 &= \varepsilon \frac{\sqrt{3\beta_2 (C_2 \beta_2 + C_1 \alpha_2)}}{\beta_2 \alpha_2}, \quad q_2 = 0, \quad p_1 = 0, \\
 H &= H, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, \\
 \beta_1 &= \beta_1, \beta_2 = \beta_2, \gamma_1 = \gamma_1, \gamma_2 = \gamma_2, \\
 K_1 &= K_1, K_2 = K_2.
 \end{aligned}$$

(78)

Case 9

$$\begin{aligned}
 v_2 &= C_2, \quad p_2 = 0, \quad p_1 = 0, \quad q_1 = q_1, \\
 k_1 &= 0, \quad k_2 = \varepsilon \frac{\sqrt{3\beta_2 (C_2 \beta_2 + C_1 \alpha_2)}}{\beta_2 \alpha_2}, \\
 u_2 &= C_1, \quad q_2 = q_2, \\
 H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
 \beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
 K_1 &= K_1, \quad K_2 = K_2.
 \end{aligned}$$

(79)

Case 10

$$\begin{aligned}
 v_2 &= C_2, \quad p_2 = 0, \\
 k_1 &= \varepsilon \frac{\sqrt{3\beta_1 (C_2 \beta_1 + C_1 \alpha_1)}}{\beta_1 \alpha_1}, \\
 p_1 &= 0, \quad q_1 = q_1, \quad k_2 = 0, \quad u_2 = C_1, \\
 q_2 &= q_2, \quad H = H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
 \beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
 K_1 &= K_1, \quad K_2 = K_2.
 \end{aligned}$$

(80)

Case 11

$$\begin{aligned}
 p_2 &= p_2, \quad q_2 = 0, \quad p_1 = 0, \quad q_1 = q_1, \\
 k_2 &= \varepsilon \frac{\sqrt{3C_1}}{\alpha_2}, \quad v_2 = C_1 \text{ or } k_2 = -\varepsilon \frac{\sqrt{3C_1}}{\alpha_2}, \\
 v_2 &= C_1, \quad k_1 = \varepsilon \frac{\sqrt{3C_1}}{\alpha_1}, \quad u_2 = 0, \\
 H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
 \beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
 K_1 &= K_1, \quad K_2 = K_2.
 \end{aligned}$$

(81)

Case 12

$$\begin{aligned}
p_2 &= p_2, \quad q_2 = 0, \quad p_1 = 0, \quad q_1 = q_1, \\
k_2 &= \varepsilon \frac{\sqrt{3C_1}}{\alpha_2} \text{ or } k_2 = -\varepsilon \frac{\sqrt{3C_1}}{\alpha_2}, \quad v_2 = C_1, \\
k_1 &= \varepsilon \frac{\sqrt{3C_1}}{\alpha_1}, \\
u_2 &= 0, \quad K_1 = K_1, \quad K_2 = K_2, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2.
\end{aligned} \tag{82}$$

Case 13

$$\begin{aligned}
v_2 &= C_2, \quad p_2 = 0, \\
k_1 &= \varepsilon \frac{\sqrt{3\beta_1(C_2\beta_1 + C_1\alpha_1)}}{\beta_1\alpha_1}, \\
q_2 &= 0, \quad p_1 = 0, \quad q_1 = q_1, \quad k_2 = k_2, \quad u_2 = C_1, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned} \tag{83}$$

Case 14

$$\begin{aligned}
p_2 &= 0, \quad v_2 = \varepsilon \frac{C_1\alpha_2}{\beta_2}, \quad k_1 = k_1, \quad q_2 = 0, \\
p_1 &= 0, \quad q_1 = q_1, \quad k_2 = 0, \quad u_2 = C_1, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned} \tag{84}$$

Case 15

$$\begin{aligned}
v_2 &= C_2, \quad p_2 = 0, \\
k_2 &= \varepsilon \frac{\sqrt{3\beta_2(C_2\beta_2 + C_1\alpha_2)}}{\beta_2\alpha_2}, \\
k_1 &= k_1, \quad q_2 = 0, \quad p_1 = 0, \quad q_1 = q_1, \\
u_2 &= C_1, \quad H = H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned} \tag{85}$$

Case 16

$$\begin{aligned}
p_2 &= 0, \quad k_1 = k_1, \quad q_2 = 0, \quad p_1 = 0, \\
q_1 &= q_1, \quad v_2 = v_2, \quad k_2 = k_2, \quad u_2 = C_1, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned} \tag{86}$$

Case 17

$$\begin{aligned}
p_2 &= 0, \quad v_2 = \varepsilon \frac{C_1\alpha_2}{\beta_2} \text{ or } v_2 = -\varepsilon \frac{C_1\alpha_2}{\beta_2}, \\
k_2 &= 0, \quad k_1 = \varepsilon \frac{\sqrt{-3\beta_2\beta_1C_1(\beta_1\alpha_2 - \beta_2\alpha_1)}}{\beta_2\beta_1\alpha_1}, \\
q_1 &= 0, \quad p_1 = p_1, \quad u_2 = C_1, \quad q_2 = q_2, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned} \tag{87}$$

Case 18

$$\begin{aligned}
p_2 &= p_2, \quad v_2 = \varepsilon \frac{C_1\alpha_2}{\beta_2} \text{ or } v_2 = -\varepsilon \frac{C_1\alpha_2}{\beta_2}, \\
k_2 &= 0, \quad k_1 = \frac{\sqrt{-3\beta_2\beta_1C_1(\beta_1\alpha_2 - \beta_2\alpha_1)}}{\beta_2\beta_1\alpha_1}, \\
q_1 &= 0, \quad p_1 = p_1, \quad u_2 = C_1, \quad q_2 = q_2, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned} \tag{88}$$

Case 19

$$\begin{aligned}
p_2 &= 0, \quad q_1 = q_1, \quad k_1 = 0, \quad v_2 = \varepsilon \frac{C_1\alpha_1}{\beta_1} \text{ or } \\
v_2 &= -\varepsilon \frac{C_1\alpha_1}{\beta_1}, \\
k_2 &= -\varepsilon \frac{\sqrt{3\beta_2\beta_1C_1(\beta_1\alpha_2 - \beta_2\alpha_1)}}{\beta_2\beta_1\alpha_2}, \quad p_1 = p_1, \\
u_2 &= C_1, \quad q_2 = q_2, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned} \tag{89}$$

Case 20

$$\begin{aligned}
p_2 &= 0, \quad v_2 = C_2, \\
k_2 &= \varepsilon \frac{\sqrt{3\beta_2(C_2\beta_2 + C_1\alpha_2)}}{\beta_2\alpha_2} \\
\text{or } k_2 &= -\varepsilon \frac{\sqrt{3\beta_2(C_2\beta_2 + C_1\alpha_2)}}{\beta_2\alpha_2}, \\
k_1 &= \varepsilon \frac{\sqrt{3\beta_1(C_2\beta_1 + C_1\alpha_1)}}{\beta_1\alpha_1}, \quad q_2 = 0, \quad q_1 = 0, \\
p_1 &= p_1, \quad u_2 = C_1, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned}$$

(90)

Case 21

$$\begin{aligned}
v_2 &= C_1, \quad k_2 = \varepsilon \frac{\sqrt{3C_1}}{\alpha_2} \text{ or } k_2 = -\varepsilon \frac{\sqrt{3C_1}}{\alpha_2}, \\
k_1 &= -\varepsilon \frac{\sqrt{3C_1}}{\alpha_1}, \quad u_2 = 0, \quad p_2 = 0, \quad q_1 = 0, \\
p_1 &= p_1, \quad q_2 = q_2, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned}$$

(91)

Case 22

$$\begin{aligned}
p_2 &= 0, \quad q_1 = q_1, \quad k_2 = \varepsilon \frac{\sqrt{3\beta_2(C_2\beta_2 + C_1\alpha_2)}}{\beta_2\alpha_2} \\
\text{or } k_2 &= -\varepsilon \frac{\sqrt{3\beta_2(C_2\beta_2 + C_1\alpha_2)}}{\beta_2\alpha_2}, \\
k_1 &= \varepsilon \frac{\sqrt{3}\sqrt{\beta_1(C_2\beta_1 + C_1\alpha_1)}}{\beta_1\alpha_1}, \quad v_2 = C_2, \\
q_2 &= 0, \quad p_1 = p_1, \quad u_2 = C_1, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned}$$

(92)

Case 23

$$\begin{aligned}
p_2 &= 0, \quad q_1 = q_1, \\
k_2 &= \varepsilon \frac{\sqrt{3\beta_2(C_2\beta_2 + C_1\alpha_2)}}{\beta_2\alpha_2}, \\
v_2 &= C_2, \quad k_1 = 0, \quad p_1 = p_1, \quad u_2 = C_1, \quad (93) \\
q_2 &= q_2, \quad H = H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned}$$

Case 24

$$\begin{aligned}
p_2 &= p_2, \quad v_2 = C_1, \quad k_1 = \varepsilon \frac{\sqrt{3C_1}}{\alpha_1} \text{ or} \\
k_1 &= -\varepsilon \frac{\sqrt{3C_1}}{\alpha_1}, \quad u_2 = 0, \quad q_2 = 0, \quad q_1 = 0, \\
p_1 &= p_1, \quad k_2 = -\varepsilon \frac{\sqrt{3C_1}}{\alpha_2}, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned}$$

(94)

Case 25

$$\begin{aligned}
p_2 &= 0, \quad k_2 = k_2, \quad k_1 = k_1, \quad v_2 = v_2, \\
q_1 &= q_1, \quad q_2 = 0, \quad p_1 = p_1, \quad u_2 = C_1, \\
H &= H, \quad \alpha_1 = \alpha_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad \beta_2 = \beta_2, \quad \gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_2, \\
K_1 &= K_1, \quad K_2 = K_2.
\end{aligned}$$

(95)

We get the following new solution (called generalized hyperbolic function solution) of Eq. (46):

$$\begin{aligned}
& K_1 k_1 \beta_1(t) \operatorname{sech}_{p_1 q_1 k_1}(\xi) (1 - 2p_1 q_1 \\
u_1 = & 2 \frac{\operatorname{sech}_{p_1 q_1 k_1}^2(\xi) + K_2 k_2 \beta_2(t) \operatorname{sech}_{p_2 q_2 k_2}(\eta) (1 - p_2 q_2 (1 + p_2 q_2) \operatorname{sech}_{p_2 q_2 k_2}^2(\eta))}{H + K_1 \tanh_{p_1 q_1 k_1}(\xi) \operatorname{sech}_{p_1 q_1 k_1}(\xi) + K_2 \tanh_{p_2 q_2 k_2}(\eta) \operatorname{sech}_{p_2 q_2 k_2}(\eta)} \\
& \frac{K_1 k_1 \alpha_1(t) \operatorname{sech}_{p_1 q_1 k_1}(\xi) (1 - 2p_1 q_1 \operatorname{sech}_{p_1 q_1 k_1}^2(\xi)) + K_2 k_2 \alpha_2(t) \operatorname{sech}_{p_2 q_2 k_2}(\eta) (1 - p_2 q_2 (1 + p_2 q_2) \operatorname{sech}_{p_2 q_2 k_2}^2(\eta))}{H + K_1 \tanh_{p_1 q_1 k_1}(\xi) \operatorname{sech}_{p_1 q_1 k_1}(\xi) + K_2 \tanh_{p_2 q_2 k_2}(\eta) \operatorname{sech}_{p_2 q_2 k_2}(\eta)} \\
& - 2 \frac{k_1^2 K_1 \alpha_1(t) \beta_1(t) \tanh_{p_1 q_1 k_1}(\xi) \operatorname{sech}_{p_1 q_1 k_1}(\xi) (1 - 6p_1 q_1 \operatorname{sech}_{p_1 q_1 k_1}^2(\xi))}{H + K_1 \tanh_{p_1 q_1 k_1}(\xi) \operatorname{sech}_{p_1 q_1 k_1}(\xi) + K_2 \tanh_{p_2 q_2 k_2}(\eta) \operatorname{sech}_{p_2 q_2 k_2}(\eta)} \\
& - 2 \frac{k_2^2 K_2 \alpha_2(t) \beta_2(t) \tanh_{p_2 q_2 k_2}(\eta) \operatorname{sech}_{p_2 q_2 k_2}(\eta) (1 - 4p_2 q_2 (1 + p_2 q_2) \operatorname{sech}_{p_2 q_2 k_2}^2(\eta))}{H + K_1 \tanh_{p_1 q_1 k_1}(\xi) \operatorname{sech}_{p_1 q_1 k_1}(\xi) + K_2 \tanh_{p_2 q_2 k_2}(\eta) \operatorname{sech}_{p_2 q_2 k_2}(\eta)} + u_2,
\end{aligned} \tag{96}$$

$$\begin{aligned}
v_1 = & 2 \frac{(K_1 k_1 \alpha_1(t) \operatorname{sech}_{p_1 q_1 k_1}(\xi) (1 - 2p_1 q_1 \operatorname{sech}_{p_1 q_1 k_1}^2(\xi)) + K_2 k_2 \alpha_2(t) \operatorname{sech}_{p_2 q_2 k_2}(\eta) (1 - p_2 q_2 (1 + p_2 q_2) \operatorname{sech}_{p_2 q_2 k_2}^2(\eta)))^2}{(H + K_1 \tanh_{p_1 q_1 k_1}(\xi) \operatorname{sech}_{p_1 q_1 k_1}(\xi) + K_2 \tanh_{p_2 q_2 k_2}(\eta) \operatorname{sech}_{p_2 q_2 k_2}(\eta))^2} \\
& - 2 \frac{k_1^2 K_1 (\alpha_1(t))^2 \tanh_{p_1 q_1 k_1}(\xi) \operatorname{sech}_{p_1 q_1 k_1}(\xi) (1 - 6p_1 q_1 \operatorname{sech}_{p_1 q_1 k_1}^2(\xi))}{H + K_1 \tanh_{p_1 q_1 k_1}(\xi) \operatorname{sech}_{p_1 q_1 k_1}(\xi) + K_2 \tanh_{p_2 q_2 k_2}(\eta) \operatorname{sech}_{p_2 q_2 k_2}(\eta)} \\
& - 2 \frac{K_2 k_2^2 (\alpha_2(t))^2 \tanh_{p_2 q_2 k_2}(\eta) \operatorname{sech}_{p_2 q_2 k_2}(\eta) (1 - 2p_2 q_2 (2 + p_2 q_2) \operatorname{sech}_{p_2 q_2 k_2}^2(\eta))}{H + K_1 \tanh_{p_1 q_1 k_1}(\xi) \operatorname{sech}_{p_1 q_1 k_1}(\xi) + K_2 \tanh_{p_2 q_2 k_2}(\eta) \operatorname{sech}_{p_2 q_2 k_2}(\eta)} + v_2,
\end{aligned} \tag{97}$$

where $\xi = \alpha_1(t)x + \beta_1(t)y + r_1(t)$, $\eta = \alpha_2(t)x + \beta_2(t)y + r_2(t)$, here $H, K_i, p_i, q_i, k_i, \alpha_i, \beta_i$ and $r_i, i = 1, 2$ satisfy Eqs. (71), (72), (73), (74), (75), (76), (77), (78), (79), (80), (81), (82), (83), (84), (85), (86), (87), (88), (89), (91), (92), (93), (94), and (95), respectively.

The Generalized F-Expansion Method and Its Application in Another(2 + 1)-Dimensional KdV Equation

In the section, we will introduce the main steps of the generalized F-expansion method for constructing exact solutions of NLPDEs which we first presented in Ren (2007a), Ren and Zhang (2006b), and Ren et al. (2006a). Then we will make use of the method to find new exact solutions of the (2 + 1)-dimensional KdV equation.

Summary of the Generalized F-Expansion Method

In the following we would like to outline the main steps of our general method (called the generalized F-expansion method) in Ren (2007a), Ren and Zhang (2006b), and Ren et al. (2006a).

Step 1 For a given nonlinear partial differential equation system with some physical fields $u_i(t, x_1, x_2, \dots, x_m)$, ($i = 1, 2, \dots, n$) in $m + 1$ independent variables $t, x_1, x_2, \dots, x_m$,

$$\begin{cases} F_1(u_1, \dots, u_n, u_{1t}, \dots, u_{nt}, u_{1x_1}, \dots, u_{nx_m}, \\ \quad u_{1tx_1}, \dots, u_{ntx_m}, \dots) = 0, \\ F_2(u_1, \dots, u_n, u_{1t}, \dots, u_{nt}, u_{1x_1}, \dots, u_{nx_m}, \\ \quad u_{1tx_1}, \dots, u_{ntx_m}, \dots) = 0, \\ \dots\dots\dots, \\ F_n(u_1, \dots, u_n, u_{1t}, \dots, u_{nt}, u_{1x_1}, \dots, u_{nx_m}, \\ \quad u_{1tx_1}, \dots, u_{ntx_m}, \dots) = 0. \end{cases} \quad (98)$$

We first consider the following general formal solutions of the Eq. (98)

$$\begin{aligned} u_i(t, x_1, x_2, \dots, x_m) &= u_i(\omega), \\ \omega &= \omega(x), \quad (i = 1, 2, \dots, n), \end{aligned} \quad (99)$$

where $x = (x_1, x_2, x_3, \dots, x_m, t)$ and $\omega(x)$ is a function to be determined later. For example, when $n = 2$, we may take $\omega(x) = p(x_2, t)x_1 + q(x_2, t)$ for convenience, here $p(x_2, t)$ and $q(x_2, t)$ are functions to be determined later.

Step 2 We introduce new and more general formal transformations of the Eq. (98), if available, in the forms which we first presented in Ren et al. (2006a):

$$\begin{cases} u_1(x) = a_{1_0}(x) + \sum_{i_1=1}^{k_1} (f(\omega(x)))^{i_1-1} (a_{i_1}(x) \\ \quad f(\omega(x)) + b_{i_1}(x)g(\omega(x)) + \frac{c_{i_1}(x)}{f(\omega(x))} + \frac{d_{i_1}(x)}{g(\omega(x))}), \\ u_2(x) = a_{2_0}(x) + \sum_{i_2=1}^{k_2} (f(\omega(x)))^{i_2-1} (a_{i_2}(x) \\ \quad f(\omega(x)) + b_{i_2}(x)g(\omega(x)) + \frac{c_{i_2}(x)}{f(\omega(x))} + \frac{d_{i_2}(x)}{g(\omega(x))}), \\ \dots\dots\dots, \\ u_n(x) = a_{n_0}(x) + \sum_{i_n=1}^{k_n} (f(\omega(x)))^{i_n-1} (a_{i_n}(x) \\ \quad f(\omega(x)) + b_{i_n}(x)g(\omega(x)) + \frac{c_{i_n}(x)}{f(\omega(x))} + \frac{d_{i_n}(x)}{g(\omega(x))}). \end{cases} \quad (100)$$

where $a_i^2(x) + b_i^2(x) + c_i^2(x) + d_i^2(x) \neq 0$, $a_{i_0}(x), a_{i_j}(x)$, $b_{i_j}(x), c_{i_j}(x), d_{i_j}(x)$ ($j = 1, 2, \dots, n, i_j = 1, 2, \dots, k_j$) and $\omega(x)$ are all functions to be determined later, k_j ($j = 1, 2, \dots, n$) is an integer which is determined by balancing the highest order derivative terms with the nonlinear terms in the given Eq. (98), and the new variables $f(\omega) = f(\omega(x))$ and $g(\omega) = g(\omega(x))$ satisfy the following relations:

$$\begin{cases} f'^2(\omega) = l_1 f^4(\omega) + m_1 f^2(\omega) + n_1 \\ g'^2(\omega) = l_2 g^4(\omega) + m_2 g^2(\omega) + n_2 \\ g^2(\omega) = \frac{l_1 f^2(\omega)}{l_2} + \frac{m_1 - m_2}{3l_2} \\ n_1 = \frac{m_1^2 - m_2^2 + 3l_2 n_2}{3l_1} \end{cases} \quad (101)$$

Step 3 Determine k_j ($j = 1, 2, \dots, n$) in Eq. (100) by balancing the highest nonlinear terms and the highest-order partial differential terms in the given Eq. (98). If k_j is a nonnegative integer, then we first make the transformation $u_j(x) = v_j^{k_j}(x)$.

Step 4 Substitute Eq. (100) into the Eq. (98) with (101) and collecting coefficients of polynomials of $f(\omega), g(\omega), \sqrt{l_1 f^4(\omega) + m_1 f^2(\omega) + n_1}$, with the aid of Maple, then setting each coefficient to zero to get a set of over-determined partial differential equations with respect to $l_1, l_2, m_1, m_2, n_1, n_2, a_{i_0}(x)$,

$a_{i_j}(x)$, $b_{i_j}(x), c_{i_j}(x), d_{i_j}(x)$ ($j = 1, 2, \dots, n, i_j = 1, 2, \dots, k_j$) and $\omega(x)$.

Step 5 Solving the over-determined partial differential equations with Maple, we then can determine $a_{i_0}(x), a_{i_j}(x), b_{i_j}(x), c_{i_j}(x), d_{i_j}(x)$ ($j = 1, 2, \dots, n, i_j = 1, 2, \dots, k_j$) and $\omega(x)$.

Step 6 Selecting the proper value of parameters $l_1, l_2, m_1, m_2, n_1, n_2$ to determine the corresponding the solutions of the Jacobi elliptic functions $f(\omega)$ and $g(\omega)$ in Eq. (101). The relations between the parameters and their corresponding Jacobi elliptic functions are known and are given in the following Table 1.

Remark 2 Relations between values of $l_1, m_1, n_1, l_2, m_2, n_2$ and corresponding $f(\omega), g(\omega)$ in ODEs (101) are in the following Table 1.

Step 7 By using the results obtained in the above step, we can derive a series of generalized solutions, such as different kinds of Jacobi elliptic function solutions in terms of Remark 2. Finally, substituting

$a_{i_0}(x), a_{i_j}(x), b_{i_j}(x), c_{i_j}(x), d_{i_j}(x)$ ($j = 1, 2, \dots, n, i_j = 1, 2, \dots, k_j$) and $\omega(x)$ into the generalized solutions with the corresponding solutions of $f(\omega), g(\omega)$, we can get the the new exact solutions of the given Eq. (98).

Different Analytical Methods for Solving the Korteweg-de Vries Equation (KdV), Table 1 The ODEs (101) and Jacobi Elliptic Functions Relation between values of l_i, m_i, n_i ($i = 1, 2$) and corresponding $f(\omega)$ and $g(\omega)$ in ODEs (101)

l_1	m_1	n_1	l_2	m_2	n_2	$f(\omega)$	$g(\omega)$
k^2	$-1 - k^2$	1	$-k^2$	$2k^2 - 1$	$1 - k^2$	$sn(\omega)$	$cn(\omega)$
k^2	$-1 - k^2$	1	-1	$2 - k^2$	$-1 + k^2$	$sn(\omega)$	$dn(\omega)$
$-k^2$	$2k^2 - 1$	$1 - k^2$	-1	$2 - k^2$	$k^2 - 1$	$cn(\omega)$	$dn(\omega)$
$1 - k^2$	$2k^2 - 1$	$-k^2$	$1 - k^2$	$2 - k^2$	1	$nc(\omega)$	$sc(\omega)$
1	$2 - k^2$	$1 - k^2$	1	$2k^2 - 1$	$-k^2(1 - k^2)$	$cs(\omega)$	$ds(\omega)$
1	$-1 - k^2$	k^2	$1 - k^2$	$2k^2 - 1$	$-k^2$	$dc(\omega)$	$nc(\omega)$
1	$-1 - k^2$	k^2	1	$2 - k^2$	$1 - k^2$	$ns(\omega)$	$cs(\omega)$
1	$-1 - k^2$	k^2	1	$2k^2 - 1$	$-k^2(1 - k^2)$	$ns(\omega)$	$ds(\omega)$
k^2	$-1 - k^2$	1	$-1 + k^2$	$2 - k^2$	-1	$cd(\omega)$	$nd(\omega)$
$-k^2(1 - k^2)$	$2k^2 - 1$	1	$-1 + k^2$	$2 - k^2$	-1	$sd(\omega)$	$nd(\omega)$
$-k^2(1 - k^2)$	$2k^2 - 1$	1	k^2	$-1 - k^2$	1	$sd(\omega)$	$cd(\omega)$
$1 - k^2$	$2 - k^2$	1	1	$-1 - k^2$	k^2	$sc(\omega)$	$dc(\omega)$

Remark 3 As is known to all, when $k \rightarrow 1$, the Jacobi elliptic functions can degenerate as hyperbolic functions, and when $k \rightarrow 0$, the Jacobi elliptic functions degenerate as trigonometric functions. So by this method we can obtain many other new exact solutions to Eq. (98).

Remark 4 The main advantages of the generalized F-expansion method are simpler, more powerful, and more convenient than the Jacobi elliptic function expansion method and the F-expansion method. First of all, with the aid of nonlinear ordinary differential equations (ODEs) (101), one only needs to calculate the functions $f(\omega)$ and $g(\omega)$ which are the solutions of the ODEs (101), instead of calculating the Jacobi elliptic functions one by one. Secondly, the values of the coefficients $l_1, m_1, n_1, l_2, m_2, n_2$ of the ODEs (101) can be selected so that the corresponding solutions of the coupled functions $f(\omega)$ and $g(\omega)$ are Jacobi elliptic functions in the above Table 1. The relations between the coefficients and the corresponding Jacobi elliptic function solutions are known and are given in Remark 2. Thus, in terms of Remark 2, one can simultaneously obtain more periodic wave solutions expressed by various Jacobi elliptic functions. Thirdly, we present a new transformations (100) that are more general than the transformations of the Jacobi elliptic function expansion method and the F-expansion method.

The Generalized F-Expansion Method to Find the Exact Solutions of the (2 + 1)-Dimensional KdV Equation

In this section, we will make use of our method (Ren 2007a; Ren and Zhang 2006b; Ren et al. 2006a) and symbolic computation to find new exact solutions of the (2 + 1)-dimension KdV equation.

The (2 + 1)-dimension KdV equation in Song and Zhang (2007) is written in the following form

$$\begin{aligned} u_t(x, y, t) - 4u(x, y, t)u_y(x, y, t) \\ - 4u_x(x, y, t)\partial_x^{-1}u_y(x, y, t) \\ - u_{xy}(x, y, t) = 0. \end{aligned} \quad (102)$$

where ∂_x^{-1} is an indefinite integrate operator $\int dx$, possessing some interesting coherent structures.

We discuss the similar solution of (102) in its potential form ($u = v_x$),

$$\begin{aligned} v_{xt}(x, y, t) - v_{xxy}(x, y, t) \\ - 4v_x(x, y, t)v_{xy}(x, y, t) \\ - 4v_{xx}(x, y, t)v_y(x, y, t) = 0. \end{aligned} \quad (103)$$

By balancing the highest nonlinear terms and the highest-order partial derivative terms in Eq. (103), we suppose Eq. (103) to have the following formal solution:

$$\begin{aligned} v = c_0(y, t) + c_1(y, t)g(\omega) \\ + c_2(y, t)f(\omega) + \frac{c_3(y, t)}{g(\omega)} + \frac{c_4(y, t)}{f(\omega)}, \end{aligned} \quad (104)$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ are all functions to be determined later. And the new variables $f(\omega)$ and $g(\omega)$ satisfy the relation (101).

Substitute Eq. (104) into the Eq. (103) with (101). Collect coefficients of polynomials of $f(\omega), g(\omega), \sqrt{l_1 f^4(\omega) + m_1 f^2(\omega) + n_1}$, with the aid of Maple, then set each coefficient to zero to get a set of over-determined partial differential equations with respect to $l_1, l_2, m_1, m_2, n_1, n_2, \alpha(y, t), p(y, t), c_i(y, t), (i = 0, 1, 2, 3, 4)$ and $q(t)$ as follows:

$$\begin{aligned}
& 120(\alpha(y, t))^2 c_4(y, t) c_1(y, t) \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_1^4 m_2^2 + 168(\alpha(y, t))^2 c_4(y, t) l_2 c_3(y, t) \left(\frac{\partial}{\partial y} p(y, t) \right) m_1^4 m_2 \\
& + 120(\alpha(y, t))^2 c_4(y, t) c_1(y, t) \left(\frac{\partial}{\partial y} p(y, t) \right) m_2^2 m_1^4 + 48(\alpha(y, t))^2 c_4(y, t) l_2 c_3(y, t) \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_1^2 m_2 \\
& - 80(\alpha(y, t))^2 c_4(y, t) c_1(y, t) \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_2^3 m_1^3 - 72(\alpha(y, t))^2 c_4(y, t) c_1(y, t) \left(\frac{\partial}{\partial y} p(y, t) \right) m_1^5 m_2 \\
& + 24(\alpha(y, t))^2 c_4(y, t) c_1(y, t) \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_2^5 m_1 + \dots = 0, \\
& 23328 c_2(y, t) (\alpha(y, t))^3 l_2 l_1^5 \left(\frac{\partial}{\partial y} p(y, t) \right) m_2 + 1458 c_2(y, t) \alpha(y, t) l_2 l_1^5 \frac{d}{dt} q(t) - 37908 c_2(y, t) \\
& (\alpha(y, t))^3 l_2 l_1^5 \left(\frac{\partial}{\partial y} p(y, t) \right) m_1 + 23328 c_2(y, t) (\alpha(y, t))^3 l_2 l_1^5 \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_2 + 1458 c_2(y, t) \\
& \alpha(y, t) l_2 l_1^5 \left(\frac{\partial}{\partial t} \alpha(y, t) \right) x - 5832 (\alpha(y, t))^2 \left(\frac{\partial}{\partial y} c_0(y, t) \right) l_2 l_1^5 c_2(y, t) \\
& - 37908 c_2(y, t) (\alpha(y, t))^3 l_2 l_1^5 \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_1 + \dots = 0 \\
& - 60 (\alpha(y, t))^3 c_1(y, t) m_1^3 \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_2^3 l_1 - 540 (\alpha(y, t))^3 c_1(y, t) l_2 n_2 \left(\frac{\partial}{\partial y} p(y, t) \right) m_2^4 l_1 \\
& + 2160 (\alpha(y, t))^3 c_1(y, t) l_2 n_2 \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_1 m_2^3 l_1 - 3240 (\alpha(y, t))^3 c_1(y, t) l_2 n_2 \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_1^2 m_2^2 l_1 \\
& + 2160 (\alpha(y, t))^3 c_1(y, t) l_2 n_2 \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_1^3 m_2 l_1 - 1296 c_3(y, t) (\alpha(y, t))^3 l_2^2 n_2 \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x l_1 m_2^3 \\
& + 180 c_3(y, t) (\alpha(y, t))^3 l_2 l_1 m_1^3 \left(\frac{\partial}{\partial y} \alpha(y, t) \right) x m_2^2 + \dots = 0, \\
& \dots
\end{aligned} \tag{105}$$

Because there are so many over-determined partial differential equations, only a few of them are shown here for convenience. With the aid of Maple symbolic computation software, we solve

the over-determined partial differential equations, and we get the following solutions of $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$:

Case 1

$$\begin{aligned}
c_4(y, t) &= \frac{C_1}{F_1(t)}, \quad p(y, t) = F_2(t), \\
c_3(y, t) &= \frac{C_2}{F_1(t)}, \quad \alpha(y, t) = F_1(t), \\
c_2(y, t) &= \frac{C_3}{F_1(t)}, \quad c_1(y, t) = \frac{C_4}{F_1(t)}, \\
q(t) &= \int 4F_1(t)F_3(t) - \left(\frac{d}{dt}F_1(t)\right)x \\
&\quad - \frac{d}{dt}F_2(t)dt + C_5, \quad c_0(y, t) = F_3(t)y + F_4(t),
\end{aligned} \tag{106}$$

where $C_i, i = 1, 2, 3, 4, 5$ are arbitrary constants and $F_i, i = 1, 2, 3, 4$ are all arbitrary functions with respect to variable t .

Case 2

$$\begin{aligned}
c_2(y, t) &= -\frac{48C_1l_1}{F_1(t)(m_1 - m_2)}, \\
q(t) &= \int 4F_1(t)F_3(t) - \left(\frac{d}{dt}F_1(t)\right)x \\
&\quad - \frac{d}{dt}F_2(t)dt + C_4, \\
\alpha(y, t) &= F_1(t), \quad c_1(y, t) = \frac{C_3}{F_1(t)}, \\
c_4(y, t) &= \frac{C_1}{F_1(t)}, \quad p(y, t) = F_2(t), \\
c_3(y, t) &= \frac{C_2}{F_1(t)}, \quad c_0(y, t) = F_3(t)y + F_4(t),
\end{aligned} \tag{107}$$

where $C_i, i = 1, 2, 3, 4$ are arbitrary constants and $F_i, i = 1, 2, 3, 4$ are all arbitrary functions with respect to variable t .

Case 3

$$\begin{aligned}
c_2(y, t) &= -\frac{48C_1l_1}{F_1(t)(m_1 - m_2)}, \quad c_4(y, t) = \frac{C_1}{F_1(t)}, \\
p(y, t) &= F_2(t), \quad c_3(y, t) = \frac{C_2}{F_1(t)}, \quad \alpha(y, t) = F_1(t), \\
c_0(y, t) &= F_3(t)y + F_4(t), \quad c_1(y, t) = 0, \\
q(t) &= \int 4F_1(t)F_3(t) - \left(\frac{d}{dt}F_1(t)\right)x \\
&\quad - \frac{d}{dt}F_2(t)dt + C_3,
\end{aligned} \tag{108}$$

where $C_i, i = 1, 2, 3, 4$ are arbitrary constants and $F_i, i = 1, 2, 3, 4$ are all arbitrary functions with respect to variable t .

Case 4

$$\begin{aligned}
c_3(y, t) &= c_3(y, t), \quad c_4(y, t) = c_4(y, t), \\
q(t) &= q(t), \quad c_0(y, t) = c_0(y, t), \\
p(y, t) &= p(y, t), \quad c_1(y, t) = c_1(y, t), \\
c_2(y, t) &= c_2(y, t), \quad \alpha(y, t) = 0,
\end{aligned} \tag{109}$$

where C_1 is an arbitrary constant and $c_3(y, t), c_4(y, t), q(t), c_0(y, t), p(y, t), c_1(y, t), c_2(y, t)$ are all arbitrary functions with respect to variable y and t .

Select the proper value of parameters $l_1, l_2, m_1, m_2, n_1, n_2$ to determine the corresponding Jacobi elliptic functions $f(\omega)$ and $g(\omega)$ in Eq. (101). The relations between the parameters and the corresponding Jacobi elliptic function are known and given in Table 1, above.

By using the results obtained in the above step, we can derive a series of generalized solutions, such as different kinds of Jacobi elliptic functions solutions. Finally, by substituting Eqs. (106), (107), (108), and (109) into the generalized solutions with the corresponding solutions of $f(\omega)$ and

$g(\omega)$, respectively, we can get the the new exact solutions of the given Eq. (103).

Type 1 If we select $l_1 = k^2$, $m_1 = -(1 + k^2)$, $n_1 = 1$, $l_2 = -k^2$, $m_2 = (2k^2 - 1)$, $n_2 = (1 - k^2)$, corresponding to Eq. (101), we can get $f = sn(\omega)$, $g = cn(\omega)$.

Thus we get the following Jacobi elliptic function solutions of the Eq. (103):

$$\begin{aligned} v_1(x, y, t) = & c_0(y, t) + c_1(y, t)cn(\omega) \\ & + c_2(y, t)sn(\omega) + \frac{c_3(y, t)}{cn(\omega)} \\ & + \frac{c_4(y, t)}{sn(\omega)}, \end{aligned} \quad (110)$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109) respectively.

For example, when we select $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ to satisfy (106), namely Case 1, we can easily get the following Jacobi elliptic function solution of Eq. (103):

$$\begin{aligned} v_2(x, y, t) = & F_3(t)y + F_4(t) + \frac{C_4cn(\omega)}{F_1(t)} \\ & + \frac{C_3sn(\omega)}{F_1(t)} + \frac{C_2}{F_1(t)cn(\omega)} \\ & + \frac{C_1}{F_1(t)sn(\omega)}, \end{aligned} \quad (111)$$

where F_i , $i = 1, 2, 3, 4$ are arbitrary functions of t , C_i ($i = 1, 2, 3, 4, 5$) are arbitrary constants, and

$$\begin{aligned} \omega = & F_1(t)x + F_2(t) + \int 4F_1(t)F_3(t) \\ & - \left(\frac{d}{dt}F_1(t) \right)x - \frac{d}{dt}F_2(t)dt + C_5. \end{aligned}$$

When $k \rightarrow 1$, $sn(\omega) \rightarrow \tanh(\omega)$, $cn(\omega) \rightarrow \text{sech}(\omega)$, so we get degenerative soliton-like solutions from the solution (111):

$$\begin{aligned} v_3(x, y, t) = & F_3(t)y + F_4(t) \\ & + \frac{C_4\text{sech}(\omega)}{F_1(t)} + \frac{C_3\tanh(\omega)}{F_1(t)} \\ & + \frac{C_2}{F_1(t)\text{sech}(\omega)} \\ & + \frac{C_1}{F_1(t)\tanh(\omega)}. \end{aligned} \quad (112)$$

When $k \rightarrow 0$, $sn(\omega) \rightarrow \sin(\omega)$, $cn(\omega) \rightarrow \cos(\omega)$, so we get the degenerative trigonometric function solutions from the solution (111):

$$\begin{aligned} v_4(x, y, t) = & F_3(t)y + F_4(t) \\ & + \frac{C_4\cos(\omega)}{F_1(t)} + \frac{C_3\sin(\omega)}{F_1(t)} \\ & + \frac{C_2}{F_1(t)\cos(\omega)} \\ & + \frac{C_1}{F_1(t)\sin(\omega)}. \end{aligned} \quad (113)$$

So from Case 1, we can obtain Jacobi elliptic function solutions, degenerative soliton-like solutions and trigonometric function solutions of Eq. (103). We can also get many other solutions if we make use of Case 2–Case 4. Therefore, through selecting $l_1 = k^2$, $m_1 = -(1 + k^2)$, $n_1 = 1$, $l_2 = -k^2$, $m_2 = (2k^2 - 1)$, $n_2 = (1 - k^2)$, we can get families of new exact solutions of Eq. (103).

Type 2 If we select $l_1 = -k^2$, $m_1 = (2k^2 - 1)$, $n_1 = 1 - k^2$, $l_2 = -1$, $m_2 = (2 - k^2)$, $n_2 = k^2 - 1$, corresponding to Eq. (101), we can get $f = cn(\omega)$, $g = dn(\omega)$.

So we get the following Jacobi elliptic function solutions of Eq. (103):

$$\begin{aligned} v_5(x, y, t) = & c_0(y, t) + c_1(y, t)dn(\omega) \\ & + c_2(y, t)cn(\omega) + \frac{c_3(y, t)}{dn(\omega)} \\ & + \frac{c_4(y, t)}{cn(\omega)}, \end{aligned} \quad (114)$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109), respectively.

For example, when we select $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ to satisfy Eq. (107), namely Case 2, we can easily get the following Jacobi elliptic function solution of Eq. (103):

$$\begin{aligned} v_6(x, y, t) = & F_3(t)y + F_4(t) + \frac{C_3 dn(\omega)}{F_1(t)} \\ & - 48 \frac{C_1 l_1 cn(\omega)}{F_1(t)(m_1 - m_2)} + \frac{C_2}{F_1(t)dn(\omega)} \\ & + \frac{C_1}{F_1(t)cn(\omega)}. \end{aligned} \quad (115)$$

where $F_i(t)$, $i = 1, 2, 3, 4$ are arbitrary functions of t , C_i ($i = 1, 2, 3, 4$) are arbitrary constants, and

$$\begin{aligned} \omega = & F_1(t)x + F_2(t) + \int 4F_1(t)F_3(t) \\ & - \left(\frac{d}{dt}F_1(t) \right)x - \frac{d}{dt}F_2(t)dt + C_4. \end{aligned}$$

When $k \rightarrow 1$, $cn(\omega) \rightarrow \operatorname{sech}(\omega)$, $dn(\omega) \rightarrow \operatorname{sech}(\omega)$, so we get a degenerative soliton-like solution from the solution (115):

$$\begin{aligned} v_7(x, y, t) = & F_3(t)y + F_4(t) + \frac{C_3 \operatorname{sech}(\omega)}{F_1(t)} \\ & - 48 \frac{C_1 l_1 \operatorname{sech}(\omega)}{F_1(t)(m_1 - m_2)} \\ & + \frac{C_1 + C_2}{F_1(t) \operatorname{sech}(\omega)}, \end{aligned} \quad (116)$$

When $k \rightarrow 0$, $cn(\omega) \rightarrow \cos(\omega)$, $dn(\omega) \rightarrow 1$, so we get the degenerative trigonometric function solutions from the solution (115):

$$\begin{aligned} v_8 = & F_3(t)y + F_4(t) + \frac{C_2 + C_3}{F_1(t)} \\ & - 48 \frac{C_1 l_1 \cos(\omega)}{F_1(t)(m_1 - m_2)} + \frac{C_1}{F_1(t) \cos(\omega)}. \end{aligned} \quad (117)$$

where $F_i(t)$, $i = 1, 2, 3, 4$ are arbitrary functions of t and C_i ($i = 1, 2, 3, 4$) are arbitrary constants.

So from Case 2, we can get the Jacobi elliptic function solutions, degenerative soliton-like solutions and trigonometric function solutions of Eq. (103). We can also get some other solutions if we make use of other cases. Therefore, through selecting $l_1 = -k^2$, $m_1 = (2k^2 - 1)$, $n_1 = 1 - k^2$, $l_2 = -1$, $m_2 = (2 - k^2)$, $n_2 = k^2 - 1$, we can get families of new exact solutions of Eq. (103).

Type 3 If we select $l_1 = 1$, $m_1 = (2 - k^2)$, $n_1 = 1 - k^2$, $l_2 = 1$, $m_2 = (2k^2 - 1)$, $n_2 = -k^2(1 - k^2)$, corresponding to Eq. (101), we can get $f = cs(\omega)$, $g = ds(\omega)$.

So we get the following Jacobi elliptic function solutions of Eq. (103):

$$\begin{aligned} v_9(x, y, t) = & c_0(y, t) + c_1(y, t)ds(\omega) \\ & + c_2(y, t)cs(\omega) + \frac{c_3(y, t)}{ds(\omega)} \\ & + \frac{c_4(y, t)}{cs(\omega)}, \end{aligned} \quad (118)$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109), respectively.

For example, when we select $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ to satisfy Eq. (108), namely Case 3, we can easily get the following Jacobi elliptic function solution of Eq. (103):

$$\begin{aligned} v_{10}(x, y, t) = & F_3(t)y + F_4(t) \\ & - 48 \frac{C_1 l_1 cs(\omega)}{F_1(t)(m_1 - m_2)} \\ & + \frac{C_2}{F_1(t)ds(\omega)} + \frac{C_1}{F_1(t)cs(\omega)}, \end{aligned} \quad (119)$$

where $F_i(t)$, $i = 1, 2, 3, 4$ are arbitrary functions of t , C_i ($i = 1, 2, 3, 4$) are arbitrary constants, and

$$\begin{aligned}\omega &= F_1(t)x + F_2(t)q(t) \\ &= \int 4F_1(t)F_3(t) - \left(\frac{d}{dt}F_1(t)\right)x - \frac{d}{dt}F_2(t)dt + C_3.\end{aligned}$$

When $k \rightarrow 1$, $cs(\omega) \rightarrow \text{sech}(\omega) \coth(\omega)$, $ds(\omega) \rightarrow \text{sech}(\omega) \coth(\omega)$, so we get a degenerative soliton-like solution from the solution (119):

$$\begin{aligned}v_{11}(x, y, t) &= F_3(t)y + F_4(t) \\ &\quad - 48 \frac{C_1 l_1 \text{sech}(\omega) \coth(\omega)}{F_1(t)(m_1 - m_2)} \\ &\quad + \frac{C_1 + C_2}{F_1(t) \text{sech}(\omega) \coth(\omega)}. \quad (120)\end{aligned}$$

When $k \rightarrow 0$, $cs(\omega) \rightarrow \cot(\omega)$, $ds(\omega) \rightarrow \csc(\omega)$, so we get the triangular function solution from the solution (119):

$$\begin{aligned}v_{12}(x, y, t) &= F_3(t)y + F_4(t) \\ &\quad - 48 \frac{C_1 l_1 \cot(\omega)}{F_1(t)(m_1 - m_2)} \\ &\quad + \frac{C_2}{F_1(t) \csc(\omega)} + \frac{C_1}{F_1(t) \cot(\omega)}. \quad (121)\end{aligned}$$

So from Case 3, we can get Jacobi elliptic function solutions, degenerative soliton-like solutions and trigonometric function solutions of Eq. (103). We can also get many other solutions if we make use of other cases. Therefore, through selecting $l_1 = 1$, $m_1 = (2 - k^2)$, $n_1 = 1 - k^2$, $l_2 = 1$, $m_2 = (2k^2 - 1)$, $n_2 = -k^2(1 - k^2)$, we can get families of new exact solutions of Eq. (103).

Type 4 If we select $l_1 = 1 - k^2$, $m_1 = (2k^2 - 1)$, $n_1 = -k^2$, $l_2 = (1 - k^2)$, $m_2 = (2 - k^2)$, $n_2 = 1$, corresponding to Eq. (101), we can get $f = nc(\omega)$, $g = sc(\omega)$.

So we get the following Jacobi elliptic function solutions of Eq. (103):

$$\begin{aligned}v_{13}(x, y, t) &= c_0(y, t) + c_1(y, t)sc(\omega) \\ &\quad + c_2(y, t)nc(\omega) + \frac{c_3(y, t)}{sc(\omega)} \\ &\quad + \frac{c_4(y, t)}{nc(\omega)}, \quad (122)\end{aligned}$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ satisfy (106), (107), (108), and (109), respectively.

So, in the same way, we can get many other Jacobi elliptic function solutions of Eq. (103). When $k \rightarrow 1$, or $k \rightarrow 0$, we can also get families of new exact solutions of Eq. (103).

Type 5 If we select $l_1 = k^2$, $m_1 = -(1 + k^2)$, $n_1 = 1$, $l_2 = -1$, $m_2 = (2 - k^2)$, $n_2 = -(1 - k^2)$, corresponding to Eq. (101), we can get $f = sn(\omega)$, $g = dn(\omega)$.

So we get the following Jacobi elliptic function solutions of Eq. (103):

$$\begin{aligned}v_{14}(x, y, t) &= c_0(y, t) + c_1(y, t)dn(\omega) \\ &\quad + c_2(y, t)sn(\omega) + \frac{c_3(y, t)}{dn(\omega)} \\ &\quad + \frac{c_4(y, t)}{sn(\omega)}, \quad (123)\end{aligned}$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109), respectively.

So, in the same way, we can get many other Jacobi elliptic function solutions of Eq. (103). When $k \rightarrow 1$, or $k \rightarrow 0$, we can also get families of new exact solutions of Eq. (103).

Type 6 If we select $l_1 = 1$, $m_1 = -(1 + k^2)$, $n_1 = k^2$, $l_2 = 1 - k^2$, $m_2 = (2k^2 - 1)$, $n_2 = -k^2$, corresponding to Eq. (101), we can get $f = dc(\omega)$, $g = nc(\omega)$.

So we get the following Jacobi elliptic function solutions of Eq. (103):

$$\begin{aligned}v_{15}(x, y, t) &= c_0(y, t) + c_1(y, t)nc(\omega) \\ &\quad + c_2(y, t)dc(\omega) + \frac{c_3(y, t)}{nc(\omega)} \\ &\quad + \frac{c_4(y, t)}{dc(\omega)}, \quad (124)\end{aligned}$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109), respectively.

So, in the same way, we can get many other Jacobi elliptic function solutions of Eq. (103). When $k \rightarrow 1$, or $k \rightarrow 0$, we can also get families of new exact solutions of Eq. (103).

Type 7 If we select $l_1 = 1, m_1 = -(1 + k^2), n_1 = k^2, l_2 = 1, m_2 = 2 - k^2, n_2 = 1 - k^2$, corresponding to Eq. (101), we can get $f = ns(\omega), g = cs(\omega)$.

So we get the following Jacobi elliptic function solutions:

$$\begin{aligned} v_{16}(x, y, t) = & c_0(y, t) + c_1(y, t)cs(\omega) \\ & + c_2(y, t)ns(\omega) + \frac{c_3(y, t)}{cs(\omega)} \\ & + \frac{c_4(y, t)}{ns(\omega)}, \end{aligned} \quad (125)$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t), p(y, t), q(t), c_0(y, t), c_1(y, t), c_2(y, t), c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109), respectively.

So, in the same way, we can get many other Jacobi elliptic function solutions of Eq. (103). When $k \rightarrow 1$, or $k \rightarrow 0$, we can also get families of new exact solutions of Eq. (103).

Type 8 If we select $l_1 = 1, m_1 = -(1 + k^2), n_1 = k^2, l_2 = 1, m_2 = 2k^2 - 1, n_2 = -k^2(1 - k^2)$, corresponding to Eq. (101), we can get $f = ns(\omega), g = ds(\omega)$.

So we get the following Jacobi elliptic function solutions:

$$\begin{aligned} v_{17}(x, y, t) = & c_0(y, t) + c_1(y, t)ds(\omega) \\ & + c_2(y, t)ns(\omega) + \frac{c_3(y, t)}{ds(\omega)} \\ & + \frac{c_4(y, t)}{ns(\omega)}, \end{aligned} \quad (126)$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t), p(y, t), q(t), c_0(y, t), c_1(y, t), c_2(y, t), c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109), respectively.

So, in the same way, we can get many other Jacobi elliptic function solutions of Eq. (103).

When $k \rightarrow 1$, or $k \rightarrow 0$, we can also get families of new exact solutions of Eq. (103).

Type 9 If we select $l_1 = k^2, m_1 = -(1 + k^2), n_1 = 1, l_2 = -(1 - k^2), m_2 = 2 - k^2, n_2 = -1$, corresponding to Eq. (101), we can get $f = cd(\omega), g = nd(\omega)$.

So we get the following Jacobi elliptic function solutions of Eq. (103):

$$\begin{aligned} v_{18}(x, y, t) = & c_0(y, t) + c_1(y, t)nd(\omega) \\ & + c_2(y, t)cd(\omega) + \frac{c_3(y, t)}{nd(\omega)} \\ & + \frac{c_4(y, t)}{cd(\omega)}, \end{aligned} \quad (127)$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t), p(y, t), q(t), c_0(y, t), c_1(y, t), c_2(y, t), c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109), respectively.

So, in the same way, we can get many other Jacobi elliptic function solutions of Eq. (103). When $k \rightarrow 1$, or $k \rightarrow 0$, we can also get families of new exact solutions of Eq. (103).

Type 10 If we select $l_1 = -k^2(1 - k^2), m_1 = 2k^2 - 1, n_1 = 1, l_2 = -(1 - k^2), m_2 = 2 - k^2, n_2 = -1$, corresponding to Eq. (101), we can get $f = sd(\omega), g = nd(\omega)$.

So we get the following Jacobi elliptic function solutions of Eq. (103):

$$\begin{aligned} v_{19}(x, y, t) = & c_0(y, t) + c_1(y, t)nd(\omega) \\ & + c_2(y, t)sd(\omega) + \frac{c_3(y, t)}{nd(\omega)} \\ & + \frac{c_4(y, t)}{sd(\omega)}, \end{aligned} \quad (128)$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t), p(y, t), q(t), c_0(y, t), c_1(y, t), c_2(y, t), c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109), respectively.

So, in the same way, we can get many other Jacobi elliptic function solutions of Eq. (103). When $k \rightarrow 1$, or $k \rightarrow 0$, we can also get families of new exact solutions of Eq. (103).

Type 11 If we select $l_1 = -k^2(1 - k^2)$, $m_1 = 2k^2 - 1$, $n_1 = 1$, $l_2 = k^2$, $m_2 = -(1 + k^2)$, $n_2 = 1$, corresponding to Eq. (101), we can get $f = sd(\omega)$, $g = cd(\omega)$.

So we get the following Jacobi elliptic function solutions of Eq. (103):

$$\begin{aligned} v_{20}(x, y, t) = & c_0(y, t) + c_1(y, t)cd(\omega) \\ & + c_2(y, t)sd(\omega) + \frac{c_3(y, t)}{cd(\omega)} \\ & + \frac{c_4(y, t)}{sd(\omega)}, \end{aligned} \quad (129)$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109), respectively.

So, in the same way, we can get many other Jacobi elliptic function solutions of Eq. (103). When $k \rightarrow 1$, or $k \rightarrow 0$, we can also get families of new exact solutions of Eqs. (103).

Type 12 If we select $l_1 = 1 - k^2$, $m_1 = 2 - k^2$, $n_1 = 1$, $l_2 = 1$, $m_2 = -(1 + k^2)$, $n_2 = k^2$, corresponding to Eq. (101), we can get $f = sc(\omega)$, $g = dc(\omega)$.

So we get the following Jacobi elliptic function solutions:

$$\begin{aligned} v_{21}(x, y, t) = & c_0(y, t) + c_1(y, t)dc(\omega) \\ & + c_2(y, t)sc(\omega) + \frac{c_3(y, t)}{dc(\omega)} \\ & + \frac{c_4(y, t)}{sc(\omega)}, \end{aligned} \quad (130)$$

where $\omega = \alpha(y, t)x + p(y, t) + q(t)$, $\alpha(y, t)$, $p(y, t)$, $q(t)$, $c_0(y, t)$, $c_1(y, t)$, $c_2(y, t)$, $c_3(y, t)$ and $c_4(y, t)$ satisfy Eqs. (106), (107), (108), and (109), respectively.

So, in the same way, we can get many other Jacobi elliptic function solutions of Eq. (103). When $k \rightarrow 1$, or $k \rightarrow 0$, we can also get families of new exact solutions of Eq. (103).

Remark 5 From the details above, it is very easy to see that we have gotten many new exact

solutions of the (2 + 1)-dimensional KdV equation (103). The solutions we get include Jacobi elliptic doubly periodic function solutions, soliton-like solutions and triangular function solutions. These solutions are more general than the solutions which the extended F-expansion method gets, and they are more abundant than the solutions in Zhou et al. (2003). Besides, our method is more convenient than the method in Zhou et al. (2003). Among the solutions, the arbitrary functions imply that these solutions have rich local structures. In this paper, we have provided some figures to describe the character of the new exact solutions of Eq. (103).

The Generalized Algebra Method and Its Application in (1 + 1)-Dimensional Generalized Variable – Coefficient KdV Equation

In this section, with the aid of the Maple symbolic computation system, we will develop the algebraic methods (Fan and Dai 2003; Yan 2004) for constructing the traveling wave solutions of NLPDEs and present the following new methods and theorems:

1. A new and general transformation, a new theorem and its proof by using Maple, are presented in Ren (2007a).
2. A new mechanization method to find the exact solutions of a first-order nonlinear ordinary differential equation with any degree. The validity and reliability of the method are tested by its application to the first-order nonlinear ordinary differential equation with six degrees, eight degrees, ten degrees, and twelve degrees in Ren (2007a) and Ren et al. (2007, 2008a).
3. A general transformation, a new generalized algebraic method and their algorithms are suggested based on a nonlinear ordinary differential equation with any degree. The (1 + 1)-Dimensional Generalized Variable – Coefficient KdV Equations are chosen to illustrate our algorithm so that more families of new exact solutions are obtained, which contain both non-traveling and traveling wave

solutions in Ren (2007a) and Ren et al. (2007, 2004). The algebraic method can obtain many types of traveling wave solutions based on a nonlinear ordinary differential equation with four degrees and the following theorem:

Recently, much research work has been concentrated on the various extensions and applications of the algebraic method with computerized symbolic computation (Fan and Dai 2003; Yan

Theorem 3 The following nonlinear ordinary differential equation with four degrees

$$\frac{d}{d\xi}\phi(\xi) = \sqrt{c_0 + c_1\phi(\xi) + c_2\phi^2(\xi) + c_3\phi^3(\xi) + c_4\phi^4(\xi)} \quad c_i = \text{constant}, \quad i = 0,1,2,3,4 \quad (131)$$

admits many kinds of fundamental solutions, some of which are listed as follows.

Case 1 If $c_0 = c_1 = c_3 = 0$, Eq. (131) admits: a bell shaped solitary wave solution

$$\phi(\xi) = \sqrt{-\frac{c_2}{c_4}} \operatorname{sech}(\sqrt{c_2}\xi), \quad c_2 > 0, \quad c_4 < 0, \quad (132)$$

a triangular type solution

$$\phi(\xi) = \sqrt{-\frac{c_2}{c_4}} \sec(\sqrt{-c_2}\xi), \quad c_2 < 0, \quad c_4 > 0, \quad (133)$$

a rational polynomial type solution

$$\phi(\xi) = -\frac{1}{\sqrt{c_4}\xi}, \quad c_2 = 0, \quad c_4 > 0. \quad (134)$$

Case 2 If $c_0 = d_2^2/(4c_4), c_1 = c_3 = 0$, Eq. (131) admits a kink shaped solitary wave solution

$$\phi(\xi) = \sqrt{-\frac{c_2}{2c_4}} \tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right), \quad c_2 < 0, \quad c_4 > 0, \quad (135)$$

a triangular type solution

$$\phi(\xi) = \sqrt{\frac{c_2}{2c_4}} \tan\left(\sqrt{\frac{c_2}{2}}\xi\right), \quad c_2 > 0, \quad c_4 > 0, \quad (136)$$

a rational polynomial type solution

$$\phi(\xi) = -\frac{1}{\sqrt{c_4}\xi}, \quad c_2 = 0, \quad c_4 > 0. \quad (137)$$

Case 3 If $c_1 = c_3 = 0$, Eq. (131) admits three Jacobian elliptic functions type solutions

$$\begin{aligned} \phi(\xi) &= \sqrt{-\frac{c_2 m^2}{c_4(2m^2 - 1)}} \operatorname{cn}\left(\sqrt{\frac{c_2}{2m^2 - 1}}\xi\right), \\ c_0 &= \frac{d_2^2 m^2 (m^2 - 1)}{c_4 (2m^2 - 1)^2}, \quad c_2 > 0, \end{aligned} \quad (138)$$

$$\begin{aligned} \phi(\xi) &= \sqrt{-\frac{c_2 m^2}{c_4(m^2 + 1)}} \operatorname{sn}\left(\sqrt{-\frac{c_2}{m^2 + 1}}\xi\right), \\ c_0 &= \frac{d_2^2 m^2}{c_4 (m^2 + 1)^2}, \quad c_2 < 0, \end{aligned} \quad (139)$$

and

$$\begin{aligned} \phi(\xi) &= \sqrt{-\frac{c_2}{c_4(2 - m^2)}} \operatorname{dn}\left(\sqrt{\frac{c_2}{2 - m^2}}\xi\right), \\ c_0 &= \frac{d_2^2 (1 - m^2)}{c_4 (2 - m^2)^2}, \quad c_2 > 0. \end{aligned} \quad (140)$$

Case 4 If $c_0 = c_1 = c_4 = 0$, Eq. (131) admits two bell shaped solitary wave solutions

$$\phi(\xi) = -\frac{c_2}{c_3} \operatorname{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right), \quad c_2 > 0, \quad (141)$$

and

$$\phi(\xi) = -\frac{c_2}{c_3} \operatorname{sech}^2\left(\frac{\sqrt{-c_2}}{2}\xi\right), \quad c_2 < 0, \quad (142)$$

a rational polynomial type solution

$$\phi(\xi) = \frac{1}{c_3 \xi^2}, \quad c_2 = 0, \quad (143)$$

Case 5 If $c_3 > 0$, $c_4 = 0$, Eq. (131) admits a Weierstrass elliptic functions type solution

$$\phi(\xi) = \wp\left(\frac{\sqrt{c_3}}{2}\xi, -\frac{4c_1}{c_3}, -\frac{4c_0}{c_3}\right). \quad (144)$$

Let's specifically see how the algebraic method works. For a given nonlinear differential equation, say, in two variables x, t

$$F(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \quad (145)$$

where F is a polynomial function with respect to the indicated variables or some function which can be reduced to a polynomial function by using some transformations.

By using the traveling wave transformation

$$u = u(\xi), \xi = x - \lambda t, \quad (146)$$

Equation (145) is reduced to an ordinary differential equation with constant coefficients

$$G(U, U', U'', U''', \dots) = 0 \quad (147)$$

A transformation was presented by Fan (Leibovitch 1970) in the form

$$u(x) = A_0 + \sum_{i=1}^n A_i \phi^{i-1}(\xi), \quad (148)$$

with the new variable $\phi(\xi)$ satisfying Eq. (131), where A_0, A_i, d_j are constants.

Substituting Eq. (148) into Eq. (147) along with (131), we can determine the parameter n in Eq. (148). And then by substituting Eq. (148) with the concrete n into Eq. (147) and equating the coefficients of these terms $\omega^i \omega^j$ ($i = 0, 1, 2, \dots; j = 0, 1$), we obtain a system of algebraic equations with respect to other parameters A_0, A_i, d_j, λ . By solving the system, if available, we may determine these parameters. Therefore we establish a transformation (147) between Eqs. (146) and (148). If we know the solutions of Eq. (148), then we can obtain the solutions of Eq. (147) (or (145)) by using Eq. (146).

A New Transformation and a New Theorem

In Yu-Jie Ren's Ph.D Dissertation of Dalian University of Technology (Ren 2007a), in order to develop the above algebraic method, we first presented the following new transformation (150) and new Theorem 4. Then we proved the theorem by means of the Maple computer algebraic system.

Theorem 4 (Ren 2007a) Suppose n and m are any integers, a first-order nonlinear ordinary differential equation with any degree in the form of

$$\frac{d}{d\xi} \phi(\xi) = \varepsilon \sqrt{c_0 + \sum_{i=1}^r c_i \cdot \phi^i(\xi)}, \quad (149)$$

$c_i = \text{consts}, i = 1, 2, \dots, r.$

If we substitute the following new transform into Eq. (149).

$$\phi(\xi) = (g(\xi))^{\frac{n}{m}}, \quad (150)$$

then Eq. (149) can be transformed into an ordinary differential equation.

$$\left(\frac{d}{d\xi} g(\xi)\right)^2 = m^2 n^{-2} \sum_{i=0}^r c_i (g(\xi))^{2 + \frac{n(i-2)}{m}}. \quad (151)$$

We give the proof of Theorem 4 (Ren 2007a) by using Maple as follows:

Proof Step 1. Importing the following Maple program at the Maple Command Window

$$eq := \text{diff}(\text{phi}(xi), xi)^2 \\ - \text{sum}(c[i] * \text{phi}(xi)^i, i = 0..r);$$

Equation (149) is displayed at a computer screen (after implementing the above program) as follows:

$$eq := \left(\frac{d}{d\xi} \phi(\xi) \right)^2 - \sum_{i=0}^r c_i (\phi(\xi))^i.$$

Step 2. Importing the following Maple program at the Maple Command Window

$$eq := \text{subs}(\text{phi}(xi) = (g(xi))(n/m), eq);$$

the following result is displayed at the screen (after running the above program):

$$eq := \left((g(\xi))^{\frac{n}{m}} \right)^2 n^2 \left(\frac{d}{d\xi} g(\xi) \right)^2 m^{-2} (g(\xi))^{-2} \\ - \sum_{i=0}^r c_i \left((g(\xi))^{\frac{n}{m}} \right)^i.$$

Step 3. Importing the following Maple program at the Maple Command Window

$$eq := \text{simplify}(eq); \quad eq := \text{expand}(eq); \quad eq := \text{numer}(eq);$$

the following result is displayed at the screen (after running the above program):

$$eq := \left((g(\xi))^{\frac{n}{m}} \right)^2 n^2 \left(\frac{d}{d\xi} g(\xi) \right)^2 \\ - \sum_{i=0}^r c_i \left((g(\xi))^{\frac{n}{m}} \right)^i m^2 (g(\xi))^2.$$

Step 4. Importing the following Maple program at the Maple Command Window

$$eq := \text{subs}(\text{diff}(g(xi), xi) = G(xi), g(xi) = g, eq) \\ eq := eq * (g(n/m))(-2); \quad eq := \text{simplify}(eq); \\ eq := n^2 * G(xi)^2 - (\text{sum}(c[i] * g((-2*(n-m)/m) \\ + i*n/m), i = 0..r)) * m^2; \\ eq := \text{simplify}(eq);$$

the following result is displayed at the screen (after running the above program):

$$eq := n^2 (G(\xi))^2 - g^{-2\frac{n-m}{m}} \sum_{i=0}^r c_i (g^{\frac{n}{m}})^i m^2 \\ eq := n^2 (G(\xi))^2 - \sum_{i=0}^r c_i g^{\frac{-2n+2m+ni}{m}} m^2$$

Step 5. Importing the following Maple program at the Maple Command Window

$$eq := \text{subs}(g = g(xi), G(xi) = \text{diff}(g(xi), xi), eq);$$

the following result is displayed at the screen (after running the above program):

$$eq := n^2 \left(\frac{d}{d\xi} g(\xi) \right)^2 - \sum_{i=0}^r c_i (g(\xi))^{\frac{-2n+2m+ni}{m}} m^2. \quad (152)$$

We can reduce Eq. (152) to (151).

Remark 6 The above transformation (150), Theorem 4, and its proof by means of Maple were first presented by us in Yu-Jie Ren's Ph.D Dissertation of Dalian University of Technology (Ren 2007a).

A New Mechanization Method to Find the Exact Solutions of a First-Order Nonlinear Ordinary Differential Equation with any Degree Using Maple and Its Application

According to Theorem 4 in subsection “The Exp-Bäcklund Transformation Method and Its Application in (1 + 1)-Dimensional KdV Equation”, in Ren (2007a) we first presented the following mechanization method to find the exact solutions of a first-order nonlinear ordinary differential equation with any degree. The validity and reliability of our method are tested by its application to a first-order nonlinear ordinary differential equation with six degrees (Ren 2007a). Now, we simply describe our mechanization method as follows:

Step 1. Import the following Maple program at the Maple Command Window

$$eq := n^2 * \text{diff}(g(xi), xi)^2 \\ - \text{sum}(c[i] \\ * g(xi)((-2*n + 2*m + n*i)/m), i = 0..r) * m^2;$$

Eq. (151) is displayed at a computer screen (after implementing the above program) as follows

$$eq:=n^2\left(\frac{d}{d\xi}g(\xi)\right)^2-\sum_{i=0}^r c_i(g(\xi))^{\frac{-2n+2m+ni}{m}}m^2$$

Step 2. Import the following Maple programs with some values of the degree r and parameter (n, m) in new transform (150) at the Maple Command Window. For example,

$$\begin{aligned} eq621 &:= \text{subs}(r = 6, m = 2, n = 1, eq); \\ eq621 &:= \text{simplify}(eq621); \end{aligned}$$

the following result is displayed at the screen (after running the above program):

$$\begin{aligned} eq621 &:= \left(\frac{d}{d\xi}g(\xi)\right)^2 - 4c_0g(\xi) \\ &\quad - 4c_1(g(\xi))^{3/2} - 4c_2(g(\xi))^2 - 4c_3(g(\xi))^{5/2} \\ &\quad - 4c_4(g(\xi))^3 - 4c_5(g(\xi))^{7/2} - 4c_6(g(\xi))^4. \end{aligned} \quad (153)$$

Step 3. According to the output results in Step 2, we choose the coefficients of c_i , $i = 1, 2, \dots, m(<n)$ to be zero. Then we import their Maple program. For example, we import the Maple program as follows:

$$\begin{aligned} eq246 &:= \text{subs}(c[0] = 0, c[1] = 0, \\ &\quad c[3] = 0, c[5] = 0, eq621); \end{aligned}$$

the following result is displayed at the screen (after running the above program):

$$\begin{aligned} eq246 &:= \left(\frac{d}{d\xi}g(\xi)\right)^2 - 4c_2(g(\xi))^2 - 4c_4(g(\xi))^3 \\ &\quad - 4c_6(g(\xi))^4. \end{aligned} \quad (154)$$

Step 4. Import the Maple program for solving the output equation in Step 3. For example, we import the Maple program for solving the output Eq. (154) as follows:

$$\text{dsolve}(eq246, g(\xi));$$

the following formal solutions of Eq. (154) are displayed at the screen (after running the above program):

$$g_1(\xi) = 1/2 \frac{-c_4 - \sqrt{c_4^2 - 4c_6c_2}}{c_6}, \quad (155)$$

$$g_2(\xi) = 1/2 \frac{-c_4 + \sqrt{c_4^2 - 4c_6c_2}}{c_6}, \quad (156)$$

$$\begin{aligned} g_3(\xi) &= 4(e^{C_1\sqrt{c_2}})^2 c_2 (e^{\xi\sqrt{c_2}})^{-2} \\ &\quad \left(-4c_6c_2 + \frac{(e^{C_1\sqrt{c_2}})^4}{(e^{\xi\sqrt{c_2}})^4} - 2 \frac{(e^{C_1\sqrt{c_2}})^2 c_4}{(e^{\xi\sqrt{c_2}})^2} + c_4^2 \right)^{-1}, \end{aligned} \quad (157)$$

$$\begin{aligned} g_4(\xi) &= -4(e^{\xi\sqrt{c_2}})^2 c_2 (e^{C_1\sqrt{c_2}})^{-2} \\ &\quad \left(4c_6c_2 - \frac{(e^{\xi\sqrt{c_2}})^4}{(e^{C_1\sqrt{c_2}})^4} + 2 \frac{(e^{\xi\sqrt{c_2}})^2 c_4}{(e^{C_1\sqrt{c_2}})^2} - c_4^2 \right)^{-1}. \end{aligned} \quad (158)$$

Import the Maple program for reducing the above solutions of Eq. (154). For example, we import the Maple program for reducing the Eqs. (157) and (158) as follows:

$$g3:=\text{simplify}(g3); g4:=\text{simplify}(g4);$$

the following results are displayed at a computer screen (after running the above program):

$$\begin{aligned} g_3(\xi) &= 4 \\ &\quad \frac{e^{2\sqrt{c_2}(C_1+\xi)}c_2}{-4c_6c_2e^{4\xi\sqrt{c_2}} + e^{4C_1\sqrt{c_2}} - 2e^{2\sqrt{c_2}(C_1+\xi)}c_4 + c_4^2e^{4\xi\sqrt{c_2}}}, \end{aligned} \quad (159)$$

$$\begin{aligned} g_4(\xi) &= 4 \\ &\quad \frac{e^{2\sqrt{c_2}(C_1+\xi)}c_2}{-4c_6c_2e^{4C_1\sqrt{c_2}} + e^{4\xi\sqrt{c_2}} - 2e^{2\sqrt{c_2}(C_1+\xi)}c_4 + c_4^2e^{4C_1\sqrt{c_2}}}. \end{aligned} \quad (160)$$

Step 5. We discuss the above solutions under different conditions in Step 3.

For example, we discuss the solutions under different conditions $c_4^2 - 4c_6c_2 < 0$ or $c_4^2 - 4c_6c_2 > 0$ or $c_4^2 - 4c_6c_2 = 0$ or $c_2 < 0$ or $c_2 > 0$ or $c_2 = 0$. When

$$c_2 > 0, 4c_6c_2 - c_4^2 < 0, \quad (161)$$

we may import the following Maple program:

```
assume(c[2] > 0, (4*c[2]*c[6] - c[4]^2) < 0);
eq246jie:=dsolve(eq246, g(xi));
```

six solutions of a first-order nonlinear ordinary differential equation with six degrees under condition (161), which includes two new solutions, are displayed at a computer screen (after running the above program). Here we omit them due to the length of our article.

Importing the following Maple program for reducing the two new solutions above,

```
eq246jie1:=subs(2*xi*sqrt(c[2])
-2*c[1]*sqrt(c[2])=eta, eq246jie1);
eq246jie2:=subs(2*xi*sqrt(c[2])
-2*c[1]*sqrt(c[2])=eta, eq246jie2);
```

the following results are displayed at a computer screen (after running the above program):

$$(\phi_{11}(\xi))^2 = -2 \frac{c_2(-c_4^2 + (\tanh(\eta))^2 c_4^2 - \tanh(\eta) \sqrt{c_4^2 - 4c_6c_2} \sqrt{c_4^2((\tanh(\eta))^2 - 1)})}{(4c_6(\tanh(\eta))^2 c_2 - c_4^2) c_4}, \quad (162)$$

$$(\phi_{12}(\xi))^2 = -2 \frac{c_2(-c_4^2 + (\tanh(\eta))^2 c_4^2 + \tanh(\eta) \sqrt{c_4^2 - 4c_6c_2} \sqrt{c_4^2((\tanh(\eta))^2 - 1)})}{(4c_6(\tanh(\eta))^2 c_2 - c_4^2) c_4}. \quad (163)$$

Importing the following Maple program for reducing Eqs. (162) and (163):

```
eq246jie3:=subs(sqrt(c[4]^2*(tanh(eta)^2-1))
=I*abs(c[4]*sech(eta)), -c[4]+tanh(eta)^2*c[4]
=-sech(eta)^2*c[4], eq246jie3);
eq246jie4:=subs(sqrt(c[4]^2*(tanh(eta)^2-1))
=I*abs(c[4]*sech(eta)), -c[4]+tanh(eta)^2*c[4]
=-sech(eta)^2*c[4], eq246jie4);
phi[1](xi):=epsilon*(eq246jie3)(1/2);
phi[2](xi):=epsilon*(eq246jie4)(1/2);
```

the following results are displayed at a computer screen (after running the above program):

$$\phi_1(\xi) = \epsilon \sqrt{\frac{2c_2(c_4^2 \operatorname{sech}^2(\eta) + i|c_4 \operatorname{sech}(\eta)| \tanh(\eta) \sqrt{c_4^2 - 4c_6c_2})}{c_4(4c_6 \tanh^2(\eta) c_2 - c_4^2)}}, \quad (164)$$

$$\phi_2(\xi) = \epsilon \sqrt{\frac{2c_2(c_4^2 \operatorname{sech}^2(\eta) - i|c_4 \operatorname{sech}(\eta)| \tanh(\eta) \sqrt{c_4^2 - 4c_6c_2})}{c_4(4c_6 \tanh^2(\eta) c_2 - c_4^2)}}. \quad (165)$$

where $\eta = 2\sqrt{c_2}(\xi - C_1)$.

Step 6. We need to further discuss the results found by sometimes adding new conditions so that the results are simpler in form. For example, we can import the following Maple program for getting rid of the absolute value sign in the above results:

```
assume(c[4]*sech(eta) < 0);
eq246jie3:=2*c[2]*(c[4]^2*sech(eta)^2+I
*tanh(eta)*sqrt(c[4]^2-4*c[6]*c[2])
*abs(c[4]*sech(eta)))/(4*c[6]*tanh(eta)^2
*c[2]-c[4]^2)/c[4];
eq246jie4:=2*c[2]*(c[4]^2*sech(eta)^2-I
*tanh(eta)*sqrt(c[4]^2-4*c[6]*c[2])*abs(c[4]*
sech(eta)))/(4*c[6]*tanh(eta)^2
*c[2]-c[4]^2)/c[4]; phi[1,1](xi):=epsilon
*(eq246jie3)(1/2);
phi[1,2](xi):=epsilon*(eq246jie4)(1/2);
```

the following results are displayed at a computer screen (after running the above program):

$$\phi_{1,1}(\xi) = \varepsilon \sqrt{\frac{2c_2 \operatorname{sech}(\eta) (c_4 \operatorname{sech}(\eta) - i \tanh(\eta) \sqrt{c_4^2 - 4c_6 c_2})}{4c_6 (\tanh(\eta))^2 c_2 - c_4^2}}, \quad (166)$$

$$\phi_{1,2}(\xi) = \varepsilon \sqrt{\frac{2c_2 \operatorname{sech}(\eta) (c_4 \operatorname{sech}(\eta) + i \tanh(\eta) \sqrt{c_4^2 - 4c_6 c_2})}{4c_6 (\tanh(\eta))^2 c_2 - c_4^2}}. \quad (167)$$

Importing the following Maple program for reducing and discussing above results:

```
assume(c[4] * sech(eta) > 0);
eq246jie3:=2*c[2] * (c[4]^2*sech(eta)^2 + I
  * tanh(eta) * sqrt(c[4]^2 - 4*c[6] * c[2]))*
  abs(c[4] * sech(eta)))/(4*c[6] * tanh(eta)^2
  * c[2] - c[4]^2)/c[4];
eq246jie4:=2*c[2] * (c[4]^2*sech(eta)^2
```

```
-I* tanh(eta) * sqrt(c[4]^2 - 4*c[6] * c[2]))*
  abs(c[4] * sech(eta)))/(4*c[6] * tanh(eta)^2
  * c[2] - c[4]^2)/c[4];
phi[2, 1](xi):=epsilon*(eq246jie3)(1/2);
phi[2, 2](xi):=epsilon*(eq246jie4)(1/2);
```

the following results are shown at a computer screen (after running the above program):

$$\phi_{2,1}(\xi) = \varepsilon \sqrt{\frac{2c_2 \operatorname{sech}(\eta) (c_4 \operatorname{sech}(\eta) + i \tanh(\eta) \sqrt{c_4^2 - 4c_6 c_2})}{4c_6 (\tanh(\eta))^2 c_2 - c_4^2}}, \quad (168)$$

$$\phi_{2,2}(\xi) = \varepsilon \sqrt{\frac{2c_2 \operatorname{sech}(\eta) (c_4 \operatorname{sech}(\eta) - i \tanh(\eta) \sqrt{c_4^2 - 4c_6 c_2})}{4c_6 (\tanh(\eta))^2 c_2 - c_4^2}}. \quad (169)$$

By using this method, we obtained some new types of general solution of a first-order nonlinear ordinary differential equation with six degrees and presented the following theorem in Ren (2007a) and Ren et al. (2007, 2008a).

Theorem 5 The following nonlinear ordinary differential equation with six degrees.

$$\frac{d\phi(\xi)}{d\xi} = \varepsilon \sqrt{c_0 + c_1\phi(\xi) + c_2\phi^2(\xi) + c_3\phi^3(\xi) + c_4\phi^4(\xi) + c_5\phi^5(\xi) + c_6\phi^6(\xi)},$$

$$c_i = \text{constant}, \quad i = 0, 1, 2, 3, 4, 5, 6$$
(170)

admits many kinds of fundamental solutions which depend on the values and constraints between c_i , $i = 0, 1, 2, 3, 4, 5, 6$.

Some of the solutions are listed in the following sections.

Case 1 If $c_0 = c_1 = c_3 = c_5 = 0$, Eq. (170) admits the following constant solutions.

$$\phi_{1,2}(\xi) = \varepsilon \sqrt{\frac{-c_4 + \sqrt{c_4^2 - 4c_6c_2}}{2c_6}}, \quad (171)$$

$$\phi_{3,4}(\xi) = \varepsilon \sqrt{\frac{-c_4 - \sqrt{c_4^2 - 4c_6c_2}}{2c_6}}, \quad (172)$$

and the following exponential type solutions.

$$\phi_{5,6}(\xi) = 2\varepsilon \sqrt{\frac{e^{2\sqrt{c_2}(\xi+C_1)}c_2}{-4c_6c_2e^{4C_1\sqrt{c_2}} + e^{4\xi\sqrt{c_2}}} - 2e^{2\sqrt{c_2}(\xi+C_1)}c_4 + c_4^2e^{4C_1\sqrt{c_2}}}},$$
(173)

$$\phi_{7,8}(\xi) = 2\varepsilon \sqrt{\frac{e^{2\sqrt{c_2}(\xi+C_1)}c_2}{-4c_6c_2e^{4\xi\sqrt{c_2}} + e^{4C_1\sqrt{c_2}}} - 2e^{2\sqrt{c_2}(\xi+C_1)}c_4 + c_4^2e^{4\xi\sqrt{c_2}}}}.$$
(174)

When we take different values and constraints of $4c_2c_6 - c_4^2$, the solutions (5.43) and (5.44) can be written in different formats as follows:

Case 1.1 If $4c_2c_6 - c_4^2 < 0$, Eq. (170) admits the following tanh-sech hyperbolic type solutions

$$\phi_{1,2}(\xi) = \varepsilon \sqrt{\frac{2c_2\text{sech}(\eta)[c_4\text{sech}(\eta) + \varepsilon i\sqrt{c_4^2 - 4c_6c_2}\tanh(\eta)]}{4c_6c_2\tanh^2(\eta) - c_4^2}},$$

$$c_2 > 0,$$
(175)

$$\phi_{3,4}(\xi) = \varepsilon \sqrt{\frac{2c_2\text{sech}(\eta)[c_4\text{sech}(\eta) - \varepsilon i\sqrt{c_4^2 - 4c_6c_2}\tanh(\eta)]}{4c_6c_2\tanh^2(\eta) - c_4^2}},$$

$$c_2 > 0,$$
(176)

and the following tan-sec triangular type solutions

$$\phi_{5,6}(\xi) = \varepsilon \sqrt{\frac{2c_2\sec(\zeta)[-c_4\sec(\zeta) + \varepsilon\sqrt{c_4^2 - 4c_6c_2}\tan(\zeta)]}{4c_6c_2\tan^2(\zeta) + c_4^2}},$$

$$c_2 < 0,$$
(177)

$$\phi_{7,8}(\xi) = \varepsilon \sqrt{\frac{2c_2\sec(\zeta)[-c_4\sec(\zeta) - \varepsilon\sqrt{c_4^2 - 4c_6c_2}\tan(\zeta)]}{4c_6c_2\tan^2(\zeta) + c_4^2}}, \quad c_2 < 0,$$
(178)

where $\eta = 2\sqrt{c_2}(\xi - C_1)$ and C_1 is any constant, $\zeta = 2\sqrt{-c_2}(\xi - C_1)$ and C_1 is any constant.

Case 1.2 If $4c_2c_6 - c_4^2 > 0$, Eq. (170) admits the following sinh-cosh hyperbolic type solutions

$$\phi_{9,10}(\xi) = \varepsilon \sqrt{\frac{2c_2 [c_4 + \varepsilon \sqrt{4c_2c_6 - c_4^2} \sinh(\eta)]}{4c_2c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)}},$$

$$c_2 > 0,$$
(179)

$$\phi_{11,12}(\xi) = \varepsilon \sqrt{\frac{2c_2 [c_4 - \varepsilon \sqrt{4c_2c_6 - c_4^2} \sinh(\eta)]}{4c_2c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)}},$$

$$c_2 > 0,$$
(180)

and the following sin-cos triangular type solutions

$$\phi_{13,14}(\xi) = \varepsilon \sqrt{\frac{2c_2 [-c_4 + \varepsilon i \sqrt{4c_2c_6 - c_4^2} \sin(\zeta)]}{4c_6c_2 \sin^2(\zeta) + c_4^2 \cos^2(\zeta)}},$$

$$c_2 < 0$$
(181)

$$\phi_{15,16}(\xi) = \varepsilon \sqrt{\frac{2c_2 [-c_4 - \varepsilon i \sqrt{4c_2c_6 - c_4^2} \sin(\zeta)]}{4c_6c_2 \sin^2(\zeta) + c_4^2 \cos^2(\zeta)}},$$

$$c_2 < 0;$$
(182)

where $\eta = 2\sqrt{c_2}(\xi - C_1)$, $\zeta = 2\sqrt{-c_2}(\xi - C_1)$ and C_1 is any constant.

Case 1.3 If $4c_2c_6 - c_4^2 = 0$, Eq. (170) admits the following tanh hyperbolic type solutions

$$\phi_{17,18,19,20}(\xi) = \varepsilon \sqrt{\frac{2c_2}{e^{-\varepsilon 2\sqrt{c_2}(\xi - C_1)} - c_4}}$$

$$= \varepsilon \sqrt{\frac{2c_2 (1 + \varepsilon \tanh(\sqrt{c_2}(\xi - C_1)))}{1 - c_4 - \varepsilon (1 + c_4) \tanh(\sqrt{c_2}(\xi - C_1))}},$$

$$c_2 > 0,$$
(183)

and the following tan triangular type solutions

$$\phi_{21,22,23,24}(\xi) = \varepsilon \sqrt{\frac{2c_2}{e^{-\varepsilon 2i\sqrt{-c_2}(\xi - C_1)} - c_4}}$$

$$= \varepsilon \sqrt{\frac{2c_2 (1 + \varepsilon \tan(i\sqrt{-c_2}(\xi - C_1)))}{1 - c_4 - \varepsilon (1 + c_4) \tan(i\sqrt{-c_2}(\xi - C_1))}},$$

$$c_2 < 0$$
(184)

where C_1 is any constant.

Case 2 If $c_5 = c_6 = 0$, Eq. (170) admits the solutions in Theorem 6.1.

Remark 7 By using our mechanization method using Maple to find the exact solutions of a first-order nonlinear ordinary differential equation with any degree, we can obtain some new types of general solution of a first-order nonlinear ordinary differential equation with $r(=7, 8, 9, 10, 11, 12, \dots)$ degree (Ren 2007a). We do not list the solutions here in order to avoid unnecessary repetition.

Summary of the Generalized Algebra Method

In this section, based on a first-order nonlinear ordinary differential equation with any degree (149) and its the exact solutions obtained by using our mechanization method via Maple, we will develop the algebraic methods (Fan and Dai 2003; Yan 2004) for constructing the traveling wave solutions and present a new generalized algebraic method and its algorithms (Ren 2007a; Ren et al. 2007, 2008a). The KdV equation is chosen to illustrate our algorithm so that more families of new exact solutions are obtained which contain both non-traveling solutions and traveling wave solutions. We outline the main steps of our generalized algebraic method as follows:

Step 1

For given nonlinear differential equations, with some physical fields $u_i(t, x_1, x_2, \dots, x_m)$, ($i = 1, 2, \dots, n$) in $m + 1$ independent variables $t, x_1, x_2, \dots, x_m$,

$$F_j(u_1, \dots, u_n, u_{1,t}, \dots, u_{n,t}, u_{1,x_1}, \dots, u_{n,x_m}, u_{1,tt}, \dots, u_{n,tt}, u_{1,tx_1}, \dots, u_{n,tx_m}, \dots) = 0, \quad (185)$$

$$j = 1, 2, \dots, n,$$

where $j = 1, 2, \dots, n$.

By using the following more general transformation, which we first present here,

$$\begin{aligned} u_i(t, x_1, x_2, \dots, x_m) &= U_i(\xi), \\ \xi &= u_i(t, x_1, x_2, \dots, x_m) = U_i(\xi), \\ \xi &= \alpha_0(t) + \sum_{i=1}^{m-1} \alpha_i(x_i, x_{i+1}, \dots, x_m, t) \cdot \beta_i(x_i), \end{aligned} \quad (186)$$

where $\alpha_0(t)$, $\alpha_i(x_i, x_{i+1}, \dots, x_m, t)$ and $\beta_i(x_i)$, $i = 1, 2, \dots, m-1$ are functions to be determined later. For example, when $n = 1$, we may take

$$\xi = \alpha_0(t) + \alpha_1(x_1 t) \cdot \beta_1(x_1),$$

where $\alpha_0(t)$, $\alpha_1(x_1 t)$ and $\beta_1(x_1)$ are undetermined functions.

Then Eq. (185) is reduced to nonlinear differential equations

$$G_j(U_1, \dots, U_n, U'_1, \dots, U'_n, U''_1, \dots, U''_n, \dots) = 0, \quad j = 1, 2, \dots, n, \quad (187)$$

where G_j ($j = 1, 2, \dots, n$) are all polynomials of U_i ($i = 1, 2, \dots, n$), $\alpha_0(t)$, $\alpha_i(x_i, x_{i+1}, \dots, x_m, t)$, $\beta_i(x_i)$, $i = 1, 2, \dots, m-1$ and their derivatives. If G_k of them is not a polynomial of U_i ($i = 1, 2, \dots, n$), $\alpha_i(x_i, x_{i+1}, \dots, x_m, t)$, $\beta_i(x_i)$, $i = 1, 2, \dots, m-1$, $\alpha_0(t)$ and their derivatives, then we may use new variable $v_i(\xi)$ ($i = 1, 2, \dots, n$) which makes G_k become a polynomial of $v_i(\xi)$ ($i = 1, 2, \dots, n$), $\alpha_0(t)$, $\alpha_i(x_i, x_{i+1}, \dots, x_m, t)$ and $\beta_i(x_i)$, $i = 1, 2, \dots, m-1$ and their derivatives. Otherwise the following transformation will fail to seek solutions of Eq. (185).

Step 2

We introduce a new variable $\phi(\xi)$ which is a solution of the following ODE

$$\begin{aligned} \frac{d\phi(\xi)}{d\xi} &= \varepsilon \sqrt{c_0 + c_1\phi(\xi) + c_2\phi^2(\xi) + c_3\phi^3(\xi) + c_4\phi^4(\xi) + \dots + c_r\phi^r(\xi)}, \\ r &= 0, 1, 2, 3, \dots \end{aligned} \quad (188)$$

Then the derivatives with respect to the variable ξ become the derivatives with respect to the variable ϕ .

Step 3

By using the new variable ϕ , we expand the solution of Eq. (185) in the form:

$$\begin{aligned} U_i &= a_{i,0}(X) + \sum_{k=1}^{n_i} \\ &\times (a_{i,k}(X)\phi^k(\xi(X)) + b_{i,k}(X)\phi^{-k}(\xi(X))). \end{aligned} \quad (189)$$

where $X = (x_1, x_2, \dots, x_m, t)$, $\xi = \xi(X)$, $a_{i,0}(X)$, $a_{i,k}(X)$, $b_{i,k}(X)$ ($i = 1, 2, \dots, n$; $k = 1, 2, \dots, n_i$) are all differentiable functions of X to be determined later.

Step 4

In order to determine n_i ($i = 1, 2, \dots, n$) and r , we may substitute Eq. (188) into (187) and balance the highest derivative term with the nonlinear terms in Eq. (187). By using the derivatives with respect to the variable ϕ , we can obtain a relation for n_i and r , from which the different possible values of n_i and r can be obtained. These values lead to the series expansions of the solutions for Eq. (185).

Step 5

Substituting Eq. (189) into the given Eq. (185) and collecting coefficients of polynomials of ϕ^k , ϕ^{-k} , and $\phi^i\phi^{-k}\sqrt{\sum_{j=1}^r c_j\phi^j(\xi)}$, with the aid of Maple, then setting each coefficient to zero, we

will get a system of over-determined partial differential equations with respect to $a_{i,0}(X)$, $a_{i,k}(X)$, $b_{i,k}(X)$, $i = 1, 2, \dots, n$; $k = 1, 2, \dots, n_i$; c_j , $j = 0, 1, \dots, r$; $\alpha_0(t)$, $\alpha_i(x_i, x_{i+1}, \dots, x_m, t)$ and $\beta_i(x_i)$, $i = 1, 2, \dots, m - 1$.

Step 6

Solving the over-determined partial differential equations with Maple, then we can determine $a_{i,0}(X)$, $a_{i,k}(X)$, $b_{i,k}(X)$, $i = 1, 2, \dots, n$; $k = 1, 2, \dots, n_i$; c_j , $j = 0, 1, \dots, r$; $\alpha_0(t)$, $\alpha_i(x_i, x_{i+1}, \dots, x_m, t)$ and $\beta_i(x_i)$, $i = 1, 2, \dots, m - 1$.

Step 7

From the constants $a_{i,0}(X)$, $a_{i,k}(X)$, $b_{i,k}(X)$ ($i = 1, 2, \dots, n$; $k = 1, 2, \dots, n_i$); c_j ($j = 0, 1, \dots, r$); $\alpha_0(t)$, $\alpha_i(x_i, x_{i+1}, \dots, x_m, t)$ and $\beta_i(x_i)$, $i = 1, 2, \dots, m - 1$ obtained in Step 6 to Eq. (188), we can then obtain all the possible solutions.

Remark 8 When $c_5 = c_6 = 0$ and $b_{i,k} = 0$, Eq. (170) and transformation (189) just become the ones used in our previous method (Fan and Dai 2003; Yan 2004). However, if $c_5 \neq 0$ or $c_6 \neq 0$, we may obtain solutions that cannot be found by using the methods (Fan and Dai 2003; Yan 2004). It should be pointed out that there is no method to find all solutions of nonlinear PDEs. But our method can be used to find more solutions of nonlinear PDEs, and with the exact solutions obtained by using our mechanization method via Maple, we will develop the algebraic methods (Fan and Dai 2003; Yan 2004) for constructing the traveling wave solutions and present a new generalized algebraic method and its algorithms (Ren 2007a; Ren et al. 2007, 2008a).

Remark 9 By the above description, we find that our method is more general than the method in Fan and Dai (2003) and Yan (2004). We have improved the method (Fan and Dai 2003; Yan 2004) in five

aspects: First, we extend the ODE with four degrees (131) into the ODE with any degree (188) and get its new general solutions by using our mechanization method via Maple (Ren 2007a; Ren et al. 2007, 2008a). Second, we change the solution of Eq. (185) into a more general solution (189) and get more types of new rational solutions and irrational solutions. Third, we replace the traveling wave transformation (146) in Fan and Dai (2003) and Yan (2004) by a more general transformation (186). Fourth, we suppose the coefficients of the transformation (186) and (189) are undetermined functions, but the coefficients of the transformation (146) in Fan and Dai (2003) and Yan (2004) are all constants. Fifth, we present a more general algebra method than the method given in Fan and Dai (2003) and Yan (2004), which is called the generalized algebra method, to find more types of exact solutions of nonlinear differential equations based upon the solutions of the ODE (188). This can obtain more general solutions of the NPDEs than the number obtained by the method in Fan and Dai (2003) and Yan (2004).

The Generalized Algebra Method to Find New Non-traveling Waves Solutions of the (1 + 1)-Dimensional Generalized Variable-Coefficient KdV Equation

In this section, we will make use of our generalized algebra method and symbolic computation to find new non-traveling waves solutions and traveling waves solutions of the following (1 + 1) – dimensional Generalized Variable – Coefficient KdV equation (Houria 2007).

Propagation of weakly nonlinear long waves in an inhomogeneous waveguide is governed by a variable – coefficient KdV equation of the form (Clarke et al. 2002).

$$u_t(x, t) + 6u(x, t)u_x(x, t) + B(t)u_{xxx}(x, t) = 0. \quad (190)$$

where $u(x, t)$ is the wave amplitude, t the propagation coordinate, x the temporal variable and $B(t)$ is the local dispersion coefficient.

The applicability of the variable-coefficient KdV equation (190) arises in many areas of physics as, for example, for the description of the propagation of gravity-capillary and interfacial-capillary waves, internal waves and Rossby waves (Clarke et al. 2002). In order to study the propagation of weakly nonlinear, weakly dispersive waves in inhomogeneous media, Eq. (190) is rewritten as follows (Houria 2007)

$$u_t(x, t) + 6A(t)u(x, t)u_x(x, t) + B(t)u_{xxx}(x, t) = 0. \quad (191)$$

which now has a variable nonlinearity coefficient $A(t)$. Here, Eq. (191) is called a (1 + 1)-dimensional generalized variable-coefficient KdV (gvcKdV) equation.

In order to find new non-traveling waves solutions and traveling waves solutions of the following (1 + 1)-dimensional gvcKdV equation (191) by using our generalized algebra method and symbolic computation, we first take the following new general transformation, which we first present here

$$u(x, t) = u(\xi), \quad \xi = \alpha(x, t)\beta(t) - r(t), \quad (192)$$

where $\alpha(x, t)$, $\beta(t)$ and $r(t)$ are functions to be determined later.

By using the new variable $\phi = \phi(\xi)$ which is a solution of the following ODE

$$\frac{d\phi(\xi)}{d\xi} = \varepsilon \sqrt{c_0 + c_1\phi(\xi) + c_2\phi^2(\xi) + c_3\phi^3(\xi) + c_4\phi^4(\xi) + c_5\phi^5(\xi) + c_6\phi^6(\xi)}. \quad (193)$$

we expand the solution of Eq. (191) in the form (Ren 2007a; Ren et al. 2007):

$$u = a_0(X) + \sum_{i=1}^n \times (a_i(X)\phi^i(\xi(X)) + b_i(X)\phi^{-i}(\xi(X))) \quad (194)$$

where $X = (x, t)$, $a_0(X)$, $a_i(X)$, $b_i(X)$ ($i = 1, 2, \dots, n$) are all differentiable functions of X to be determined later.

Balancing the highest derivative term with the nonlinear terms in Eq. (191) by using the derivatives with respect to the variable ϕ , we can determine the parameter $n = 2$ in Eq. (194). In addition, we take $a_0(X) = a_0(t)$, $a_i(X) = a_i(t)$, $b_i(X) = b_i(t)$, $i = 1, 2$ in Eq. (194) for simplicity, then substituting them and Eq. (192) into (194) along with $n = 2$, leads to:

$$u(x, t) = a_0(t) + a_1(t)\phi(\xi) + a_2(t)(\phi(\xi))^2 + \frac{b_1(t)}{\phi(\xi)} + \frac{b_2(t)}{(\phi(\xi))^2}. \quad (195)$$

where $\xi = \alpha(x, t)\beta(t) - r(t)$, and $\alpha(x, t)$, $\beta(t)$, $r(t)$, $a_0(t)$, $a_i(t)$ and $b_i(t)$, $i = 1, 2$ are all differentiable functions of x or t to be determined later.

By substituting Eq. (195) into the given Eq. (191) along with (193) and the derivatives of ϕ , and collecting coefficients of polynomials of ϕ^k and $\phi^\tau \sqrt{\sum_{j=1}^6 c_j \phi^j(\xi)}$, with the aid of Maple, then setting each coefficient to zero, we will get a system of over-determined partial differential equations with respect to $A(t)$, $B(t)$, $\alpha(x, t)$, $\beta(t)$, $r(t)$, $a_0(t)$, $a_i(t)$ and $b_i(t)$, $i = 1, 2$ as follows:

$$\begin{aligned}
& 21B(t)(\beta(t))^3 - \varepsilon a_1(t) \left(\frac{\partial}{\partial x} \alpha(x, t) \right)^3 c_4 c_6 + 18A(t) \\
& - \varepsilon \left(\frac{\partial}{\partial x} \alpha(x, t) \right) \beta(t) a_1(t) a_2(t) c_6 - 3B(t)(\beta(t))^3 \\
& - \varepsilon b_1(t) \left(\frac{\partial}{\partial x} \alpha(x, t) \right)^3 c_6^2 = 0, \\
& 12b_2(t) - \varepsilon \left(\frac{d}{dt} r(t) \right) c_2 - 8B(t)(\beta(t))^3 \\
& - \varepsilon b_2(t) \left(\frac{\partial}{\partial x} \alpha(x, t) \right)^3 c_2^2 - 2b_2(t) \\
& - \varepsilon \left(\frac{\partial}{\partial t} \alpha(x, t) \right) \beta(t) c_2 - 2B(t)\beta(t) \\
& - \varepsilon b_2(t) \left(\frac{\partial^3}{\partial x^3} \alpha(x, t) \right) c_2 - 2b_2(t) \\
& - \varepsilon \alpha(x, t) \left(\frac{d}{dt} \beta(t) \right) c_2 - 6A(t) \\
& - \varepsilon \left(\frac{\partial}{\partial x} \alpha(x, t) \right) \beta(t) (b_1(t))^2 c_2 \\
& - 12A(t) - \varepsilon \left(\frac{\partial}{\partial x} \alpha(x, t) \right) \beta(t) a_0(t) b_2(t) c_2 - 12A(t) \\
& - \varepsilon \left(\frac{\partial}{\partial x} \alpha(x, t) \right) \beta(t) (b_2(t))^2 c_4 = 0, \\
& b_1(t) - \varepsilon \left(\frac{d}{dt} r(t) \right) c_6 - a_1(t) - \varepsilon \left(\frac{d}{dt} r(t) \right) c_4 \\
& + 6A(t) - \varepsilon \left(\frac{\partial}{\partial x} \alpha(x, t) \right) \beta(t) a_2(t) b_1(t) c_4 \\
& + a_1(t) - \varepsilon \alpha(x, t) \left(\frac{d}{dt} \beta(t) \right) c_4 - b_1(t) \\
& - \varepsilon \alpha(x, t) \left(\frac{d}{dt} \beta(t) \right) c_6 + 6A(t) \\
& - \varepsilon \left(\frac{\partial}{\partial x} \alpha(x, t) \right) \beta(t) a_0(t) a_1(t) c_4 - 6A(t) \\
& - \varepsilon \left(\frac{\partial}{\partial x} \alpha(x, t) \right) \beta(t) a_0(t) b_1(t) c_6 \\
& - 4B(t)(\beta(t))^3 - \varepsilon b_1(t) \left(\frac{\partial}{\partial x} \alpha(x, t) \right)^3 c_2 c_6 \\
& + B(t)\beta(t) - \varepsilon a_1(t) \left(\frac{\partial^3}{\partial x^3} \alpha(x, t) \right) c_4 \\
& + a_1(t) - \varepsilon \left(\frac{\partial}{\partial t} \alpha(x, t) \right) \beta(t) c_4 - b_1(t) \\
& - \varepsilon \left(\frac{\partial}{\partial t} \alpha(x, t) \right) \beta(t) c_6 - 6A(t) \\
& - \varepsilon \left(\frac{\partial}{\partial x} \alpha(x, t) \right) \beta(t) a_1(t) b_2(t) c_6 + 7B(t)(\beta(t))^3 \\
& - \varepsilon a_1(t) \left(\frac{\partial}{\partial x} \alpha(x, t) \right)^3 c_2 c_4 + 18A(t) \\
& - \varepsilon \left(\frac{\partial}{\partial x} \alpha(x, t) \right) \beta(t) a_1(t) a_2(t) c_2 - B(t)\beta(t) \\
& - \varepsilon b_1(t) \left(\frac{\partial^3}{\partial x^3} \alpha(x, t) \right) c_6 = 0,
\end{aligned}$$

(196)

because there are so many over-determined partial differential equations, only a few of them are shown here for convenience. Solving the over-determined partial differential equations with Maple, we have the following solutions.

Case 1

$$\begin{aligned}
A(t) &= A(t), \quad B(t) = B(t), \quad \alpha(x, t) = F_1(t), \\
\beta(t) &= \beta(t), \quad a_2(t) = C_1, \quad a_1(t) = C_2, \\
b_2(t) &= C_3, a_0(t) = C_5, \\
r(t) &= \int F_1(t) \frac{d}{dt} \beta(t) + \left(\frac{d}{dt} F_1(t) \right) \beta(t) dt + C_6, \\
b_1(t) &= C_4,
\end{aligned} \tag{197}$$

where $A(t)$, $B(t)$, $\beta(t)$, $F_1(t)$ are arbitrary functions of t , and C_i , $i = 1, 2, 3, 4, 5$ are arbitrary constants.

Case 2

$$\begin{aligned}
b_1(t) &= 0, \quad A(t) = A(t), \quad B(t) = B(t), \\
\alpha(x, t) &= F_1(t), \quad \beta(t) = \beta(t), \quad a_2(t) = C_1, \\
a_0(t) &= C_4, \quad a_1(t) = C_2, \\
r(t) &= \int F_1(t) \frac{d}{dt} \beta(t) + \left(\frac{d}{dt} F_1(t) \right) \beta(t) dt + C_5, \\
b_2(t) &= C_3,
\end{aligned} \tag{198}$$

where $A(t)$, $B(t)$, $\beta(t)$, $F_1(t)$ are arbitrary functions of t , and C_i , $i = 1, 2, 3, 4$ are arbitrary constants.

Case 3

$$\begin{aligned}
\beta(t) &= 0, \quad A(t) = A(t), \quad B(t) = B(t), \\
a_2(t) &= C_1, \quad a_1(t) = C_2, \quad b_2(t) = C_3, \\
a_0(t) &= C_5, \quad b_1(t) = C_4, \quad r(t) = C_6, \\
\alpha(x, t) &= \alpha(x, t),
\end{aligned} \tag{199}$$

where $\alpha(x, t)$ are arbitrary functions of x and t , $A(t)$, $B(t)$ are arbitrary functions of t , and C_i , $i = 1, 2, 3, 4, 5$ are arbitrary constants.

Case 4

$$\begin{aligned}
b_2(t) &= 0, \quad A(t) = A(t), \quad B(t) = B(t), \\
\alpha(x, t) &= F_1(t), \quad \beta(t) = \beta(t), \quad a_2(t) = C_1, \\
a_0(t) &= C_4, a_1(t) = C_2, \\
r(t) &= \int F_1(t) \frac{d}{dt} \beta(t) + \left(\frac{d}{dt} F_1(t) \right) \beta(t) dt + C_5, \\
b_1(t) &= C_3,
\end{aligned} \tag{200}$$

where $A(t), B(t), F_1(t), \beta(t)$ are arbitrary functions of t , and $C_i, i = 1, 2, 3, 4, 5$ are arbitrary constants.

Case 5

$$\begin{aligned}
b_2(t) &= 0, \quad \beta(t) = 0, \quad A(t) = A(t), \\
B(t) &= B(t), \quad a_2(t) = C_1, \quad a_0(t) = C_4, \\
a_1(t) &= C_2, \quad r(t) = C_5, \quad b_1(t) = C_3, \\
\alpha(x, t) &= \alpha(x, t),
\end{aligned} \tag{201}$$

where $\alpha(x, t)$ are arbitrary functions of x and t , $A(t), B(t)$ are arbitrary functions of t , and $C_i, i = 1, 2, 3, 4, 5$ are arbitrary constants.

Case 6

$$\begin{aligned}
b_1(t) &= \frac{C_2 C_4}{C_6}, \quad \beta(t) = 0, \quad b_2(t) = 0, \\
A(t) &= A(t), \quad B(t) = B(t), \quad a_2(t) = C_1, \\
a_1(t) &= C_2, \quad r(t) = C_4, \quad a_0(t) = C_3, \\
\alpha(x, t) &= \alpha(x, t),
\end{aligned} \tag{202}$$

where $\alpha(x, t)$ are arbitrary functions of x and t , $A(t), B(t)$ are arbitrary functions of t , and $C_i, i = 1, 2, 3, 4$ are arbitrary constants.

Case 7

$$\begin{aligned}
B(t) &= B(t), \quad b_1(t) = 0, \quad a_0(t) = C_3, \\
A(t) &= 0, \quad b_2(t) = C_2, \quad a_1(t) = 0, \quad a_2(t) = 0, \\
\alpha(x, t) &= F_1(t)x + F_2(t), \quad \beta(t) = \frac{C_1}{F_1(t)}, \\
&C_1, (F_1(t) \frac{d}{dt} F_2(t) - \left(\frac{d}{dt} F_1(t) \right) \\
r(t) &= \int \frac{F_2(t) + 4C_2 B(t) C_1^2 (F_1(t))^2}{(F_1(t))^2} dt + C_4,
\end{aligned} \tag{203}$$

where $F_1(t), B(t)$ are arbitrary functions of t , and $C_i, i = 1, 2, 3, 4$ are arbitrary constants.

Case 8

$$\begin{aligned}
B(t) &= B(t), \quad b_1(t) = 0, \quad \alpha(x, t) = F_1(t), \\
\beta(t) &= \beta(t), \quad A(t) = 0, \quad a_1(t) = 0, \\
a_2(t) &= 0, \\
r(t) &= \int F_1(t) \frac{d}{dt} \beta(t) + \left(\frac{d}{dt} F_1(t) \right) \beta(t) dt + C_3, \\
a_0(t) &= C_2, \quad b_2(t) = C_1,
\end{aligned} \tag{204}$$

where $A(t), B(t), \beta(t), F_1(t)$ are arbitrary functions of t , and $C_i, i = 1, 2, 3$ are arbitrary constants.

Case 9

$$\begin{aligned}
A(t) &= A(t), \quad B(t) = B(t), \quad b_1(t) = 0, \\
\alpha(x, t) &= F_1(t), \quad \beta(t) = \beta(t), \\
a_2(t) &= C_1, \quad a_0(t) = C_3, \quad b_2(t) = C_2, \\
a_1(t) &= 0, \\
r(t) &= \int F_1(t) \frac{d}{dt} \beta(t) + \left(\frac{d}{dt} F_1(t) \right) \beta(t) dt + C_4,
\end{aligned} \tag{205}$$

where $A(t), B(t), \beta(t), F_1(t)$ are arbitrary functions of t , and $C_i, i = 1, 2, 3, 4$ are arbitrary constants.

Case 10

$$\begin{aligned}
A(t) &= A(t), \quad B(t) = B(t), \quad b_1(t) = 0, \\
\alpha(x, t) &= F_1(t), \quad \beta(t) = \beta(t), \\
a_2(t) &= C_1, \quad b_2(t) = 0, \quad a_1(t) = 0, \\
r(t) &= \int F_1(t) \frac{d}{dt} \beta(t) + \left(\frac{d}{dt} F_1(t) \right) \beta(t) dt + C_3, \\
a_0(t) &= C_2,
\end{aligned} \tag{206}$$

where $A(t), B(t), \beta(t), F_1(t)$ are arbitrary functions of t , and $C_i, i = 1, 2, 3, 4$ are arbitrary constants.

Case 11

$$\begin{aligned}
A(t) &= A(t), \quad B(t) = B(t), \quad b_1(t) = 0, \\
a_2(t) &= C_1, \quad b_2(t) = 0, \quad a_1(t) = 0, \\
\beta(t) &= 0, \quad r(t) = C_3, \\
\alpha(x, t) &= \alpha(x, t), \quad a_0(t) = C_2,
\end{aligned} \tag{207}$$

where $\alpha(x, t)$ are arbitrary functions of x and t , $A(t), B(t)$ are arbitrary functions of t , and $C_i, i = 1, 2$ are arbitrary constants. Because there are so many solutions, only a few of them are shown here for convenience.

So we get new general forms of solutions of equations (191):

$$\begin{aligned}
u(x, t) &= a_0(t) + a_1(t)\phi(\alpha(x, t)\beta(t) - r(t)) + a_2(t)(\phi(\alpha(x, t)\beta(t) - r(t)))^2 \\
&\quad + \frac{b_1(t)}{\phi(\alpha(x, t)\beta(t) - r(t))} + \frac{b_2(t)}{(\phi(\alpha(x, t)\beta(t) - r(t)))^2}.
\end{aligned} \tag{208}$$

where $\alpha(x, t), \beta(t), r(t), a_0(t), a_i(t), b_i(t), i = 1, 2$ satisfy Eqs. (197), (198), (199), (200), (201), (202), (203), (204), (205), (206), and (207) respectively, and the variable $\phi(\alpha(x, t)\beta(t) - r(t))$ takes the solutions of the Eq. (193). For example, we may take the variable $\phi(\alpha(x, t)\beta(t) - r(t))$ as follows:

Type 1 If $4c_2c_6 - c_4^2 < 0$, corresponding to Eq. (193), $\phi(\alpha(x, t)\beta(t) - r(t))$ is taken, we get the following four tanh-sech hyperbolic type solutions

$$\phi_{1,2}((\alpha(x, t)\beta(t) - r(t))) = \varepsilon \sqrt{\frac{2c_2 \operatorname{sech}(\eta) [c_4 \operatorname{sech}(\eta) + \varepsilon i \sqrt{c_4^2 - 4c_6c_2} \tanh(\eta)]}{4c_6c_2 \tanh^2(\eta) - c_4^2}}, \quad c_2 > 0, \tag{209}$$

$$\phi_{3,4}((\alpha(x, t)\beta(t) - r(t))) = \varepsilon \sqrt{\frac{2c_2 \operatorname{sech}(\eta) [c_4 \operatorname{sech}(\eta) - \varepsilon i \sqrt{c_4^2 - 4c_6c_2} \tanh(\eta)]}{4c_6c_2 \tanh^2(\eta) - c_4^2}}, \quad c_2 > 0, \tag{210}$$

and the following tan-sec triangular type solutions

$$\phi_{5,6}((\alpha(x, t)\beta(t) - r(t))) = \varepsilon \sqrt{\frac{2c_2 \sec(\zeta) [-c_4 \sec(\zeta) + \varepsilon \sqrt{c_4^2 - 4c_6c_2} \tan(\zeta)]}{4c_6c_2 \tan^2(\zeta) + c_4^2}}, \quad c_2 < 0, \tag{211}$$

$$\phi_{7,8}((\alpha(x, t)\beta(t) - r(t))) = \varepsilon \sqrt{\frac{2c_2 \sec(\zeta) [-c_4 \sec(\zeta) - \varepsilon \sqrt{c_4^2 - 4c_6c_2} \tan(\zeta)]}{4c_6c_2 \tan^2(\zeta) + c_4^2}}, \quad c_2 < 0, \tag{212}$$

where $\eta = 2\sqrt{c_2}((\alpha(x, t)\beta(t) - r(t)) - C_1)$ and C_1 is any constant, $\zeta = 2\sqrt{-c_2} \times ((\alpha(x, t)\beta(t) - r(t)) - C_1)$ and C_1 is any constant.

Type 2 If $4c_2c_6 - c_4^2 > 0$, corresponding to Eq. (193), $\phi(\alpha(x, t)\beta(t) - r(t))$ is taken, we get the following four sinh-cosh hyperbolic type solutions

$$\begin{aligned} \phi_{9,10}((\alpha(x, t)\beta(t) - r(t))) \\ = \varepsilon \sqrt{\frac{2c_2 [c_4 + \varepsilon \sqrt{4c_2c_6 - c_4^2} \sinh(\eta)]}{4c_2c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)}}, \\ c_2 > 0, \end{aligned} \quad (213)$$

$$\begin{aligned} \phi_{11,12}((\alpha(x, t)\beta(t) - r(t))) \\ = \varepsilon \sqrt{\frac{2c_2 [c_4 - \varepsilon \sqrt{4c_2c_6 - c_4^2} \sinh(\eta)]}{4c_2c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)}}, \\ c_2 > 0, \end{aligned} \quad (214)$$

$$\begin{aligned} \phi_{13,14}((\alpha(x, t)\beta(t) - r(t))) \\ = \varepsilon \sqrt{\frac{2c_2 [-c_4 + \varepsilon i \sqrt{4c_2c_6 - c_4^2} \sin(\zeta)]}{4c_2c_6 \sin^2(\zeta) + c_4^2 \cos^2(\zeta)}}, \\ c_2 < 0 \end{aligned} \quad (215)$$

$$\begin{aligned} \phi_{15,16}((\alpha(x, t)\beta(t) - r(t))) \\ = \varepsilon \sqrt{\frac{2c_2 [-c_4 - \varepsilon i \sqrt{4c_2c_6 - c_4^2} \sin(\zeta)]}{4c_2c_6 \sin^2(\zeta) + c_4^2 \cos^2(\zeta)}}, \\ c_2 < 0, \end{aligned} \quad (216)$$

where $\eta = 2\sqrt{c_2}((\alpha(x, t)\beta(t) - r(t)) - C_1)$, $\zeta = 2\sqrt{-c_2}((\alpha(x, t)\beta(t) - r(t)) - C_1)$, and C_1 is any constant.

Type 3 If $4c_2c_6 - c_4^2 = 0$, corresponding to Eq. (193), $\phi(\alpha(x, t)\beta(t) - r(t))$ is taken, we get the following two sech-tanh hyperbolic type solutions $4c_2c_6 - c_4^2 = 0$, Eq. (170) admits the following tanh hyperbolic type solutions

and the following sin-cos triangular type solutions

$$\phi_{17,18,19,20}(\alpha(x, t)\beta(t) - r(t)) = \varepsilon \sqrt{\frac{2c_2 \{1 + \varepsilon \tanh[\sqrt{c_2}(\alpha(x, t)\beta(t) - r(t) - C_1)]\}}{1 - c_4 \varepsilon (1 + c_4) \tanh[\sqrt{c_2}(\alpha(x, t)\beta(t) - r(t) - C_1)]}}, \quad c_2 > 0, \quad (217)$$

and the following tan triangular type solutions

$$\begin{aligned} \phi_{21,22,23,24}(\alpha(x, t)\beta(t) - r(t)) = \varepsilon \sqrt{\frac{2c_2 \{1 + \varepsilon \tan[i\sqrt{-c_2}(\alpha(x, t)\beta(t) - r(t) - C_1)]\}}{1 - c_4}} \\ \sqrt{-\varepsilon(1 + c_4) \tan[i\sqrt{-c_2}(\alpha(x, t)\beta(t) - r(t) - C_1)]} \end{aligned} \quad c_2 < 0, \quad (218)$$

where C_1 is any constant.

Substituting Eqs. (209), (210), (211), (212), (213), (214), (215), (216), (217), (218) and (197), (198), (199), (200), (201), (202), (203), (204), (205), (206), (207) into (208), respectively, we get many new irrational solutions and rational solutions of combined hyperbolic type or triangular type solutions of Eq. (191).

For example, when we select $A(t)$, $B(t)$, $\alpha(x, t)$, $\beta(t)$, $r(t)$, $a_0(t)$, $a_i(t)$ and $b_i(t)$, $i = 1, 2$ to satisfy Case 1, we can easily get the following irrational solutions of combined hyperbolic type or triangular type solutions of Eq. (191):

$$\begin{aligned} u(x, t) = & C_5 + C_2 \phi(F_1(t)\beta(t) - r(t)) \\ & + C_1 (\phi(F_1(t)\beta(t) - r(t)))^2 \\ & + \frac{C_4}{\phi(F_1(t)\beta(t) - r(t))} \\ & + \frac{C_3}{(\phi(F_1(t)\beta(t) - r(t)))^2}, \quad (219) \end{aligned}$$

where $A(t)$, $B(t)$, $\beta(t)$, $F_1(t)$ are arbitrary functions of t , C_i , $i = 1, 2, 3, 4, 5$ are arbitrary constants, and

$$r(t) = \int F_1(t) \frac{d}{dt} \beta(t) + \left(\frac{d}{dt} F_1(t) \right) \beta(t) dt + C_6.$$

Substituting Eqs. (209), (210), (211), (212), (213), (214), (215), (216), (217), and (218) into (219), respectively, we get the following new irrational solutions and rational solutions of combined hyperbolic type or triangular type solutions of Eq. (191).

Case 1

If $4c_2c_6 - c_4^2 < 0$, $c_2 > 0$, corresponding to Eqs. (209), (210) and (197), Eq. (191) admits the following four tanh-sech hyperbolic type solutions

$$\begin{aligned} u_{1,2}(x, t) = & C_5 + \frac{C_4}{\varepsilon \sqrt{\frac{2c_2 \operatorname{sech}(\eta) [c_4 \operatorname{sech}(\eta) + \varepsilon i \sqrt{c_4^2 - 4c_6c_2} \tanh(\eta)]}{4c_6c_2 \tanh^2(\eta) - c_4^2}}} \\ & + \frac{C_3 (4c_6c_2 \tanh^2(\eta) - c_4^2)}{2c_2 \operatorname{sech}(\eta) [c_4 \operatorname{sech}(\eta) + \varepsilon i \sqrt{c_4^2 - 4c_6c_2} \tanh(\eta)]} \\ & + C_2 \varepsilon \sqrt{\frac{2c_2 \operatorname{sech}(\eta) [c_4 \operatorname{sech}(\eta) + \varepsilon i \sqrt{c_4^2 - 4c_6c_2} \tanh(\eta)]}{4c_6c_2 \tanh^2(\eta) - c_4^2}} \\ & + \frac{2c_2 C_1 \operatorname{sech}(\eta) [c_4 \operatorname{sech}(\eta) + \varepsilon i \sqrt{c_4^2 - 4c_6c_2} \tanh(\eta)]}{4c_6c_2 \tanh^2(\eta) - c_4^2}, \quad (220) \end{aligned}$$

and

$$\begin{aligned}
 u_{3,4}(x, t) = & C_5 + \frac{C_3(4c_6c_2 \tanh^2(\eta) - c_4^2)}{2c_2 \operatorname{sech}(\eta) [c_4 \operatorname{sech}(\eta) - \varepsilon i \sqrt{c_4^2 - 4c_6c_2} \tanh(\eta)]} \\
 & + \frac{C_4}{\varepsilon \sqrt{\frac{2c_2 \operatorname{sech}(\eta) [c_4 \operatorname{sech}(\eta) - \varepsilon i \sqrt{c_4^2 - 4c_6c_2} \tanh(\eta)]}{4c_6c_2 \tanh^2(\eta) - c_4^2}}} \\
 & + C_2 \varepsilon \sqrt{\frac{2c_2 \operatorname{sech}(\eta) [c_4 \operatorname{sech}(\eta) - \varepsilon i \sqrt{c_4^2 - 4c_6c_2} \tanh(\eta)]}{4c_6c_2 \tanh^2(\eta) - c_4^2}} \\
 & + \frac{2c_2 C_1 \operatorname{sech}(\eta) [c_4 \operatorname{sech}(\eta) - \varepsilon i \sqrt{c_4^2 - 4c_6c_2} \tanh(\eta)]}{4c_6c_2 \tanh^2(\eta) - c_4^2},
 \end{aligned} \tag{221}$$

where $\eta = 2\sqrt{c_2}((\alpha(x, t)\beta(t) - r(t)) - C_1)$, $r(t) = \int F_1(t) \frac{d}{dt} \beta(t) + \left(\frac{d}{dt} F_1(t) \right) \beta(t) dt + C_6$.
 $A(t)$, $B(t)$, $\beta(t)$, $F_1(t)$ are arbitrary functions of t ,
 C_i , $i = 1, 2, 3, 4, 5$ are arbitrary constants, and

Case 2

If $4c_2c_6 - c_4^2 < 0$, $c_2 < 0$, corresponding to Eqs. (211), (212) and (197), Eq. (191) admits the following four tan-sec triangular type solutions

$$\begin{aligned}
 u_{5,6}(x, t) = & C_5 + \frac{C_4}{\varepsilon \sqrt{\frac{2c_2 \sec(\zeta) [-c_4 \sec(\zeta) + \varepsilon \sqrt{c_4^2 - 4c_6c_2} \tan(\zeta)]}{4c_6c_2 \tan^2(\zeta) + c_4^2}}} \\
 & + \frac{C_3(4c_6c_2 \tan^2(\zeta) + c_4^2)}{2c_2 \sec(\zeta) [-c_4 \sec(\zeta) + \varepsilon \sqrt{c_4^2 - 4c_6c_2} \tan(\zeta)]} \\
 & + C_2 \varepsilon \sqrt{\frac{2c_2 \sec(\zeta) [-c_4 \sec(\zeta) + \varepsilon \sqrt{c_4^2 - 4c_6c_2} \tan(\zeta)]}{4c_6c_2 \tan^2(\zeta) + c_4^2}} \\
 & + \frac{2c_2 C_1 \sec(\zeta) [-c_4 \sec(\zeta) + \varepsilon \sqrt{c_4^2 - 4c_6c_2} \tan(\zeta)]}{4c_6c_2 \tan^2(\zeta) + c_4^2},
 \end{aligned} \tag{222}$$

and

$$\begin{aligned}
 u_{7,8}(x, t) = C_5 + & \frac{C_4}{\varepsilon \sqrt{\frac{2c_2 \sec(\zeta) [-c_4 \sec(\zeta) - \varepsilon \sqrt{c_4^2 - 4c_6c_2 \tan(\zeta)}]}{4c_6c_2 \tan^2(\zeta) + c_4^2}}} \\
 & + \frac{C_3(4c_6c_2 \tan^2(\zeta) + c_4^2)}{2c_2 \sec(\zeta) [-c_4 \sec(\zeta) - \varepsilon \sqrt{c_4^2 - 4c_6c_2 \tan(\zeta)}]} \\
 & + C_2 \varepsilon \sqrt{\frac{2c_2 \sec(\zeta) [-c_4 \sec(\zeta) - \varepsilon \sqrt{c_4^2 - 4c_6c_2 \tan(\zeta)}]}{4c_6c_2 \tan^2(\zeta) + c_4^2}} \\
 & + \frac{2c_2 C_1 \sec(\zeta) [-c_4 \sec(\zeta) - \varepsilon \sqrt{c_4^2 - 4c_6c_2 \tan(\zeta)}]}{4c_6c_2 \tan^2(\zeta) + c_4^2},
 \end{aligned} \tag{223}$$

where $\zeta = 2\sqrt{-c_2}((\alpha(x, t)\beta(t) - r(t)) - C_1)$
 and the rest of the parameters are the same as
 with Case 1.

Case 3

If $4c_2c_6 - c_4^2 > 0$, $c_2 > 0$, corresponding to
 Eqs. (213), (214) and (197), Eq. (191) admits the
 following four sinh-cosh hyperbolic type solutions

$$\begin{aligned}
 u_{9,10}(x, t) = C_5 + C_2 \varepsilon \sqrt{\frac{2c_2 [c_4 + \varepsilon \sqrt{4c_2c_6 - c_4^2} \sinh(\eta)]}{4c_2c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)}} \\
 + \frac{2c_2 C_1 [c_4 + \varepsilon \sqrt{4c_2c_6 - c_4^2} \sinh(\eta)]}{4c_2c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)} \\
 + \frac{C_4}{\varepsilon \sqrt{\frac{2c_2 [c_4 + \varepsilon \sqrt{4c_2c_6 - c_4^2} \sinh(\eta)]}{4c_2c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)}}} \\
 + \frac{2c_2 C_3 [c_4 + \varepsilon \sqrt{4c_2c_6 - c_4^2} \sinh(\eta)]}{4c_2c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)},
 \end{aligned} \tag{224}$$

and

$$\begin{aligned}
 u_{11,12}(x, t) = & C_5 + C_2 \varepsilon \sqrt{\frac{2c_2 [c_4 + \varepsilon \sqrt{4c_2 c_6 - c_4^2} \sinh(\eta)]}{4c_2 c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)}} \\
 & + \frac{2c_2 C_1 [c_4 + \varepsilon \sqrt{4c_2 c_6 - c_4^2} \sinh(\eta)]}{4c_2 c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)} \\
 & + \frac{C_4}{\varepsilon \sqrt{\frac{2c_2 [c_4 + \varepsilon \sqrt{4c_2 c_6 - c_4^2} \sinh(\eta)]}{4c_2 c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)}}} \\
 & + \frac{2c_2 C_3 [c_4 + \varepsilon \sqrt{4c_2 c_6 - c_4^2} \sinh(\eta)]}{4c_2 c_6 \sinh^2(\eta) - c_4^2 \cosh^2(\eta)},
 \end{aligned} \tag{225}$$

where $\eta = 2\sqrt{c_2}((\alpha(x, t)\beta(t) - r(t)) - C_1)$ and the rest of the parameters are the same as with Case 1.

Case 4

If $4c_2 c_6 - c_4^2 > 0$, $c_2 < 0$, corresponding to Eqs. (215), (216) and (197), Eq. (191) admits the following four sin-cos triangular type solutions

$$\begin{aligned}
 u_{13,14}(x, t) = & C_5 + \frac{C_4}{\varepsilon \sqrt{\frac{2c_2 [-c_4 + \varepsilon i \sqrt{4c_2 c_6 - c_4^2} \sin(\zeta)]}{4c_6 c_2 \sin^2(\zeta) + c_4^2 \cos^2(\zeta)}}} \\
 & + \frac{C_3 (4c_6 c_2 \sin^2(\zeta) + c_4^2 \cos^2(\zeta))}{2c_2 [-c_4 + \varepsilon i \sqrt{4c_2 c_6 - c_4^2} \sin(\zeta)]} \\
 & + C_2 \varepsilon \sqrt{\frac{2c_2 [-c_4 + \varepsilon i \sqrt{4c_2 c_6 - c_4^2} \sin(\zeta)]}{4c_6 c_2 \sin^2(\zeta) + c_4^2 \cos^2(\zeta)}} \\
 & + \frac{2c_2 C_1 [-c_4 + \varepsilon i \sqrt{4c_2 c_6 - c_4^2} \sin(\zeta)]}{4c_6 c_2 \sin^2(\zeta) + c_4^2 \cos^2(\zeta)},
 \end{aligned} \tag{226}$$

and

$$\begin{aligned}
 u_{15,16}(x, t) = & C_5 + \frac{C_4}{\varepsilon \sqrt{\frac{2c_2 \left[-c_4 - \varepsilon i \sqrt{4c_2 c_6 - c_4^2} \sin(\zeta) \right]}{4c_6 c_2 \sin^2(\zeta) + c_4^2 \cos^2(\zeta)}}} \\
 & + \frac{C_3 (4c_6 c_2 \sin^2(\zeta) + c_4^2 \cos^2(\zeta))}{2c_2 \left[-c_4 - \varepsilon i \sqrt{4c_2 c_6 - c_4^2} \sin(\zeta) \right]} \\
 & + C_2 \varepsilon \sqrt{\frac{2c_2 \left[-c_4 - \varepsilon i \sqrt{4c_2 c_6 - c_4^2} \sin(\zeta) \right]}{4c_6 c_2 \sin^2(\zeta) + c_4^2 \cos^2(\zeta)}} \\
 & + \frac{2c_2 C_1 \left[-c_4 - \varepsilon i \sqrt{4c_2 c_6 - c_4^2} \sin(\zeta) \right]}{4c_6 c_2 \sin^2(\zeta) + c_4^2 \cos^2(\zeta)},
 \end{aligned} \tag{227}$$

where $\zeta = 2\sqrt{-c_2}((\alpha(x, t)\beta(t) - r(t)) - C_1)$ and the rest of the parameters are the same as with Case 1.

Case 5

If $4c_2 c_6 - c_4^2 = 0$, $c_2 > 0$, corresponding to Eqs. (217) and (197), Eq. (191) admits the following four tanh hyperbolic type solutions

$$\begin{aligned}
 u_{17,18,19,20}(x, t) = & C_5 + C_2 - \varepsilon \sqrt{\frac{2c_2 \left[1 + \varepsilon \tanh\left(\frac{\eta}{2}\right) \right]}{1 - c_4 - \varepsilon(1 + c_4) \tanh\left(\frac{\eta}{2}\right)}} \\
 & + \frac{2c_2 C_1 \left[1 + \varepsilon \tanh\left(\frac{\eta}{2}\right) \right]}{1 - c_4 - \varepsilon(1 + c_4) \tanh\left(\frac{\eta}{2}\right)} \\
 & + \frac{C_4}{- \varepsilon \sqrt{\frac{2c_2 \left[1 + \varepsilon \tanh\left(\frac{\eta}{2}\right) \right]}{1 - c_4 - \varepsilon(1 + c_4) \tanh\left(\frac{\eta}{2}\right)}}} \\
 & + \frac{C_3 \left[1 - c_4 - \varepsilon(1 + c_4) \tanh\left(\frac{\eta}{2}\right) \right]}{2c_2 \left[1 + \varepsilon \tanh\left(\frac{\eta}{2}\right) \right]},
 \end{aligned} \tag{228}$$

where $\eta = 2\sqrt{c_2}((\alpha(x, t)\beta(t) - r(t)) - C_1)$ and the rest of the parameters are the same as with Case 1.

Case 6

If $4c_2 c_6 - c_4^2 = 0$, $c_2 < 0$, corresponding to Eqs. (218) and (197), Eq. (191) admits the following two sec-tan triangular type solutions

$$\begin{aligned}
u_{21,22,23,24}(x, t) = & C_5 + C_2 - \varepsilon \sqrt{\frac{2c_2 \left[1 + \varepsilon \tan \left(\frac{i\zeta}{2} \right) \right]}{1 - c_4 - \varepsilon(1 + c_4) \tan \left(\frac{i\zeta}{2} \right)}} \\
& - \frac{2c_2 C_1 \left[1 + \varepsilon \tan \left(\frac{i\zeta}{2} \right) \right]}{1 - c_4 - \varepsilon(1 + c_4) \tan \left(\frac{i\zeta}{2} \right)} \\
& + \frac{C_4}{- \varepsilon \sqrt{\frac{2c_2 \left[1 + \varepsilon \tan \left(\frac{i\zeta}{2} \right) \right]}{1 - c_4 - \varepsilon(1 + c_4) \tan \left(\frac{i\zeta}{2} \right)}}} \\
& - \frac{C_3 \left(1 - c_4 - \varepsilon(1 + c_4) \tan \left(\frac{i\zeta}{2} \right) \right)}{2c_2 \left[1 + \varepsilon \tan \left(\frac{i\zeta}{2} \right) \right]},
\end{aligned} \tag{229}$$

where $\zeta = 2\sqrt{-c_2}(\alpha(x, t)\beta(t) - r(t)) - C_1$ and the rest of the parameters are the same as with Case 1.

Remark 10 We may further generalize Eq. (194) as follows:

$$\begin{aligned}
u = a_0(X) + \sum_{i=0}^m \frac{\sum_{r_{i1}+\dots+r_{mi}=i} a_{r_{i1},\dots,r_{mi}}(X) G_{1i}^{r_{i1}}(\xi_{1i}(X)) \dots G_{mi}^{r_{mi}}(\xi_{mi}(X))}{\left[\sum_{l_{i1}+\dots+l_{mi}=i} b_{l_{i1},\dots,l_{mi}}(X) G_{1i}^{l_{i1}}(\xi_{1i}(X)) \dots G_{mi}^{l_{mi}}(\xi_{mi}(X)) + c_i(X)^t \right]}
\end{aligned} \tag{230}$$

where $\sum_{l_{i1}+\dots+l_{mi}=i} b_{l_{i1},\dots,l_{mi}}^2(X) + c_i^2(X) \neq 0$, $a_0(X)$, $\xi_{1i}(X)$, $\dots$, $\xi_{mi}(X)$, $a_{r_{i1},\dots,r_{mi}}(X)$, $b_{l_{i1},\dots,l_{mi}}(X)$ and $c_i(X)$, $i = 0, 1, 2, \dots, m$ are all differentiable functions to be determined later, t is a constant, $G_{1i}(\xi_{1i}(X))$, $\dots$, $G_{mi}(\xi_{mi}(X))$ are all $\phi(\xi_{ki}(X))$ or $\phi^{-1}(\xi_{ki}(X))$ or some derivatives $\phi^{(j)}(\xi_{ki}(X))$, $k = 1, 2, \dots, n$; $i = 0, 1, 2, \dots, m$; $j = \pm 1, \pm 2, \dots$. We can get many new explicit solutions of Eq. (191).

A New Exp-N Solitary-Like Method and Its Application in the (1 + 1)-Dimensional Generalized KdV Equation

In this section, in order to develop the Exp-function method (He and Wu 2006), we present two new generic transformations, a new Exp-N solitary-like method, and its algorithm (Ren 2007a; Ren et al. 2008c). In addition, we apply our method to construct new exact solutions of the (1 + 1)-dimensional classical generalized KdV(gKdV) equation.

Summary of the Exp-N Solitary-Like Method

In the following we would like to outline the main steps of our Exp-N solitary-like method (Ren 2007a; Ren et al. 2008c):

Step 1

For a given NLEE system with some physical fields $u_m(t, x_1, x_2, \dots, x_{n-1})$, ($m = 1, 2, \dots, n$) in n independent variables $t, x_1, x_2, \dots, x_{n-1}$,

$$F_m(u_1, u_2, \dots, u_n, u_{1,t}, u_{2,t}, \dots, u_{n,t}, \\ u_{1,x_1}, u_{2,x_1}, \dots, u_{n,x_1}, u_{1,tt}, u_{2,tt}, \dots, u_{n,tt}, \dots) = 0, \\ (m = 1, 2, \dots, n). \quad (231)$$

We introduce a new generic transformation

$$\begin{cases} u_m(t, x_1, \dots, x_{n-1}) = u_m(\xi_1, \xi_2) & (m = 1, 2, \dots, n), \\ \xi_j = p_{0j}(t) + \\ \sum_{i=1}^{n-1} p_{ij}(t, x_1, x_2, \dots, x_{i-1}) \cdot q_{ij}(x_i), & j = 1, 2, \end{cases} \quad (232)$$

where we let $p_{1j}(t, x_0) = p_{1j}(t)$, here $p_{ij}(t, x_1, x_2, \dots, x_{i-1})$, $q_{ij}(x_i)$, $j = 1, 2$; $i = 1, 2, \dots, n-1$ and $p_{0j}(t)$ are functions to be determined later. Then the nonlinear partial differentials of Eq. (231) are reduced to a nonlinear partial differential equation with respect to ξ_j , $j = 1, 2$, $p_{ij}(t, x_1, x_2, \dots, x_{i-1})$, $q_{ij}(x_i)$, $j = 1, 2$; $i = 1, 2, \dots, n-1$ and $p_{0j}(t)$.

Step 2

Let $X = (t, x_1, \dots, x_{n-1})$. Then we introduce a new transformation in terms of Exp-function rational formal expansion in the following form based on the idea of the Exp-function method (Ren and Zhang 2006b) by the new generic transformation (232).

$$u_m(\xi_1, \xi_2) = g_m(X) + \frac{\sum_{j=-c}^d a_{mj}(X) \exp(j\xi_1)}{\sum_{k=-p}^q b_{mk}(X) \exp(k\xi_2)}, \\ (m = 1, 2, \dots, n), \quad (233)$$

where c, d, p and q are any positive integers, ξ_1, ξ_2 , $g_m(X)$, $a_{mj}(X)$, $j = -c, \dots, d$, and $b_{mk}(X)$, $k = -p, \dots, q$; $m = 1, 2, \dots, n$ are functions to be determined later.

Step 3

We take a numerical value of c, d, p and q , substitute Eq. (233) into the partial differential equations obtained in Step 1, and then set all coefficients of $\exp(k\eta)$, $k = 0, 1, 2, \dots$ to zero to get an over-determined partial differential equation with respect to $g_m(X)$, $a_{mj}(X)$, $j = -c, \dots, d$, $b_{mk}(X)$, $k = -p, \dots, q$; $m = 1, 2, \dots, n$, $p_{ij}(t, x_1, x_2, \dots, x_{i-1})$, $q_{ij}(x_i)$, $j = 1, 2$; $i = 1, 2, \dots, n-1$ and $p_{0j}(t)$.

Step 4

Solving the over-determined partial differential equations in Step 3 by use of Maple, we would end up with explicit expressions for $g_m(X)$, $a_{mj}(X)$, $j = -c, \dots, d$, $b_{mk}(X)$, $k = -p, \dots, q$; $m = 1, 2, \dots, n$, $p_{ij}(t, x_1, x_2, \dots, x_{i-1})$, $q_{ij}(x_i)$, $j = 1, 2$; $i = 1, 2, \dots, n-1$ and $p_{0j}(t)$.

Step 5

Substituting the results obtained in Step 4 into Eqs. (232) and (233), we can then obtain new rational solutions of Exp-functions as follows

$$u_m(X) = g_m(X) + \frac{\sum_{j=-c}^d a_{mj}(X) \exp \left(j \left(p_{01}(t) + \sum_{i=1}^{n-1} p_{i1}(t, x_1, x_2, \dots, x_{i-1}) \cdot q_{i1}(x_i) \right) \right)}{\sum_{k=-p}^q b_{mk}(X) \exp \left(k \left(p_{02}(t) + \sum_{i=1}^{n-1} p_{i2}(t, x_1, x_2, \dots, x_{i-1}) \cdot q_{i2}(x_i) \right) \right)}, \quad (m = 1, 2, \dots, n) \quad (234)$$

Step 6

If we again take a numerical value of c , d , p and q and repeat Step 3, Step 4, and Step 5, then we can obtain other rational solutions of Exp-functions Eq. (231). Now, repeat the process. We can obtain a family of new N solitary-like solutions of Eq. (231).

Remark 11 The new transformations (232) and (233) proposed here are more general than the transformation in the Exp-function method (He and Wu 2006). We have obtained many families of non-traveling waves solutions and traveling waves solutions of the $(2 + 1)$ -dimensional generalized KdV–Burgers equation by using the two transformations in Ren (2007a) and Ren et al. (2008c).

Remark 12 Our methods are more powerful, simpler, and more convenient than the Exp-function method (He and Wu 2006). The Exp-function method (He and Wu 2006) is used only to obtain a traveling waves solution of lower dimensional NLPDE because it makes use of the homogeneous balance method. In this paper we improve the method to be more powerful such that it can be used to obtain many families of non-traveling waves solutions and traveling waves solutions of higher dimensional NLPDEs because we don't use

the homogeneous balance method and make use of our transformations (232) and (233).

The Application of the Exp-N Solitary-Like Method in the $(1 + 1)$ -Dimensional Generalized KdV Equation

In this section, we will present a new transformation, and then make use of the transformation and our method via symbolic computation to find the solutions of the $(1 + 1)$ -dimensional classical generalized Korteweg–de Vries (gKdV) equation.

The $(1 + 1)$ -dimensional gKdV equation has the following form (Helal and Mehanna 2007; Hamdia et al. 2005; Sarma 2008):

$$u_t(x, t) + \alpha u^p(x, t) u_x(x, t) + \beta u_{xxx}(x, t) = 0, \quad (235)$$

where the coefficient of the nonlinear term α and the coefficient of the dispersive term β are independent of x and t .

According to the above steps, to seek traveling wave solutions of Eq. (235), we present the following new transformation here. When $p \geq 2$

$$u(x, t) = f^{\frac{2}{p}}(x, t). \quad (236)$$

By substituting Eq. (236) into (235), we get the following equation:

$$\begin{aligned}
& p^2 f^2(x, t) f_t(x, t) + \alpha p^2 f^4(x, t) f_x(x, t) \\
& + 4\beta f_x^3(x, t) + 6p\beta f(x, t) f_x(x, t) f_{xx}(x, t) \\
& - 6p\beta f_x^3(x, t) + \beta p^2 f_{xxx}(x, t) f^2(x, t) \\
& - 3\beta p^2 f(x, t) f_x(x, t) f_{xx}(x, t) + 2\beta p^2 f_x^3(x, t) = 0.
\end{aligned} \quad (237)$$

We introduce a new generic transformation (Ren 2007a; Ren et al. 2008c).

$$f(x, t) = g + \frac{\sum_{i=-m_1}^{m_2} k_i e^{i(\alpha_1 x + \beta_1 t + \gamma_1)}}{\sum_{j=-n_1}^{n_2} h_j e^{j(\alpha_2 x + \beta_2 t + \gamma_2)}}, \quad (238)$$

where m_1, m_2, n_1, n_2 are any positive integers, $g, \alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \gamma_2, k_i, i = -m_1, \dots, m_2$, and $h_j, j = -n_1, \dots, n_2$ are all constants to be determined later.

With the aid of Maple, substituting Eq. (238) into (237) and setting $\xi = \alpha_1 x + \beta_1 t + \gamma_1$, $\eta = \alpha_2 x + \beta_2 t + \gamma_2$, yields the following equation (because the equation is so long, just part of the equation is shown here for simplification)

$$\alpha p^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^4 \alpha_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \sum_{j=-n_1}^{n_2} h_j e^{j\eta}$$

$$-p^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^4 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \beta_2$$

$$\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \beta p^2 g^2$$

$$+ \alpha_1^3 \sum_{i=-m_1}^{m_2} k_i i^3 e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^5 g^2$$

$$- \beta p^2 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2^3$$

$$\sum_{j=-n_1}^{n_2} h_j j^3 e^{j\eta} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^4 g^2$$

$$+ \alpha p^2 g^4 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^5 \alpha_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi}$$

$$+ 6\beta p \alpha_1^3 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^4$$

$$\sum_{i=-m_1}^{m_2} k_i i^2 e^{i\xi} g$$

$$+ 6\beta p \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \alpha_2^3 \sum_{j=-n_1}^{n_2} h_j j e^{j\eta} \sum_{j=-n_1}^{n_2} h_j j^2 e^{j\eta} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 g$$

$$- 12\beta p \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \alpha_2^3 \left(\sum_{j=-n_1}^{n_2} h_j j e^{j\eta} \right)^3$$

$$g \sum_{j=-n_1}^{n_2} h_j e^{j\eta} + 12\beta \alpha_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \sum_{j=-n_1}^{n_2} h_j e^{j\eta} \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \alpha_2^2 \left(\sum_{j=-n_1}^{n_2} h_j j e^{j\eta} \right)^2$$

$$- 12\beta \alpha_1^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2$$

$$\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2 \sum_{j=-n_1}^{n_2} h_j j e^{j\eta} + 4\alpha p^2 g \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3$$

$$\alpha_1 \sum_{i=-m_1}^{m_2} k_i i e^{i\xi} - 4\alpha p^2 g^3 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \alpha_2$$

$$\alpha_1 \sum_{i=-m_1}^{m_2} k_i i e^{i\xi} - 2p^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \beta_2$$

$$\sum_{j=-n_1}^{n_2} h_j j e^{j\eta} g - 3\beta p^2 \alpha_1^3 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^4$$

$$\sum_{i=-m_1}^{m_2} k_i i^2 e^{i\xi} g - 4\alpha p^2 g \sum_{j=-n_1}^{n_2} h_j e^{j\eta} \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^4$$

$$\alpha_2 \sum_{j=-n_1}^{n_2} h_j e^{j\eta} + 2\beta p^2 \alpha_1^3 \sum_{i=-m_1}^{m_2} k_i i^3 e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^4$$

$$\begin{aligned}
& g \sum_{i=-m_1}^{m_2} k_i e^{i\xi} - 6\beta p^2 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2^3 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 g^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \\
& + 3\beta p^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \alpha_2^3 \sum_{j=-n_1}^{n_2} h_j e^{j\eta} \\
& \sum_{j=-n_1}^{n_2} h_j^2 e^{j\eta} \sum_{j=-n_1}^{n_2} h_j e^{j\eta} + 6\beta p \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \alpha_2^3 \sum_{j=-n_1}^{n_2} h_j e^{j\eta} \\
& \sum_{j=-n_1}^{n_2} h_j^2 e^{j\eta} \sum_{j=-n_1}^{n_2} h_j e^{j\eta} + 6\beta p^2 \alpha_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \\
& \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 g^2 - 3\beta p^2 \alpha_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2^2 \sum_{j=-n_1}^{n_2} h_j^2 e^{j\eta} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^4 g^2 \\
& + 6\beta p \alpha_1^3 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \\
& \sum_{i=-m_1}^{m_2} k_i^2 e^{i\xi} \sum_{i=-m_1}^{m_2} k_i e^{i\xi} - 6\beta p \alpha_1^3 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \\
& - 2\beta p^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \alpha_2^3 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 - 6\beta p \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \alpha_2^3 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \\
& + 2\beta p^2 \alpha_1^3 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 - 6\alpha p^2 g^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \\
& \alpha_2 \sum_{j=-n_1}^{n_2} h_j e^{j\eta} + 6\beta p \alpha_1^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \\
& \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2 \sum_{j=-n_1}^{n_2} h_j^2 e^{j\eta} + 9\beta p^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \alpha_2^3 \sum_{j=-n_1}^{n_2} h_j e^{j\eta} \\
& \sum_{j=-n_1}^{n_2} h_j^2 e^{j\eta} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 g - 6\beta p \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \alpha_2 \\
& \sum_{j=-n_1}^{n_2} h_j e^{j\eta} \alpha_1^2 \sum_{i=-m_1}^{m_2} k_i^2 e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \\
& + 6\beta p^2 \alpha_1^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \alpha_1^2 \sum_{i=-m_1}^{m_2} k_i^2 e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \\
& + 6\beta p^2 \alpha_1^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \\
& \alpha_2 \sum_{j=-n_1}^{n_2} h_j e^{j\eta} g - 6\beta p \alpha_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \\
& \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \alpha_2^2 \sum_{j=-n_1}^{n_2} h_j^2 e^{j\eta} + 6\beta p \alpha_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \sum_{j=-n_1}^{n_2} h_j e^{j\eta} \\
& \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \alpha_2^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 - 2\beta p^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \alpha_2^3 \sum_{j=-n_1}^{n_2} h_j^3 e^{j\eta} \\
& \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 g
\end{aligned}$$

$$\begin{aligned}
& +4\alpha p^2 g^3 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \\
& \alpha_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} + 2p^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^4 \beta_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \\
& g \sum_{i=-m_1}^{m_2} k_i e^{i\xi} - 6\beta p \alpha_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \\
& \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2^2 \sum_{j=-n_1}^{n_2} h_{jj}^2 e^{j\eta} g - 6\beta p^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \alpha_2^3 \left(\sum_{j=-n_1}^{n_2} h_{jj} e^{j\eta} \right)^3 g \sum_{j=-n_1}^{n_2} h_j e^{j\eta} \\
& -3\beta p^2 \alpha_1^3 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2^2 \sum_{j=-n_1}^{n_2} h_{jj}^2 e^{j\eta} g + 24\beta p \alpha_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \\
& \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2^2 \left(\sum_{j=-n_1}^{n_2} h_{jj} e^{j\eta} \right)^2 g - 6\beta p \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2 \sum_{j=-n_1}^{n_2} h_{jj} e^{j\eta} \alpha_1^2
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=-m_1}^{m_2} k_i i^2 e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^8 g - 3\beta p^2 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2 \sum_{j=-n_1}^{n_2} h_{jj} e^{j\eta} \alpha_1^2 \\
& \sum_{i=-m_1}^{m_2} k_i i^2 e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 g - 3\beta p^2 \alpha_1^2 \sum_{i=-m_1}^{m_2} k_i i^2 e^{i\xi} \alpha_2 \\
& \sum_{j=-n_1}^{n_2} h_{jj} e^{j\eta} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^4 g^2 - 12\beta p_1^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \\
& \alpha_2 \sum_{j=-n_1}^{n_2} h_{jj} e^{j\eta} g + 6\beta p^2 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2^3 \sum_{j=-n_1}^{n_2} h_{jj} e^{j\eta}
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=-n_1}^{n_2} h_{jj}^2 e^{j\eta} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 g^2 + p^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \beta_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \\
& \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 + \beta p^2 \alpha_1^3 \sum_{i=-m_1}^{m_2} k_i i^3 e^{i\xi} \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \\
& \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^2 - \beta p^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \alpha_2^3 \sum_{j=-n_1}^{n_2} h_{jj}^3 e^{j\eta}
\end{aligned}$$

$$\begin{aligned}
& \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 - 4\beta \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \alpha_2^3 \left(\sum_{j=-n_1}^{n_2} h_{jj} e^{j\eta} \right)^3 \\
& +4\beta \alpha_1^3 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^3 \\
& -p^2 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^3 \\
& \beta_2 \sum_{j=-n_1}^{n_2} h_{jj} e^{j\eta} - \alpha p^2 \left(\sum_{i=-m_1}^{m_2} k_i e^{i\xi} \right)^5 \alpha_2 \sum_{j=-n_1}^{n_2} h_{jj} e^{j\eta} + p^2
\end{aligned}$$

$$\begin{aligned}
& \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^5 \beta_1 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} g^2 \\
& - \alpha p^2 g^4 \left(\sum_{j=-n_1}^{n_2} h_j e^{j\eta} \right)^4 \sum_{i=-m_1}^{m_2} k_i e^{i\xi} \alpha_2 \sum_{j=-n_1}^{n_2} h_{ji} e^{j\eta} \\
& = 0
\end{aligned}
\tag{239}$$

where m_1, m_2, n_1, n_2 are all any positive integers which we may take arbitrarily.

For example, we can take $n_2 = 2, m_2 = 2, n_1 = 1, m_1 = 1$ and substitute them into Eq. (239), and then set all coefficients of $\exp(k\eta)$ and $\exp(j\xi)$, $j, k = 0, 1, 2, \dots$ to zero to get an over-determined equation with respect to $h_{-1}, h_0, h_1, h_2, k_0, k_1, k_2, k_{-1}, \alpha_1, \beta_1, \alpha_2, \beta_2$ and g . Solving the over-determined equations by use of Maple, we obtain the following solutions of $h_{-1}, h_0, h_1, h_2, k_0, k_1, k_2, k_{-1}, \alpha_1, \beta_1, \alpha_2, \beta_2$ and g :

Case 1

$$\begin{aligned}
\beta_2 &= -\frac{4\beta\alpha_2^3}{p^2}, \quad h_1 = \frac{p^2\alpha k_0^2}{8\alpha_2^2\beta h_{-1}(3p+2+p^2)}, \\
k_{-1} &= 0, \quad g = 0, \gamma_2 = \gamma_2, \quad \alpha_2 = \alpha_2, \\
k_0 &= k_0, \quad \alpha_1 = \alpha_1, \quad \beta_1 = \beta_1, \quad \gamma_1 = \gamma_1, \\
h_{-1} &= h_{-1}, \quad h_0 = 0, \quad k_2 = 0, \quad k_1 = 0, \\
h_2 &= 0,
\end{aligned}
\tag{240}$$

where $\alpha, \beta, \alpha_2, \gamma_2, \alpha_2, k_0, \alpha_1, \beta_1, \gamma_1$ and k_0 are any constants.

Case 2

$$\begin{aligned}
\gamma_2 &= \gamma_2, \quad k_0 = k_0, \quad \alpha_1 = \alpha_1, \quad \beta_1 = \beta_1, \\
\gamma_1 &= \gamma_1, \quad k_2 = k_2, \quad k_1 = k_1, \quad h_{-1} = 0, \\
\beta_2 &= \beta_2, \quad \alpha_2 = \alpha_2, \quad k_{-1} = k_{-1}, \quad h_2 = 0, \\
h_0 &= 0, \quad h_1 = 0, \quad g = g,
\end{aligned}
\tag{241}$$

where $\alpha, \beta, \gamma_2, k_0, \alpha_1, \beta_1, \gamma_1, k_2, k_1, \beta_2, \alpha_2, k_{-1}$ and g are any constants.

Case 3

$$\begin{aligned}
\alpha_2 &= -2\alpha_1, \quad \beta_2 = -2\beta_1, \quad \gamma_2 = \gamma_2, \quad \alpha_1 = \alpha_1, \\
\beta_1 &= \beta_1, \quad \gamma_1 = \gamma_1, \quad h_{-1} = h_{-1}, \quad h_0 = 0, \\
k_1 &= 0, \quad h_2 = 0, \quad k_{-1} = 0, \quad k_0 = 0, \quad h_1 = 0, \\
k_2 &= k_2, g = g,
\end{aligned}
\tag{242}$$

where $\alpha, \beta, \alpha_1, \beta_1, \gamma_2, \beta_1, \gamma_1, h_{-1}, k_2$ and g are any constants.

Case 4

$$\begin{aligned}
\gamma_2 &= \gamma_2, \quad \alpha_2 = \alpha_2, \quad \alpha_1 = \alpha_1, \quad \beta_1 = \beta_1, \\
\gamma_1 &= \gamma_1, \quad h_{-1} = h_{-1}, \quad k_2 = 0, \quad k_1 = 0, \\
k_{-1} &= 0, \quad \beta_2 = \beta_2, \quad h_1 = h_1, \quad h_2 = h_2, \\
h_0 &= h_0, \quad k_0 = 0, \quad g = g,
\end{aligned}
\tag{243}$$

where $\gamma_2, \alpha_2, \alpha_1, \beta_1, \gamma_1, h_{-1}, \beta_2, h_1, h_2$ and g are any constants.

Case 5

$$\begin{aligned}
\gamma_2 &= \gamma_2, k_0 = k_0, \alpha_1 = \alpha_1, \quad \beta_1 = \beta_1, \\
\gamma_1 &= \gamma_1, \quad h_{-1} = h_{-1}, \quad k_2 = 0, \quad k_1 = 0, \\
k_{-1} &= 0, \quad h_1 = h_1, \quad h_2 = h_2, \quad h_0 = h_0, \\
\alpha_2 &= 0, \beta_2 = 0, \quad g = g,
\end{aligned}
\tag{244}$$

where $\alpha, \beta, \gamma_2, k_0, \alpha_1, \beta_1, \gamma_1, h_{-1}, h_1, h_2, h_0$ and g are any constants.

Case 6

$$\begin{aligned}
\beta_2 &= -\beta_1, \quad \alpha_2 = -\alpha_1, \quad \gamma_2 = \gamma_2, \quad \alpha_1 = \alpha_1, \\
\beta_1 &= \beta_1, \quad \gamma_1 = \gamma_1, \quad h_{-1} = h_{-1}, \quad h_0 = 0, \\
k_2 &= 0, \quad h_2 = 0, \quad k_{-1} = 0, \quad k_0 = 0, \\
h_1 &= 0, \quad g = g, \quad k_1 = k_1,
\end{aligned}
\tag{245}$$

where $\alpha, \beta, \beta_1, \alpha_1, \gamma_2, \gamma_1, h_{-1}, g$ and k_1 are any constants.

Case 7

$$\begin{aligned}
\gamma_2 &= \gamma_2, \quad \gamma_1 = \gamma_1, \quad h_{-1} = h_{-1}, \quad k_2 = 0, \\
k_1 &= 0, \quad \beta_2 = \beta_2, \quad \alpha_2 = \alpha_2, \quad k_{-1} = k_{-1}, \\
\beta_1 &= \beta_2, \quad h_2 = 0, \quad h_0 = 0, \quad h_1 = 0, \\
\alpha_1 &= \alpha_2, \quad k_0 = 0, \quad g = g,
\end{aligned}
\tag{246}$$

where $\alpha, \beta, \gamma_2, \gamma_1, h_{-1}, \beta_2, \alpha_2, k_{-1}, \alpha_1$ and g are any constants.

Case 8

$$\begin{aligned}
\gamma_2 &= \gamma_2, \quad k_0 = k_0, \quad \gamma_1 = \gamma_1, \quad h_0 = h_0, \\
\beta_1 &= 0, \quad \alpha_1 = 0, \quad k_2 = k_2, \quad k_1 = k_1, \\
h_{-1} &= 0, \quad \beta_2 = \beta_2, \quad \alpha_2 = \alpha_2, \quad k_{-1} = k_{-1}, \\
h_2 &= 0, \quad h_1 = 0, \quad g = g,
\end{aligned}
\tag{247}$$

where $\alpha, \beta, \gamma_2, k_0, \gamma_1, h_0, k_2, k_1, \beta_2, \alpha_2, k_{-1}$ and g are any constants.

Case 9

$$\begin{aligned}
\gamma_2 &= \gamma_2, \quad \gamma_1 = \gamma_1, \quad k_2 = 0, \quad k_1 = 0, \\
\alpha_1 &= -\alpha_2, \quad \beta_1 = -\beta_2, \quad g = 0, \quad h_1 = h_1, \\
h_{-1} &= 0, \quad \beta_2 = \beta_2, \quad \alpha_2 = \alpha_2, \quad k_{-1} = k_{-1}, \\
h_2 &= 0, \quad h_0 = 0, \quad k_0 = 0,
\end{aligned}
\tag{248}$$

where $\alpha, \beta, \gamma_2, \gamma_1, \alpha_2, \beta_2, h_1, \beta_2, \alpha_2$ and k_{-1} are any constants.

Case 10

$$\begin{aligned}
\gamma_2 &= \gamma_2, \beta_1 = \beta_1, \gamma_1 = \gamma_1, \quad k_2 = 0, \\
k_1 &= 0, \quad \alpha_1 = -\alpha_2, h_1 = h_1, \beta_2 = -\beta_1, \\
h_{-1} &= 0, \alpha_2 = \alpha_2, \quad k_{-1} = k_{-1}, h_2 = 0, \\
h_0 &= 0, k_0 = 0, g = g,
\end{aligned}
\tag{249}$$

where $\alpha, \beta, \gamma_2, \beta_1, \gamma_1, \alpha_2, h_1, k_{-1}$ and g are any constants.

Case 11

$$\begin{aligned}
\beta_1 &= \frac{1}{2}\beta_2, \quad \gamma_2 = \gamma_2, \quad \gamma_1 = \gamma_1, \quad k_1 = 0, \\
k_{-1} &= 0, \quad h_1 = h_1, \quad \alpha_2 = 0, \quad \alpha_1 = 0, \\
k_2 &= k_2, \quad h_{-1} = 0, \quad \beta_2 = \beta_2, \quad h_2 = 0, \\
h_0 &= 0, \quad k_0 = 0, \quad g = g,
\end{aligned}
\tag{250}$$

where $\beta_1 = \frac{1}{2}\beta_2, \alpha, \beta, \gamma_2, \gamma_1, h_1, k_2, \beta_2$ and g are any constants.

Case 12

$$\begin{aligned}
\beta_2 &= 2\beta_1, \quad \alpha_1 = \frac{1}{2}\alpha_2, \quad \gamma_2 = \gamma_2, \quad \gamma_1 = \gamma_1, \\
k_1 &= 0, \quad k_{-1} = 0, \quad h_1 = h_1, \quad \alpha_2 = \alpha_2, \\
\beta_1 &= \beta_1, \quad k_2 = k_2, \quad h_{-1} = 0, \quad h_2 = 0, \\
h_0 &= 0, \quad k_0 = 0, \quad g = g,
\end{aligned}
\tag{251}$$

where $\alpha, \beta, \gamma_2, \gamma_1, h_1, \alpha_2, \beta_1, k_2$ and g are any constants.

Case 13

$$\begin{aligned}
\beta_1 &= -2\beta_2, \quad h_1 = 0, \quad k_2 = 0, \quad \alpha_2 = -1/2\alpha_1, \\
g &= 0, \quad h_2 = h_2, \quad k_{-1} = k_{-1}, \quad \alpha_1 = \alpha_1, \\
\gamma_2 &= \gamma_2, \quad \gamma_1 = \gamma_1, \quad k_1 = 0, \quad h_{-1} = 0, \\
\beta_2 &= \beta_2, \quad h_0 = 0, \quad k_0 = 0,
\end{aligned}
\tag{252}$$

where $\alpha, \beta, \beta_2, \alpha_1, h_2, \gamma_2, \gamma_1, \beta_2$ and k_{-1} are any constants.

Case 14

$$\begin{aligned}
\alpha_1 &= 2\alpha_2, \quad \beta_1 = 2\beta_2, \quad \alpha_2 = \alpha_2, \quad h_1 = 0, \\
k_2 &= 0, h_2 = h_2, \quad \gamma_2 = \gamma_2, \quad \gamma_1 = \gamma_1, \\
k_{-1} &= 0, \quad k_1 = k_1, \quad h_{-1} = 0, \quad \beta_2 = \beta_2, \\
h_0 &= 0, \quad k_0 = 0, \quad g = g,
\end{aligned}
\tag{253}$$

where $\alpha, \beta, \alpha_2, \beta_2, h_2, \gamma_2, \gamma_1, k_1, \beta_2$ and g are any constants.

Case 15

$$\begin{aligned}
 \beta_1 &= -2\beta_2, \quad h_1 = 0, \quad k_2 = 0, \quad \alpha_2 = -1/2\alpha_1, \\
 g &= 0, \quad h_2 = h_2, \quad k_{-1} = k_{-1}, \quad \alpha_1 = \alpha_1, \\
 \gamma_2 &= \gamma_2, \quad \gamma_1 = \gamma_1, \quad k_1 = 0, \quad h_{-1} = 0, \\
 \beta_2 &= \beta_2, \quad h_0 = 0, \quad k_0 = 0,
 \end{aligned}
 \tag{254}$$

where $\alpha, \beta, \beta_2, \alpha_1, h_2, \alpha_1, \gamma_2, \gamma_1, \beta_2$ and k_{-1} are any constants.

Case 16

$$\begin{aligned}
 \beta_1 &= 2\beta_2, \quad h_1 = 0, \quad k_2 = 0, \quad h_2 = h_2, \\
 \gamma_2 &= \gamma_2, \quad \gamma_1 = \gamma_1, \quad k_{-1} = 0, \quad k_1 = k_1, \\
 \alpha_1 &= 2\alpha_2, \quad \alpha_2 = \alpha_2, \quad h_{-1} = 0, \quad \beta_2 = \beta_2, \\
 h_0 &= 0, \quad k_0 = 0, \quad g = g,
 \end{aligned}
 \tag{255}$$

where $\alpha, \beta, \beta_2, \gamma_2, \gamma_1, k_1, \alpha_2, \beta_2$ and g are any constants.

Case 17

$$\begin{aligned}
 \alpha_1 &= -2\alpha_2, \quad h_1 = 0, \quad k_2 = 0, \quad h_2 = h_2, \\
 \gamma_2 &= \gamma_2, \quad \gamma_1 = \gamma_1, \quad \alpha_2 = \alpha_2, \quad \beta_1 = -2\beta_2, \\
 k_{-1} &= k_{-1}, \quad k_1 = 0, \quad h_{-1} = 0, \quad \beta_2 = \beta_2, \\
 h_0 &= 0, \quad k_0 = 0, \quad g = g,
 \end{aligned}
 \tag{256}$$

where $\alpha, \beta, \alpha_2, h_2, \gamma_2, \gamma_1, \beta_2, k_{-1}, \beta_2$ and g are any constants.

Case 18

$$\begin{aligned}
 h_2 &= h_2, \quad \gamma_2 = \gamma_2, \quad \gamma_1 = \gamma_1, \quad h_1 = 0, \\
 k_{-1} &= 0, \quad \alpha_2 = \alpha_2, \quad \alpha_1 = \alpha_2, \quad k_2 = k_2, \\
 \beta_1 &= \beta_2, \quad k_1 = 0, \quad h_{-1} = 0, \quad \beta_2 = \beta_2, \\
 h_0 &= 0, \quad k_0 = 0, \quad g = g,
 \end{aligned}
 \tag{257}$$

where $\alpha, \beta, h_2, \gamma_2, \gamma_1, \alpha_2, k_2, \beta_2$ and g are any constants.

Case 19

$$\begin{aligned}
 h_1 &= 0, \quad k_0 = k_0, \quad h_0 = h_0, \quad h_2 = 0, \\
 \beta_1 &= 0, \quad \alpha_1 = 0, \quad \gamma_2 = \gamma_2, \quad \gamma_1 = \gamma_1, \\
 k_1 &= k_1, \quad \alpha_2 = \alpha_2, \quad k_2 = k_2, \quad k_{-1} = k_{-1}, \\
 h_{-1} &= 0, \quad \beta_2 = \beta_2, \quad g = g,
 \end{aligned}
 \tag{258}$$

where $\alpha, \beta, k_0, h_0, \gamma_2, \gamma_1, k_1, \alpha_2, k_2, k_{-1}, \beta_2$ and g are any constants.

Substituting Eqs. (240), (241), (242), (243), (244), (245), (246), (247), (248), (249), (250), (251), (252), (253), (254), (255), (256), (257), and (258) into (236) and (238), respectively, we can obtain the exact solutions of Eq. (235).

$$u(x, t) = \sqrt[2/p]{g + \frac{k_0 + 2 \cosh_{k_1, k_{-1}, 1}(\alpha_1 x + \beta_1 t + \gamma_1) + k_2 e^{2(\alpha_1 x + \beta_1 t + \gamma_1)}}{h_0 + 2 \cosh_{h_1, h_{-1}, 1}(\alpha_2 x + \beta_2 t + \gamma_2) + h_2 e^{2(\alpha_2 x + \beta_2 t + \gamma_2)}}},
 \tag{259}$$

where $\cosh_{k_1, k_{-1}, 1}(\alpha_1 x + \beta_1 t + \gamma_1)$ and $\cosh_{h_1, h_{-1}, 1}(\alpha_2 x + \beta_2 t + \gamma_2)$ are all the generalized hyperbolic cosine functions, here $k_i, h_i, i = 0, \pm 1, 2; \alpha_j, \beta_j, \gamma_j, j = 1, 2$ and g satisfy Eqs. (240), (241), (242), (243), (244), (245), (246), (247), (248), (249), (250), (251), (252), (253), (254), (255), (256), (257), and (258), respectively.

For example, if we substitute Eq. (240) into (236) and (238), we obtain the following solutions of Eq. (235).

$$u_1(x, t) = \sqrt[2/p]{\frac{k_0 + 2 \cosh_{0, 0, 1}(\alpha_1 x + \beta_1 t + \gamma_1)}{2 \cosh_{h_1, h_{-1}, 1}\left(\alpha_2 x - 4 \frac{\beta \alpha_2^3 t}{p^2} + \gamma_2\right)}},
 \tag{260}$$

where $\cosh_{0, 0, 1}(\alpha_1 x + \beta_1 t + \gamma_1)$ and $\cosh_{h_1, h_{-1}, 1}(\alpha_2 x - 4(\beta \alpha_2^3 t)/p^2 + \gamma_2)$ are all the generalized hyperbolic cosine function, here $h_1 = (p^2 \alpha k_0^2)/(8 \alpha_2^2 \beta h_{-1}(3p + 2 + p^2))$, and $\alpha, \beta, \alpha_2, \gamma_2, \alpha_2, k_0, \alpha_1, \beta_1, \gamma_1, k_0$ are any constants.

Remark 13 When $m_1, m_2, n_1, n_2 = 1, 2, \dots$, we can obtain a family of N solitary-like solutions of Eq. (235).

In summary, we suggest two new generic transformations, a new Exp- N solitary-like method and its algorithm, to find new non-traveling waves solutions and traveling waves solutions of higher dimensional NLPDEs by our transformations. The suggested method is more powerful than the method proposed by Dr. He (He and Wu 2006) to seek more exact solutions of higher dimensional NLPDEs. The $(1 + 1)$ -dimensional generalized KdV equation is chosen to illustrate our algorithm such that new exact solutions are found.

The Exp-Bäcklund Transformation Method and Its Application in $(1 + 1)$ -Dimensional KdV Equation

In this section, a new method and its algorithm, which is called Exp-Bäcklund transformation

method, is presented to find more exact solutions of NLPDEs based on the idea of the Exp-function method (He and Wu 2006). We choose the $(1 + 1)$ -dimensional KdV equations to illustrate the effectiveness and convenience of our algorithm. As a result, many new solutions are obtained.

Summary of the Exp-Bäcklund Transformation Method

In the following we would like to outline the main steps of our method:

Step 1

For a given NLEEs, with $u_i(t, x_1, x_2, \dots, x_{n-1})$ ($i = 1, 2, \dots, n$) in n independent variables $t, x_1, x_2, \dots, x_{n-1}$,

$$F_j(u_1, \dots, u_n, u_{1,t}, \dots, u_{n,t}, u_{1,x_1}, \dots, u_{n,x_{n-1}}, u_{1,tt}, \dots, u_{n,tt}, u_{1,tx_1}, \dots, u_{n,tx_{n-1}}, \dots) = 0, \quad (261)$$

where $j = 1, 2, \dots, n$.

We obtain an auto-Bäcklund transformation of Eq. (261) using our method of constructing auto-Bäcklund transformation (if it exist)

$$u_i(t, x_1, \dots, x_{n-1}) = \sum_{j=0}^{m_i} u_{ij}(t, x_1, \dots, x_{n-1}) f^{j-m_i}(t, x_1, \dots, x_{n-1}), \quad i = 1, 2, \dots, n. \quad (262)$$

where $u_{ij}(t, x_1, \dots, x_{n-1})$, $i = 1, 2, \dots, n$; $j = 0, 1, 2, \dots, m_i - 1$ are differentiable functions which have already been obtained. But $f(t, x_1, \dots, x_{n-1})$ are any differentiable functions, u_{i,m_i} , $i = 1, 2, \dots, n$ are the seed solutions of Eq. (261), and m_i , $i = 1, 2, \dots, n$ are positive integers which have already been obtained.

Step 2

Substituting Eq. (262) into (261), and setting the coefficients of these terms $1/(f^k(t, x, y))$, $k = 0, 1, 2, \dots$ to zero yields a set of over-determined partial differential equations with respect to $f(t, x, y)$, u_{i,m_i} , $i = 1, 2, \dots, n$.

Step 3

We consider the variable in the form

$$\eta_i = p_{i0}(t) + \sum_{k=1}^{n-1} p_{ik}(t, x_1, \dots, x_{k-1}). \quad (263)$$

$$x_k, \quad i = 1, 2$$

and make the transformation

$$f(t, x_1, \dots, x_{n-1}) = f(\eta_1, \eta_2). \quad (264)$$

Then we express the $f(t, x_1, \dots, x_{n-1})$ in Eqs. (262) and (264) to be the following forms

$$f(\eta_1, \eta_2) = \frac{\sum_{n=-c}^d a_n \exp(n\eta_1)}{\sum_{m=-p}^q b_m \exp(m\eta_2)}, \quad (265)$$

where $\sum_{k=0}^{n-1} p_k \neq 0$, c, d, p and q are any positive integers, $a_n, n = -c, \dots, d, b_m, m = -p, \dots, q$, and $p_{ik}, k = 0, 1, 2, \dots, n-1$ are functions or constants to be determined later.

Step 4

If we substitute Eqs. (262), (263), (264), and (265) into the over-determined partial differential equations in Step 2 and yield a set of over-determined partial differential equations with respect to $\exp(k\eta_i), i = 1, 2; k = 0, \pm 1, \pm 2, \dots, a_n, n = -c, \dots, d, b_m, m = -p, \dots, q, u_{i,m_i}, i = 1, 2, \dots, n$,

and $p_{ik}, k = 0, 1, 2, \dots, n-1$. Then we collect the coefficients of the polynomials with respect to $\exp(k\eta_i), i = 1, 2; k = 0, \pm 1, \pm 2, \dots$ and set each coefficient to zero, we will get a system of over-determined partial differential of $a_n, n = -c, \dots, d, b_m, m = -p, \dots, q$, and $p_{ik}, k = 0, 1, 2, \dots, n-1, u_{i,m_i}, i = 1, 2, \dots, n$.

Step 5

We solve the over-determined partial differential in Step 4 and obtain $a_n, n = -c, \dots, d, b_m, m = -p, \dots, q, u_{i,m_i}, i = 1, 2, \dots, n$, and $p_{ik}, k = 0, 1, 2, \dots, n-1$.

Step 6

Substituting the results obtained in Step 5 into Eqs. (262), (263), (264), and (265), then we can obtain a family of rational solutions of Exp-functions as follows

$$u_i(t, x_1, \dots, x_{n-1}) = \sum_{j=0}^{m_i} u_{ij}(t, x_1, \dots, x_{n-1}) \left(\frac{\sum_{n=-c}^d a_n \exp(np_{10}(t)) + n \sum_{k=1}^{n-1} p_{1k}(t, x_1, \dots, x_{k-1}) \cdot x_k}{\sum_{m=-p}^q b_m \exp(mp_{20}(t)) + m \sum_{k=1}^{n-1} p_{2k}(t, x_1, \dots, x_{k-1}) \cdot x_k} \right)^{j-m_i}, \quad (266)$$

where c, d, p and q are any positive integers.

The Application of the Exp-Bäcklund Transformation Method in (1 + 1)-Dimensional KdV Equation

In this section, we will make use of our Exp-Bäcklund transformation method (Ren 2007a; Ren 2007b) and symbolic computation to find the exact solutions of the following (1 + 1)-dimensional KdV equation (Korteweg and de Vries 1895; Khaters et al. 1997; Helal 2001; Brugarino and Sciacca 2008).

$$u_t(x, t) + \alpha u(x, t)u_x(x, t) + \beta u_{xxx}(x, t) = 0, \quad (267)$$

where the coefficient of the nonlinear term α and the coefficient of the dispersive term β is independent of x and t .

Set the auto-Bäcklund transformation of Eq. (267) to have the following form:

$$u(x, t) = \sum_{j=1}^m u_j(x, t) f^{j-m}(x, t), \quad (268)$$

where $f(x, t)$ and $u_j(x, t), j = 0, 1, 2, \dots, m-1$ are all differential functions to be determined later, and $u_m(x, t)$ is the trivial seed solution of Eq. (267).

By balancing the highest order linear term and nonlinear terms in Eq. (267), we get $m = 2$ and (268) has the following formal

$$u(x, t) = u_0(x, t)f^{-2}(x, t) + u_1(x, t)f^{-1}(x, t) + u_2(x, t), \quad (269)$$

We take the trivial seed solution as

$$u_2 = u_2(x, t). \quad (270)$$

With the aid of Maple symbolic computation software, substituting Eqs. (269) and (270) into (267), and collecting all terms with $f^{-i}(x, t), i = 0, 1, 2, \dots$, we obtain

$$\begin{aligned} & [-2\alpha u_0^2(x, t)f_x(x, t) - 24\beta u_0(x, t)f^3(x, t)]\frac{1}{f^5} \\ & + \{-\beta[-18u_0(x, t)f_x(x, t)f_{xx}(x, t) - 18u_{0x}(x, t)f_x^2(x, t) \\ & + 6u_1(x, t)f_x^3(x, t)] \\ & + \alpha u_0(x, t)[u_{0x}(x, t) - u_1(x, t)f_x(x, t)] \\ & - 2\alpha u_1(x, t)u_0(x, t)f_x(x, t)]\frac{1}{f^4} \\ & + \{-\beta[-6u_1(x, t)f_x(x, t)f_{xx}(x, t) \\ & + 6u_{0xx}(x, t)f_x(x, t) + 2u_0(x, t)f_{xxx}(x, t) \\ & + 6u_{0x}(x, t)f_{xx}(x, t) - 6u_{1x}(x, t)f_x^2(x, t)] \\ & - 2u_0(x, t)f_t(x, t) + \alpha u_0(x, t)u_{1x}(x, t) \\ & + \alpha u_1(x, t)[u_{0x}(x, t) - u_1(x, t)f_x(x, t)] \\ & - 2\alpha u_2(x, t)u_0(x, t)f_x(x, t)]\frac{1}{f^3} \\ & + \{-\beta[-u_{0xxx}(x, t) + 3u_{1x}(x, t)f_{xx}(x, t) \\ & + 3u_{1xx}(x, t)f_x(x, t) + u_1(x, t)f_{xxx}(x, t)] \\ & + \alpha u_0(x, t)u_{2x}(x, t) + \alpha u_1(x, t)u_{1x}(x, t) \\ & + \alpha u_2(x, t)[u_{0x}(x, t) - u_1(x, t)f_x(x, t)] \\ & + u_{0t}(x, t) - u_1(x, t)f_t(x, t)]\frac{1}{f^2} \\ & + (u_{1t}(x, t) + \alpha u_1(x, t)u_{2x}(x, t) + \alpha u_2(x, t)u_{1x}(x, t) \\ & + \beta u_{1xxx}(x, t))\frac{1}{f} + u_{2t}(x, t) + \alpha u_2(x, t)u_{2x}(x, t) \\ & + \beta u_{2xxx}(x, t) = 0, \end{aligned} \quad (271)$$

Setting the coefficient of f^{-5} in Eq. (271) to be zero, we obtain a differential equation

$$-2\alpha u_0^2(x, t)f_x(x, t) - 24\beta u_0(x, t)f^3(x, t) = 0, \quad (272)$$

which has solution

$$u_0(x, t) = -\frac{12\beta}{\alpha}f_x^2(x, t). \quad (273)$$

Setting the coefficient of f^{-4} in Eq. (271) to be zero, we obtain a differential equation

$$\begin{aligned} & -\beta[-18u_0(x, t)f_x(x, t)f_{xx}(x, t) \\ & - 18u_{0x}(x, t)f_x^2(x, t) + 6u_1(x, t)f_x^3(x, t)] \\ & + \alpha u_0(x, t)[u_{0x}(x, t) - u_1(x, t)f_x(x, t)] \\ & - 2\alpha u_1(x, t)u_0(x, t)f_x(x, t) = 0, \end{aligned} \quad (274)$$

we can get the following expressions:

$$u_1(x, t) = \frac{12\beta}{\alpha}f_{xx}(x, t). \quad (275)$$

By Eqs. (273), (275), (270) and (269), we obtain an auto-Bäcklund transformation of Eq. (267) as follows

$$u(x, t) = -\frac{12\beta}{\alpha}\left(\frac{f_{xx}(x, t)}{f(x, t)} - \frac{f_x^2(x, t)}{f^2(x, t)}\right) + u_2(x, t) \quad (276)$$

where $u_2(x, t)$ is a seed solution of Eq. (267), $f(x, t)$ is any function that satisfies Eqs. (272), (274), and the following equations

$$\begin{aligned} & -\beta[-6u_1(x, t)f_x(x, t)f_{xx}(x, t) + 6u_{0xx}(x, t)f_x(x, t) \\ & + 2u_0(x, t)f_{xxx}(x, t) + 6u_{0x}(x, t)f_{xx}(x, t) \\ & - 6u_{1x}(x, t)f_x^2(x, t)] \\ & - 2u_0(x, t)f_t(x, t) + \alpha u_0(x, t)u_{1x}(x, t) \\ & + \alpha u_1(x, t)[u_{0x}(x, t) - u_1(x, t)f_x(x, t)] \\ & - 2\alpha u_2(x, t)u_0(x, t)f_x(x, t) = 0, \\ & -\beta[-u_{0xxx}(x, t) + 3u_{1x}(x, t)f_{xx}(x, t) \\ & + 3u_{1xx}(x, t)f_x(x, t) + u_1(x, t)f_{xxx}(x, t)] \\ & + \alpha u_0(x, t)u_{2x}(x, t) + \alpha u_1(x, t)u_{1x}(x, t) \\ & + \alpha u_2(x, t)[u_{0x}(x, t) - u_1(x, t)f_x(x, t)] + u_{0t}(x, t) \\ & - u_1(x, t)f_t(x, t) = 0, \\ & u_{1t}(x, t) + \alpha u_1(x, t)u_{2x}(x, t) \\ & + \alpha u_2(x, t)u_{1x}(x, t) + \beta u_{1xxx}(x, t) = 0. \end{aligned} \quad (277)$$

We take $f(x, t)$ to be of the following form

$$f(x, t) = \frac{\sum_{i=-m_1}^{m_2} k_i(t) e^{i(p_1(t)x + q_1(t))}}{\sum_{j=-n_1}^{n_2} h_j(t) e^{j(p_2(t)x + q_2(t))}}, \quad (278)$$

where n_1, n_2, m_1 and m_2 are any positive integers, $p_l(t), q_l(t), l = 1, 2; k_i(t), i = -m_1, \dots, m_2$ and $h_j(t), j = -n_1, \dots, n_2$ are the coefficients to be determined later.

By substituting (278) into the given Eq. (277) with (273) and (275), and then collecting the coefficients of the polynomials of $e^{i(p_1(t)x + q_1(t))}$ and $e^{j(p_2(t)x + q_2(t))}, i, j = 0, 1, 2, \dots$, then setting each coefficient to zero, we will get a system of over-determined partial differential equations with respect to $k_i(t), i = -m_1, \dots, m_2; h_j(t), j = -n_1, \dots, n_2; p_k(t), q_k(t), k = 1, 2$ and $u_2(x, t)$. We solve the over-determined partial differential equations and obtain many families of solutions of $k_i(t), i = -m_1, \dots, m_2; h_j(t), j = -n_1, \dots, n_2; p_k(t), q_k(t), k = 1, 2$ and $u_2(x, t)$. Finally, substituting them into (276) with (278), we can obtain the following solutions of Eq. (267)

$$\begin{aligned} u = & -\frac{12\beta \left(-\sum_{i=-m_1}^{m_2} k_i(t) i p_1(t) e^{i\xi_1} \sum_{j=-n_1}^{n_2} h_j(t) e^{j\xi_2} \right.}{\alpha \left(\sum_{j=-n_1}^{n_2} h_j(t) e^{j\xi_2} \right)^2 \left(\sum_{i=-m_1}^{m_2} k_i(t) e^{i\xi_1} \right)^2} \\ & + \left. \sum_{i=-m_1}^{m_2} k_i(t) e^{i\xi_1} \sum_{j=-n_1}^{n_2} h_j(t) j p_2(t) e^{j\xi_2} \right)^2 \\ & - \frac{12\beta p_1^2(t) \left(\sum_{i=-m_1}^{m_2} k_i(t) i e^{i\xi_1} \right)^2}{\alpha \left(\sum_{i=-m_1}^{m_2} k_i(t) e^{i\xi_1} \right)^2} \\ & + \frac{12\beta p_1^2(t) \sum_{i=-m_1}^{m_2} k_i(t) i^2 e^{i\xi_1}}{\alpha \sum_{i=-m_1}^{m_2} k_i(t) e^{i\xi_1}} \\ & + \frac{12\beta p_2^2(t) \left(\sum_{j=-n_1}^{n_2} h_j(t) j e^{j\xi_2} \right)^2}{\left(\sum_{j=-n_1}^{n_2} h_j(t) e^{j\xi_2} \right)^2 \alpha} \\ & - \frac{12\beta p_2^2(t) \sum_{j=-n_1}^{n_2} h_j(t) j^2 e^{j\xi_2}}{\sum_{j=-n_1}^{n_2} h_j(t) e^{j\xi_2} \alpha} + u_2(t), \end{aligned} \quad (279)$$

where n_1, n_2, m_1 and m_2 are any positive integers, and $\xi_1 = p_1(t)x + q_1(t)$ and $\xi_2 = p_2(t)x + q_2(t)$.

For example, when we take $n_1 = 1, n_2 = 1, m_1 = 1$ and $m_2 = 1$, substituting Eq. (278) into the given Eq. (277) with (273) and (275), and then collecting the coefficients of the polynomials of $e^{i(p_1(t)x + q_1(t))}$ and $e^{j(p_2(t)x + q_2(t))}, i, j = 0, 1, 2, \dots$, then setting each coefficient to zero, we will get a

system of over-determined partial differential equations with respect to $p_l(t), q_l(t), l = 1, 2; k_i(t), i = -1, 0, 1$ and $h_j(t), j = -1, 0, 1$ as follows

$$\begin{aligned} & -p_2(t)k_0(t)h_1^4(t)k_{-1r}(t) \\ & -p_2(t)k_{-1}(t)h_1^4(t)k_{0r}(t) \\ & +4\beta p_2(t)k_0(t)h_1^4(t)p_1^3(t)k_{-1}(t) \\ & +2p_2(t)k_{-1}(t)h_1^3(t)k_0(t)h_{1r}(t) \\ & +p_2(t)k_0(t)h_1^4(t)k_{-1}(t)q_{1r}(t) \\ & +p_1(t)k_{-1}(t)h_1^4(t)k_0(t)q_{2r}(t) \\ & +2u_2(x, t)\alpha p_1(t)k_{-1}(t)h_1^4(t)p_2(t)k_0(t) \\ & +2u_2(x, t)\alpha p_2^2(t)k_{-1}(t)h_1^4(t)k_0(t) \\ & +2p_2(t)k_{-1}(t)h_1^4(t)k_0(t)q_{2r}(t) \\ & +p_1(t)k_{-1}(t)h_1^3(t)k_0(t)h_{1r}(t) \\ & -p_1(t)k_{-1}(t)h_1^4(t)k_{0r}(t) \\ & +4\beta p_1(t)k_{-1}(t)h_1^4(t)p_2^3(t)k_0(t) \\ & +2\beta p_2^4(t)k_{-1}(t)h_1^4(t)k_0(t) \\ & +6\beta p_2^2(t)k_0(t)h_1^4(t)p_1^2(t)k_{-1}(t) = 0, \\ & p_1(t)k_1(t)h_1^4(t)k_{0r}(t) - 4\beta p_2(t)k_0(t)h_1^4(t)p_1^3(t)k_1(t) \\ & -4\beta p_1(t)k_1(t)h_1^4(t)p_2^3(t)k_0(t) \\ & +2p_2(t)k_0(t)h_1^4(t)k_1(t)q_{2r}(t) + 2\beta p_2^4(t)k_1(t)h_1^4(t)k_0(t) \\ & +2u_2(x, t)\alpha p_2^2(t)k_0(t)h_1^4(t)k_1(t) \\ & -2u_2(x, t)\alpha p_1(t)k_1(t)h_1^4(t)p_2(t)k_0(t) \\ & -p_2(t)k_1(t)h_1^4(t)k_{0r}(t) + 6\beta p_2^2(t)k_0(t)h_1^4(t)p_1^2(t)k_1(t) \\ & +2p_2(t)k_0(t)h_1^3(t)k_1(t)h_{1r}(t) \\ & -p_1(t)k_1(t)h_1^3(t)k_0(t)h_{1r}(t) - p_2(t)k_0(t)h_1^4(t)k_1(t)q_{1r}(t) \\ & -p_2(t)k_0(t)h_1^4(t)k_{1r}(t) \\ & -p_1(t)k_1(t)h_1^4(t)k_0(t)q_{2r}(t)4\beta p_1(t)k_1(t)h_{-1}^4(t)p_2^3(t)k_0(t) \\ & +p_2(t)k_0(t)h_{-1}^4(t)k_{1r}(t) + p_2(t)k_0(t)h_{-1}^4(t)k_1(t)q_{1r}(t) \\ & +2p_2(t)k_0(t)h_{-1}^4(t)k_1(t)q_{2r}(t) \\ & +p_1(t)k_1(t)h_{-1}^4(t)k_0(t)q_{2r}(t) \\ & +6\beta p_2^2(t)k_0(t)h_{-1}^4(t)p_1^2(t)k_1(t) \\ & +2u_2(x, t)\alpha p_1(t)k_1(t)h_{-1}^4(t)p_2(t)k_0(t) \\ & -p_1(t)k_1(t)h_{-1}^3(t)k_0(t)h_{-1r}(t) \\ & +2u_2(x, t)\alpha p_2^2(t)k_0(t)h_{-1}^4(t)k_1(t) \\ & -2p_2(t)k_0(t)h_{-1}^3(t)k_1(t)h_{-1r}(t) \\ & +2\beta p_2^4(t)k_1(t)h_{-1}^4(t)k_0(t) \\ & +4\beta p_2(t)k_0(t)h_{-1}^4(t)p_1^3(t)k_1(t) + p_1(t)k_1(t)h_{-1}^4(t)k_{0r}(t) \\ & +p_2(t)k_1(t)h_{-1}^4(t)k_{0r}(t) = 0, \\ & \dots\dots\dots \end{aligned} \quad (280)$$

We solve the over-determined partial differential equations and obtain the following many families of solutions:

Case 1

$$u_1(x, t) = -\frac{12\beta(k_{-1}(t))^2(p_1(t))^2(e^{-p_1(t)x-q_1(t)})^2}{\alpha(k_{-1}(t)e^{-p_1(t)x-q_1(t)} + C_1)^2} + \frac{12\beta k_{-1}(t)(p_1(t))^2 e^{-p_1(t)x-q_1(t)}}{\alpha(k_{-1}(t)e^{-p_1(t)x-q_1(t)} + C_1)} + u_2(t), \quad (281)$$

where $q_1(t) = \int \frac{\frac{d}{dt}k_{-1}(t) - (p_1(t))^3 k_{-1}(t) \beta - p_1(t) u_2(t) \alpha k_{-1}(t)}{k_{-1}(t)} dt$ Case 2
 $+C_3, k_{-1}(t), p_1(t)$ are any functions of t , and $\alpha, \beta, C_i, i = 1, 3$ are any constants.

$$u_2(x, t) = -\frac{12\beta(C_1 p_2(t) + k_{-1}(t)(p_2(t) - p_1(t))e^{-\xi_1} + C_2(p_2(t) + p_1(t))e^{\xi_1})^2}{\alpha(k_{-1}(t)e^{-\xi_1} + C_1 + C_2 e^{\xi_1})^2} + \frac{12\beta(C_1(p_2(t))^2 + C_2(p_2(t) + p_1(t))^2 e^{\xi_1} + (p_1(t) - p_2(t))^2 k_{-1}(t)e^{-\xi_1})}{\alpha(k_{-1}(t)e^{-\xi_1} + C_1 + C_2 e^{\xi_1})}, \quad (282)$$

where $\xi_1 = p_1(t)x + q_1(t)$, $k_{-1}(t), q_1(t), p_1(t)$ are any functions of t , and $\alpha, \beta, C_i, i = 1, 2$ are any constants. Case 4

$$u_4(x, t) = -\frac{12\beta(p_1(t))^2(k_1(t))^2(e^{p_1(t)x+q_1(t)})^2}{\alpha(C_1 + k_1(t)e^{p_1(t)x+q_1(t)})^2} + \frac{12\beta(p_1(t))^2 k_1(t)e^{p_1(t)x+q_1(t)}}{\alpha(C_1 + k_1(t)e^{p_1(t)x+q_1(t)})} + u_2(t), \quad (284)$$

Case 3

$$u_3(x, t) = -\frac{12\beta(k_{-1}(t)(p_2(t) - p_1(t))e^{-\xi_1} + k_1(t)(p_2(t) + p_1(t))e^{\xi_1} + C_1 p_2(t))^2}{\alpha(k_{-1}(t)e^{-\xi_1} + C_1 + k_1(t)e^{\xi_1})^2} + \frac{12\beta(C_1(p_2(t))^2 + k_1(t)(p_2(t) + p_1(t))^2 e^{\xi_1} + (p_1(t) - p_2(t))^2 k_{-1}(t)e^{-\xi_1})}{\alpha(k_{-1}(t)e^{-\xi_1} + C_1 + k_1(t)e^{\xi_1})}, \quad (283)$$

where $q_1(t) = -\int \frac{\beta(p_1(t))^3 k_1(t) + u_2(t) \alpha p_1(t) k_1(t) + \frac{d}{dt}k_1(t)}{k_1(t)} dt$
 $+C_3, k_1(t), p_1(t)$ are the functions of t , and $\alpha, \beta, C_i, i = 1, 3$ are any constants.

Case 5

where $\xi_1 = p_1(t)x + q_1(t)$, $k_{-1}(t), k_1(t), q_1(t), p_1(t)$ are any functions of t , and α, β, C_1 are any constants.

$$u_5(x, t) = \frac{2C_1 h_0(t) \left(u_2(t) \alpha - 6\beta(p_2(t))^2 \right) e^{-\xi_2} + u_2(t) \alpha C_1^2 e^{-2\xi_2} + u_2(t) \alpha (h_0(t))^2}{(C_1 e^{-\xi_2} + h_0(t))^2 \alpha}, \quad (285)$$

where $\xi_2 = p_2(t)x + q_2(t)$, $h_0(t)$, $p_2(t)$, $q_2(t)$ are the functions of t , and α , β , C_1 are any constants. Case 6

$$u_6(x, t) = - \frac{2C_1 h_0(t) \left(6\beta(p_2(t))^2 - u_2(t)\alpha \right) e^{-\xi_2} + 2C_1 h_1(t) \left(24\beta(p_2(t))^2 - u_2(t)\alpha \right)}{(C_1 e^{-\xi_2} + h_0(t) + h_1(t) e^{\xi_2})^2 \alpha} - \frac{2h_0(t) h_1(t) \left(6\beta(p_2(t))^2 - u_2(t)\alpha \right) e^{\xi_2} - u_2(t)\alpha \left((h_1(t))^2 + C_1^2 \right) e^{-2\xi_2} - u_2(t)\alpha (h_0(t))^2}{(C_1 e^{-\xi_2} + h_0(t) + h_1(t) e^{\xi_2})^2 \alpha}, \quad (286)$$

where $\xi_2 = p_2(t)x + q_2(t)$, $h_0(t)$, $h_1(t)$, $q_1(t)$, $q_2(t)$ are the functions of t , and α , β , C_1 are any constants. Case 7

$$u_7(x, t) = \frac{2h_0(t) h_1(t) \left(u_2(t)\alpha - 6\beta(p_2(t))^2 \right) e^{\xi_2} + 2h_0(t) h_{-1}(t) \left(u_2(t)\alpha - 6\beta(p_2(t))^2 \right) e^{-\xi_2}}{(h_{-1}(t) e^{-\xi_2} + h_0(t) + h_1(t) e^{\xi_2})^2 \alpha} + \frac{u_2(t)\alpha \left((h_{-1}(t))^2 e^{-2\xi_2} + (h_1(t))^2 e^{2\xi_2} \right) - 48\beta h_{-1}(t) (p_2(t))^2 h_1(t) + u_2(t)\alpha \left(2h_{-1}(t) h_1(t) + (h_0(t))^2 \right)}{(h_{-1}(t) e^{-\xi_2} + h_0(t) + h_1(t) e^{\xi_2})^2 \alpha}, \quad (287)$$

where $\xi_2 = p_2(t)x + q_2(t)$, $h_{-1}(t)$, $h_0(t)$, $h_1(t)$, $q_1(t)$, $q_2(t)$ are the functions of t , and α , β are any constants. Case 8

$$u_8(x, t) = \frac{2C_1 k_0(t) \left(6\beta(p_1(t))^2 + u_2(t)\alpha \right) e^{-\xi_1} + u_2(t)\alpha C_1^2 e^{-2\xi_1} + u_2(t)\alpha (k_0(t))^2}{\alpha (C_1 e^{-\xi_1} + k_0(t))^2}, \quad (288)$$

where $\xi_1 = p_1(t)x + q_1(t)$, $k_0(t)$, $q_1(t)$, $q_2(t)$ are the functions of t , and α , β , C_1 are any constants. Case 9

$$u_9(x, t) = \frac{u_2(t)\alpha\left((k_0(t))^2 + 2C_1k_1(t)\right) + 48\beta C_1(p_1(t))^2k_1(t) + u_2(t)\alpha(k_1(t))^2e^{2\xi_1}}{\alpha(C_1e^{-\xi_1} + k_0(t) + k_1(t)e^{\xi_1})^2} + \frac{2k_0(t)k_1(t)\left(6\beta(p_1(t))^2 + u_2(t)\alpha\right)e^{\xi_1} + 2C_1k_0(t)\left(6\beta(p_1(t))^2 + u_2(t)\alpha\right)e^{-\xi_1} + u_2(t)\alpha C_1^2e^{-2\xi_1}}{\alpha(C_1e^{-\xi_1} + k_0(t) + k_1(t)e^{\xi_1})^2}, \quad (289)$$

where $\xi_1 = p_1(t)x + q_1(t)$, $k_0(t)$, $k_1(t)$, $q_1(t)$, $q_2(t)$ are the functions of t , and α , β , C_1 are any constants. Case 10

$$u_{10}(x, t) = -\frac{12\beta(p_1(t))^2(k_1(t))^2(e^{p_1(t)x+q_1(t)})^2}{\alpha(k_0(t) + k_1(t)e^{p_1(t)x+q_1(t)})^2} + \frac{12\beta(p_1(t))^2k_1(t)e^{p_1(t)x+q_1(t)}}{\alpha(k_0(t) + k_1(t)e^{p_1(t)x+q_1(t)})} + u_2(t), \quad (290)$$

$$k_0(t)\frac{d}{dt}k_1(t) - k_1(t)\frac{d}{dt}k_0(t) + k_0(t)k_1(t) \quad \text{Case 11}$$

where $q_1(t) = \int \frac{(p_1(t))^3\beta + u_2(t)2p_1(t)k_1(t)k_0(t)}{k_0(t)k_1(t)} dt + C_2$, $k_0(t)$, $k_1(t)$, $q_2(t)$ are the functions of t , and α , β , C_2 are any constants.

$$u_{11}(x, t) = -\frac{12\beta(k_{-1}(t)(p_2(t) - p_1(t))e^{-\xi_1} + k_1(t)(p_2(t) + p_1(t))e^{\xi_1} + k_0(t)p_2(t))^2}{\alpha(k_{-1}(t)e^{-\xi_1} + k_0(t) + k_1(t)e^{\xi_1})^2} + \frac{12\beta(k_0(t)(p_2(t))^2 + k_1(t)(p_2(t) + p_1(t))^2e^{\xi_1} + (p_1(t) - p_2(t))^2k_{-1}(t)e^{-\xi_1})}{\alpha(k_{-1}(t)e^{-\xi_1} + k_0(t) + k_1(t)e^{\xi_1})} + u_2(t), \quad (291)$$

where $\xi_1 = p_1(t)x + q_1(t)$, $k_{-1}(t)$, $k_0(t)$, $k_1(t)$, $q_1(t)$, $q_2(t)$ are the functions of t , and α , β are any constants.

Case 12

$$u_{12}(x, t) = -\frac{12\beta(k_{-1}(t))^2(p_1(t))^2(e^{-p_1(t)x-q_1(t)})^2}{\alpha(k_{-1}(t)e^{-p_1(t)x-q_1(t)} + k_0(t))^2} + \frac{12k_{-1}(t)(p_1(t))^2e^{-p_1(t)x-q_1(t)}\beta}{\alpha(k_{-1}(t)e^{-p_1(t)x-q_1(t)} + k_0(t))} + u_2(t), \quad (292)$$

$$k_0(t) \frac{d}{dt} k_{-1}(t) - u_2(t) \alpha p_1(t) k_{-1}(t) k_0(t)$$

where $q_1(t) = \int \frac{-k_0(t)(p_1(t))^3 k_{-1}(t) \beta - k_{-1}(t) \frac{d}{dt} k_0(t)}{k_0(t) k_{-1}(t)} dt + C_2$, $k_{-1}(t)$, $k_0(t)$, $q_2(t)$ are the functions of t , and α , β , C_2 are any constants.

Remark 14 From the above application, it is easily seen that our method is more powerful, more convenient, and more general than the method presented by He (He and Wu 2006).

Firstly, our method is convenient to obtain the exact solutions of higher order and higher dimensional NLEEs because we do not use the homogeneous balance method to get the value of c , d , p , q in Eq. (265). Secondly, we can obtain more general exact solutions including non-traveling wave solutions and traveling wave solutions by using our method. For example, the solution (282) of Eq. (267) is a more general exact solution including the non-traveling wave solutions with constant coefficients, the non-traveling wave solutions with variable coefficients, the traveling wave solutions with constant coefficients, and the traveling wave solutions with variable coefficients of Eq. (267).

Example 1 The solution (282) of Eq. (267) includes abundant non-traveling wave solutions with variable coefficients. For example, when we take $C_1 = 1/2$, $C_2 = 1$, $p_2(t) = \cos(t^2)$, $u_2(t) = 1/2$, $p_1(t) = \tanh^2(t)$, $q_1(t) = \sin(t^4)$, $k_{-1}(t) = \cosh(t)$, $k_0(t) = \cos(t^2)$, $\beta = 2$, $\alpha = 3$, we can obtain a non-traveling wave solution $u_{2,1}(x, t)$ with variable coefficients as follows:

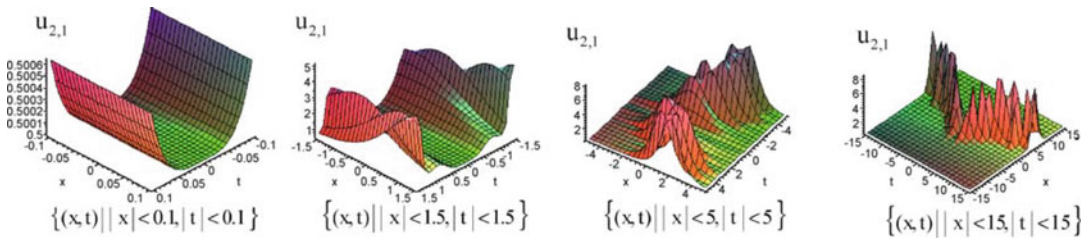
$$u_{2,1}(x, t) = \frac{8 \cos^2(t^2)(2 \cosh(t)e^{-\xi_1} + 1 + 2e^{\xi_1}) + 32 \tanh^2(t) \cos(t^2)(e^{\xi_1} - \cosh(t)e^{-\xi_1}) + 16 \tanh^4(t)(\cosh(t)e^{-\xi_1} + e^{\xi_1})}{2 \cosh(t)e^{-\xi_1} + 1 + 2e^{\xi_1}} - 8 \frac{(2 \tanh^2(t)(e^{\xi_1} - \cosh(t)e^{-\xi_1}) + \cos(t^2)(2 \cosh(t)e^{-\xi_1} + 1 + 2e^{\xi_1}))^2}{(2 \cosh(t)e^{-\xi_1} + 1 + 2e^{\xi_1})^2} + 1/2, \quad (293)$$

where $\xi_1 = \tanh^2(t)x + \sin(t^4)$.

Some figures of the long-term behavior of the $u_{2,1}(x, t)$ from $\{(x, t) | |x| < 0.1, |t| < 0.1, x \in R, t \in R\}$ to $\{(x, t) | |x| < 10^9, |t| < 10^9, x \in R, t \in R\}$ are shown by following Figs. 1 and 2. From Figs. 1 and 2, it is easy to find the global existence, asymptotic behaviors as $t \rightarrow +\infty$ and $|x| \rightarrow +\infty$ and the scattering properties of the solution

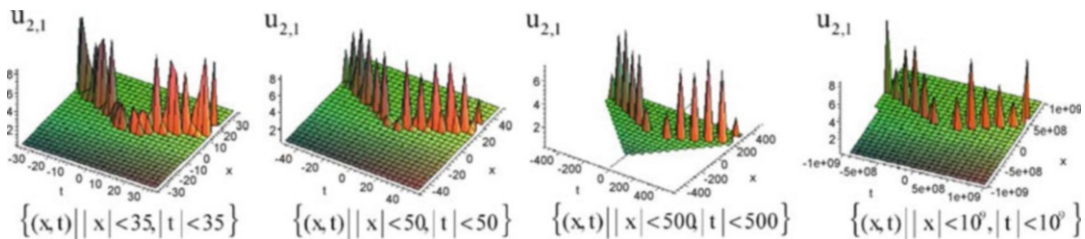
$u_{2,1}(x, t)$ of Eq. (267). It is worth noting that the shape of $u_{2,1}(x, t)$ degenerates gradually from a set of waves to a set of solitons on a trigonal curve as the $|x|$ and $|t|$ in the $u_{2,1}(x, t)$ continue to increase.

Example 2 The solution (282) of Eq. (267) includes abundant non-traveling wave solutions



Different Analytical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 1 The evolution of a non-traveling wave solution $u_{2,1}(x, t)$ with

variable coefficients of Eq. (267) from $\{(x, t) | |x| < 0.1, |t| < 0.1, x \in R, t \in R\}$ to $\{(x, t) | |x| < 15, |t| < 15, x \in R, t \in R\}$



Different Analytical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 2 The evolution of a non-traveling wave solution $u_{2,1}(x, t)$ with

variable coefficients of Eq. (267) from $\{(x, t) | |x| < 35, |t| < 35, x \in R, t \in R\}$ to $\{(x, t) | |x| < 10^9, |t| < 10^9, x \in R, t \in R\}$

with constant coefficients. For example, when we take $k_{-1}(t) = 1/2, \beta = 2, \alpha = 3, k_0(t) = 2, C_2 = 1, C_1 = 1/2, p_1(t) = 1/2, p_2(t) = 1/3, u_2(t) = 1/2$, and

$q_1(t) = \sin(t^4)$, we can obtain a non-traveling wave solution $u_{2,2}(x, t)$ with constant coefficients as follows:

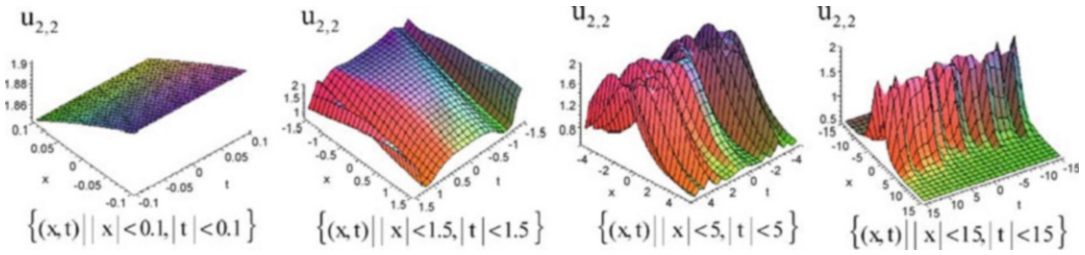
$$u_{2,2}(x, t) = -\frac{2(e^{-x/2 - \sin(t^4)} - 10e^{x/2 + \sin(t^4)} - 2)^2}{9(e^{-x/2 - \sin(t^4)} + 1 + 2e^{x/2 + \sin(t^4)})^2} + \frac{8 + 100e^{x/2 + \sin(t^4)} + 2e^{-x/2 - \sin(t^4)}}{9e^{-x/2 - \sin(t^4)} + 9 + 18e^{x/2 + \sin(t^4)}} + 1/2. \quad (294)$$

Some figures of the long-term behavior of the $u_{2,2}(x, t)$ from $\{(x, t) | |x| < 0.1, |t| < 0.1, x \in R, t \in R\}$ to $\{(x, t) | |x| < 10^9, |t| < 10^9, x \in R, t \in R\}$ are shown by following Figs. 3 and 4.

Example 3 The solution (282) of Eq. (267) includes abundant traveling wave solutions with constant coefficients. For example, when $q_1(t) = \sin(t^4)$ in Eq. (294) is replaced by $q_1(t) = t/3$, we can obtain a traveling wave solution $u_{2,3}(x, t)$ with constant coefficients of Eq. (267) as follows:

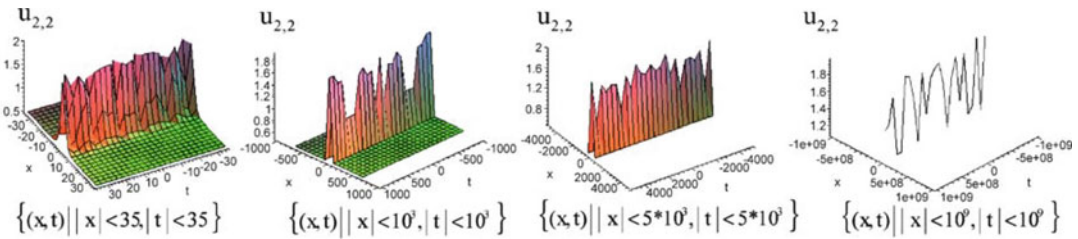
$$u_{2,3}(x, t) = -\frac{2(e^{-x/2 - t/3} - 10e^{x/2 + t/3} - 2)^2}{9(e^{-x/2 - t/3} + 1 + 2e^{x/2 + t/3})^2} + \frac{8 + 100e^{x/2 + t/3} + 2e^{-x/2 - t/3}}{9e^{-x/2 - t/3} + 9 + 18e^{x/2 + t/3}} + 1/2. \quad (295)$$

Some figures of the long-term behavior of the $u_{2,3}(x, t)$ from $\{(x, t) | |x| < 0.1, |t| < 0.1,$



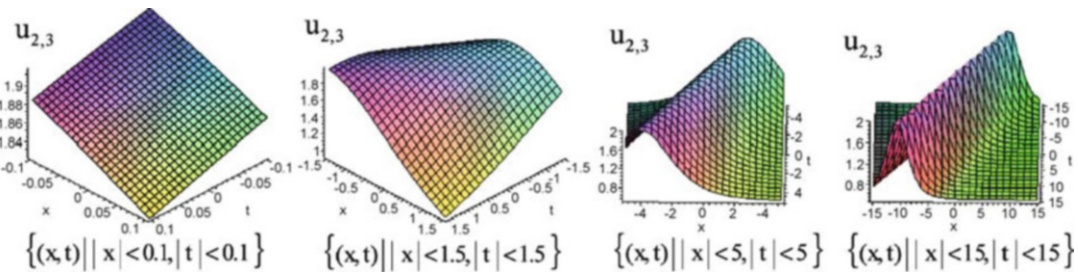
Different Analytical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 3 The evolution of a non-traveling wave solution $u_{2,2}(x, t)$ with

constant coefficients of Eq. (267) from $\{(x, t) | |x| < 0.1, |t| < 0.1, x \in R, t \in R\}$ to $\{(x, t) | |x| < 15, |t| < 15, x \in R, t \in R\}$



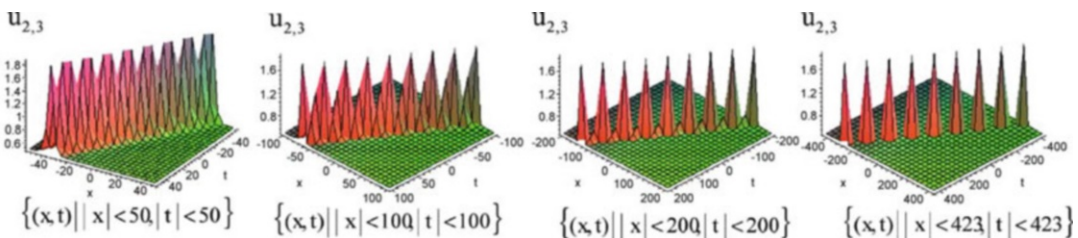
Different Analytical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 4 The evolution of a non-traveling wave solution $u_{2,2}(x, t)$ with

constant coefficients of Eq. (267) from $\{(x, t) | |x| < 35, |t| < 35, x \in R, t \in R\}$ to $\{(x, t) | |x| < 10^9, |t| < 10^9, x \in R, t \in R\}$



Different Analytical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 5 The evolution of a traveling wave solution $u_{2,3}(x, t)$ with constant

coefficients of Eq. (267) from $\{(x, t) | |x| < 0.1, |t| < 0.1, x \in R, t \in R\}$ to $\{(x, t) | |x| < 15, |t| < 15, x \in R, t \in R\}$



Different Analytical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 6 The evolution of a traveling wave solution $u_{2,3}(x, t)$ with

constant coefficients of Eq. (267) from $\{(x, t) | |x| < 50, |t| < 50, x \in R, t \in R\}$ to $\{(x, t) | |x| < 423, |t| < 423, x \in R, t \in R\}$

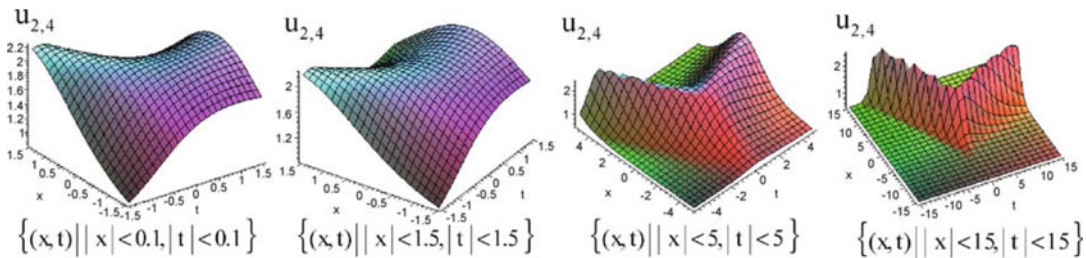
$x \in R, t \in R$ to $\{(x, t) | |x| < 423, |t| < 423, x \in R, t \in R\}$ are shown by following Figs. 5 and 6. It is worth noting that the shape of the $u_{2,3}(x, t)$ degenerates gradually from a wave to a row of solitons on a straight line as the $|x|$ and $|t|$ in $u_{2,3}(x, t)$ continue to increase.

Example 4 The solution (282) of Eq. (267) includes abundant traveling wave solutions with variable coefficients. For example, when $p_1(t) = \tanh^2(t), q_1(t) = \sin(t^4)$ in Eq. (293) is replaced by $p_1(t) = 1/2, q_1(t) = t/3$, we can obtain a traveling wave solution $u_{2,4}(x, t)$ with variable coefficients of Eq. (267) as follows:

$$u_{2,4}(x, t) = \frac{1 + 12 \cosh(t) e^{-x/2-t/3} + 12 e^{x/2+t/3} + 4 e^{x+2t/3} + 72 \cosh(t) + 4 \cosh^2(t) e^{-x-2t/3}}{2(2 \cosh(t) e^{-x/2-t/3} + 1 + 2 e^{x/2+t/3})^2}. \quad (296)$$

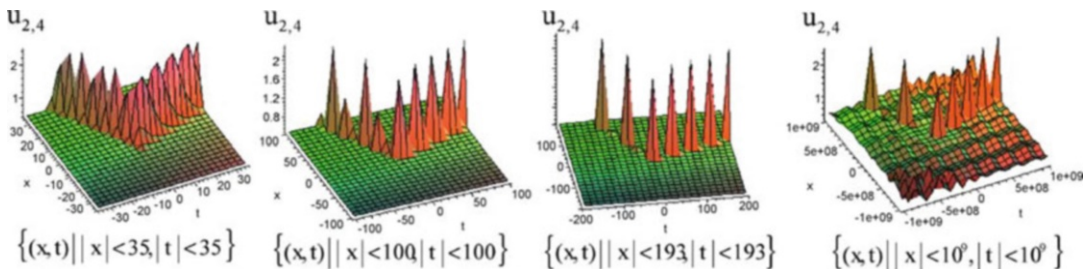
Some figures of the long-term behavior of the $u_{2,4}(x, t)$ from $\{(x, t) | |x| < 0.1, |t| < 0.1, x \in R, t \in R\}$ to $\{(x, t) | |x| < 10^9, |t| < 10^9, x \in R, t \in R\}$ are shown by following Figs. 7 and 8. From Figs. 7 and 8, it is easy to find the global existence, asymptotic behaviors as $t \rightarrow +\infty$ and $|x| \rightarrow +\infty$ and

the scattering properties of the solution $u_{2,4}(x, t)$ of Eq. (267). It is worth noting that the shape of $u_{2,4}(x, t)$ degenerates gradually from a trigonal wave to a set of solitons on a trigonal curve as $|x|$ and $|t|$ in $u_{2,4}(x, t)$ continue to increase. But the plane has become a scraggly surface over $\{(x, t) | |x|$



Different Analytical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 7 The evolution of a traveling wave solution $u_{2,4}(x, t)$ with variable

coefficients of Eq. (267) from $\{(x, t) | |x| < 0.1, |t| < 0.1, x \in R, t \in R\}$ to $\{(x, t) | |x| < 15, |t| < 15, x \in R, t \in R\}$



Different Analytical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 8 The evolution of a traveling wave solution $u_{2,4}(x, t)$ with variable

coefficients of Eq. (267) from $\{(x, t) | |x| < 35, |t| < 35, x \in R, t \in R\}$ to $\{(x, t) | |x| < 10^9, |t| < 10^9, x \in R, t \in R\}$

$<10^9, |t| < 10^9, x \in R, t \in R\}$, namely, the $u_{2,4}(x, t)$ appears a mild disorder.

Remark 15 From the above long-term behavior of the $u_{2,1}(x, t), u_{2,2}(x, t), u_{2,3}(x, t), u_{2,4}(x, t)$, we find easily that the stabilization of the traveling wave solutions $u_{2,3}(x, t)$ with constant coefficients is best as $t \rightarrow +\infty$ and $|x| \rightarrow +\infty$. In addition, we still discover that the longterm behavior of a solution, which includes the global existence, asymptotic behaviors as $t \rightarrow +\infty$ and $|x| \rightarrow +\infty$, and the scattering properties of the solution, depends mainly on the functions that constitute the solution and the selection of all arbitrary constants and the parameters in the solution. We choose other nonlinear evolution equations (Ren and Zhang 2006a; Ren 2007a; Ren et al. 2008b, c) to test and obtain similar results. Therefore, we devise a new guess as follows.

R-guess The long-term behavior of a solution for a given NLEEs (261), which includes the global existence, asymptotic behaviors as $t \rightarrow +\infty$ and $|x| \rightarrow +\infty$, and the scattering properties of the solution, depends mainly on the functions that constitute the solution and the selection of all arbitrary constants and parameters in the solution.

Future Directions

In recent years, directly searching for exact solutions of NLPDEs has become more and more attractive partly due to the availability of computersymbolic systems like Maple or mathematica which allow us to perform complicated and tedious algebraic calculation on a computer, as well as helpingus to find new exact solutions of NLPDEs in mathematical physics. Although many powerful analytical methods for solving NLPDEs have been presented, themethods fail to satisfy the developmental needs in physics, mechanics, chemistry, biology, computer science, etc. There is much new work to be done onanalytical methods for solving NLPDEs. The main future directions are as follows:

1. Researching new and uniform analytical methods based on the ideas of unification

methods, algorithm realization and mechanization for solving NLPDEs (we have done some new work in section “[The Generalized Hyperbolic Function–Bäcklund Transformation Method and Its Application in the \(2 + 1\)-Dimensional KdV Equation](#)” of this entry).

2. Researching mechanization methods and developing their computer software for showing the long-playing traveling state of the exact solutions of NLPDEs (we have done some new work in Ren 2007a).
3. Developing and improving existing analytical methods, computer algebraic systems, and software so that analytical methods can be actualized with complete mechanization by their computer software at a familiar computer (we have done some new work in Ren and Zhang 2006a, b; Ren 2007a, b, c; Ren et al. 2006a, 2007, 2008a, b, c).
4. Researching new analytical and numerical complex methods and their computer software for solving those NLPDEs that can’t make use of analytical methods. The complex methods can be actualized with complete mechanization by their computer software at a familiar computer (we have done some new work in Ren 2007a; Ren et al. 2006b).
5. One can calculate the solutions of NLPDEs as easily as counting or addition by using a small calculator with the development of mathematics and computer science (we have done some new work in Ren and Zhang 2006a, b; Ren 2005, 2007a, b, c; Ren et al. 2006a, b, c, 2007, 2008a, b, c; Wang et al. 2005; Ren and Sun 2004).

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History, Exact N-Soliton Solutions and Further Properties of the Korteweg-de Vries Equation (KdV)

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Article Outline

Glossary

Definition of the Subject

Introduction

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Glossary

Soliton A soliton is a solitary wave which asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves, or more generally, with another (arbitrary) localized disturbance.

Korteweg–de Vries equation In mathematics, the Korteweg-de Vries (KdV) equation is a mathematical model of waves on shallow water surfaces. It is particularly famous as the prototypical example of an exactly solvable model, that is, a nonlinear partial differential equation whose solutions can be exactly and precisely specified.

Inverse scattering transform The identification of the KdV equation as an isospectral flow of the Schrödinger operator enabled Gardner, Greene, Kruskal and Miura (GGKM) to devise a method of solving the KdV equation (with ‘rapidly decreasing’ boundary conditions),

called the inverse scattering or inverse spectral transform (IST). This is a direct generalization of the Fourier transform used to solve linear equations and can be represented by essentially the same scheme.

Hirota bilinear method In 1971, Hirota developed a direct method for finding N -soliton solutions of nonlinear evolution equations, also named Hirota bilinear method. The key step is to transform the equation into a bilinear form, from which we can get soliton solutions successively by means of a kind of perturbational technique.

Definition of the Subject

Among integrable equations is the celebrated KdV equation, which serves as a model equation governing weakly nonlinear long waves whose phase speed attains a simple maximum for waves of infinite length. It motivates us to explore beauty hidden in nonlinear differential (and difference) equations. The equation is named for Diederik Korteweg and Gustav de Vries. The remarkable and exceptional discovery of the inverse scattering transform is one of important developments in the field of applied mathematics, which comes from the study of the KdV equation. There are various algebraic and geometric characteristics that the KdV equation possesses, for example, infinitely many symmetries and infinitely many conserved densities, the Lax representation, bi-Hamiltonian structure, loop group, and the Darboux–Bäcklund transformation. More significantly, many physically important solutions to the KdV equation can be presented explicitly through a simple, specific form, called the Hirota bilinear form.

Introduction

A nice story about the history and the underlying physical properties of KdV equation can be found

at an Internet page of the Herriot-Watt University in Edinburgh (Scotland). The following text is taken from that page:

Over one hundred and fifty years ago, which conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808–1882) made a remarkable scientific discovery. Here his original text as he described it in Russell (1845):

I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now very generally bears.

He followed this observation with a number of experiments during which he determined the shape of a solitary wave to be that of a $\text{sech}^2()$ function. He also determined the relationship of the speed of the wave to its amplitude. At that time there was no equation describing such water waves and having such a solution. Thus John Scott Russell discovered the solution to an as yet unknown equation!

Further investigations were undertaken by Airy, Stokes, Boussinesq and Rayleigh in an attempt to understand this phenomenon. Boussinesq and Rayleigh independently obtained approximate descriptions of the solitary wave; Boussinesq derived a one-dimensional nonlinear evolution equation, which now bears his name, in order to obtain his results.

These investigations provoked much lively discussion and controversy as to whether the

inviscid equations of water waves would possess such solitary wave solutions. The issue was finally resolved by Korteweg and de Vries (1895). After performing a Galilean and a variety of scaling transformations, the KdV equation can be written in simplified form:

$$u_t - 6uu_x - u_{xxx} = 0, \quad (1)$$

where subscripts denote partial differentiations. In particular, if we now assume a solution in the form of a traveling wave $u(x, t) = f(x + ct)$, then Eq. (1) can be integrated: imposing the boundary conditions at large distances that $u(x, t)$ tends to 0 sufficiently fast as $x \rightarrow \pm \infty$, we find the exact solution

$$u(x, t) = \frac{c}{2} \text{sech}^2 \frac{1}{2} \sqrt{c}(x + ct + \delta), \quad (2)$$

where δ is the phase. This clearly represents the solitary wave observed by John Scott Russell and shows that the peak amplitude is exactly half the speed. Thus larger solitary waves have greater speeds. This suggests a numerical experiment: start with two solitary wave solutions, with centers well separated and the larger to the right. Initially, with negligible overlap, they will evolve independently as solitary wave solutions. However, the larger, faster one will start to overtake the smaller and the non linearity will play a significant role. For most dispersive evolution equations these solitary waves would scatter in elastically and lose ‘energy’ to radiation. Not so for the KdV equation: after a fully nonlinear interaction, the solitary waves reemerge, retaining their identities (same speed and form), suffering nothing more than a phase shift (modified δ ’s, representing a displacement of their centers). It was after a similar numerical experiment that Kruskal and Zabusky (Zabusky and Kruskal 1965) coined the name ‘soliton’, to reflect the particle-like behavior of the solitary waves under interaction.

It was not until the mid 1960’s when applied scientists began to use modern digital computers to study nonlinear wave propagation that the soundness of Russell’s early ideas began to be

appreciated. He viewed the solitary wave as a self-sufficient dynamic entity, a “thing” displaying many properties of a particle. From the modern perspective it is used as a constructive element to formulate the complex dynamical behavior of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasmas to shock waves, from tornado’s to the Great Red Spot of Jupiter, from the elementary particles of matter to the elementary particles of thought. For a more detailed and technical account of the solitary wave, see (Bullough 1988; Scott et al. 1973).

Inverse Scattering Transform for the KdV Equation

We now briefly discuss the inverse scattering transform for the KdV Eq. (1). First consider the Miura map $u = -v_x - v^2 + \lambda$, which can be viewed as a Riccati equation for v and thus linearized by the substitution $v = \psi_x/\psi$, giving

$$L\psi \equiv \psi_{xx} + u\psi = \lambda\psi. \quad (3)$$

This is the time-independent Schrödinger equation with $u(x, t)$ playing the role of potential and λ the energy. It is important to realize that t is *not* the time of the time-independent Schrödinger equation. We think of x as the spatial variable and t as a parameter. Considered as a Sturm–Liouville eigenvalue problem it is natural to ask how λ and ψ change with t as $u(x, t)$ evolves from some initial state according to the KdV equation. GGKM (Gardner et al. 1967, 1974) discovered the remarkable fact that the (discrete part of the) spectrum necessarily remains constant in ‘time’ while the

corresponding wave functions ψ evolve according to a very simple *linear differential equation*.

Today (following Lax (1968)) we usually take the opposite route. We postulate that ψ evolves through a linear differential equation:

$$\psi_t = P\psi. \quad (4)$$

Equations (3) and (4) form an overdetermined system, whose integrability conditions can be written:

$$L_t = [P, L] = PL - LP. \quad (5)$$

A consequence of (5) is that all eigenvalues corresponding to the function $u(x, t)$ remain constant. When P is given by:

$$P = -(4\partial^3 + 6u\partial + 3u_x), \quad (6)$$

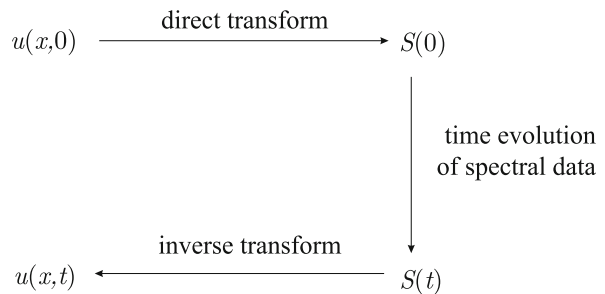
(where ∂ is a differential operator)

(5) reduces to the KdV Eq. (1). Since, under these conditions, the spectrum of L remains constant, the KdV equation is referred to as an *isospectral flow*. Eq. (5) is called the *Lax representation* of the KdV equation.

The identification of the KdV equation as an isospectral flow of the Schrödinger operator enabled GGKM to devise a method of solving the KdV equation (with ‘rapidly decreasing’ boundary conditions), called the inverse scattering or inverse spectral transform (IST). This is a direct generalization of the Fourier transform used to solve linear equations and can be represented by essentially the same scheme (Fig. 1).

For these boundary conditions, the solutions of (3) are characterized by their asymptotic

History, Exact N-Soliton Solutions and Further Properties of the Korteweg-de Vries Equation (KdV),
Fig. 1 Inverse Scattering Transform



properties as $x \rightarrow \pm \infty$. For a given potential function, the continuous and discrete spectrum are treated separately. Corresponding to the continuous spectrum ($\lambda = -k^2$) the solutions are asymptotically oscillatory, characterized by two coefficients:

$$\psi \sim \begin{cases} T(k)e^{-ikx} & x \rightarrow -\infty \\ e^{-ikx} + R(k)e^{ikx} & x \rightarrow +\infty \end{cases} \quad (7)$$

subject to the condition $|R|^2 + |T|^2 = 1$. The constants $R(k)$ and $T(k)$ are respectively called the *reflection and transmission* coefficients, from their quantum mechanical interpretation. Under mild conditions on the potential function, the Schrödinger operator L has only a finite number of discrete eigenvalues $\{\kappa_n^2\}$. The corresponding eigenfunctions are square integrable and are the ‘bound states’ of quantum mechanics with asymptotic properties (for $\kappa_n > 0$):

$$\psi_n \sim \begin{cases} \tilde{c}_n e^{\kappa_n x} & x \rightarrow -\infty \\ c_n e^{-\kappa_n x} & x \rightarrow +\infty \end{cases}. \quad (8)$$

The *direct scattering transform* constructs the quantities $\{T(k), R(k), \kappa_n, c_n\}$ from a given potential function. The important inversion formula were derived by Gel’fand and Levitan in 1955 (1951). These enable the potential u to be constructed out of the spectral or scattering data $S = \{R(k), \kappa_n, c_n\}$. This is considerably more complicated than the Inverse Fourier Transform, involving the solution of a nontrivial integral equation, whose kernel is built out of the scattering data (see (Ablovitz and Clarkson 1991a; Fordy 1990; Newell 1985; Novikov et al. 1984) for descriptions of this).

To solve the KdV equation we first construct the scattering data $S(0)$ from the initial condition $u(x, 0)$. As a consequence of (4) (with an additional constant) with the given boundary conditions, the scattering data evolves in a very simple way. Indeed, we can give explicit formula:

$$\begin{aligned} R(k, t) &= R(k, 0)e^{8ik^3 t}, \quad c_n(t) \\ &= c_n(0)e^{-4\kappa_n^3 t}. \end{aligned} \quad (9)$$

Using the inverse scattering transform on the scattering data $S(t)$, we obtain the potential $u(x, t)$

and thus the solution to the initial value problem for the KdV equation.

This process cannot be carried out *explicitly* for arbitrary initial data, although, in this case, it gives a great deal of information about the solution $u(x, t)$. However, whenever the reflection coefficient is zero, the kernel of Gel’fand-Levitan integral equation becomes separable and explicit solutions can be found. It is in this way that the N -soliton solution is constructed by IST from the initial condition:

$$u(x, 0) = N(N+1)\text{sech}^2 x. \quad (10)$$

The general formula for the multi-soliton solution is given by:

$$u(x, t) = 2(\ln \det M)_{xx}, \quad (11)$$

where M is a matrix built out of the discrete scattering data.

Exact N-soliton Solutions of the KdV Equation

Besides the IST, there are several analytical methods for obtaining solutions of the KdV equation, such as Hirota bilinear method (Hirota 1971, 1980; Hirota and Satsuma 1977), Bäcklund transformation (Wahlquist and Estabrook 1973), Darboux transformation (Matveev and Salle 1991), and so on. The existence of such analytical methods reflects a rich algebraic structure of the KdV equation. In Hirota’s method, we transform the equation into a bilinear form, from which we can get soliton solutions successively by means of a kind of perturbational technique. The Bäcklund transformation is also employed to obtain solutions from a known solution of the concerned equation. In what follows, we will mainly discuss the Hirota bilinear method to derive the N -soliton solutions of the KdV equation.

It is well known that Hirota developed a direct method for finding N -soliton solutions of nonlinear evolution equations. In particular, we shall discuss the KdV bilinear form

$$D_x(D_t + D_x^3)f \cdot f = 0, \quad (12)$$

by the dependent variable transformation

$$u(x, t) = 2(\ln f)_{xx}. \quad (13)$$

Here the Hirota bilinear operators are defined by

$$D_x^m D_t^n a \cdot b = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n a(x, t) b(x', t')|_{x'=x, t'=t}. \quad (14)$$

We expand f as power series in a parameter ε

$$f(x, t) = 1 + f^{(1)}\varepsilon + f^{(2)}\varepsilon^2 + \dots + f^{(j)}\varepsilon^j + \dots \quad (15)$$

Substituting (15) into (12) and equating coefficients of powers of ε gives the following recursion relations for the $f^{(n)}$

$$\varepsilon : f_{xxx}^{(1)} + f_{xt}^{(1)} = 0, \quad (16)$$

$$\varepsilon^2 : f_{xxx}^{(2)} + f_{xt}^{(2)} = -\frac{1}{2}(D_x D_t + D_x^4)f^{(1)} \cdot f^{(1)}, \quad (17)$$

$$\varepsilon^3 : f_{xxx}^{(3)} + f_{xt}^{(3)} = -(D_x D_t + D_x^4)f^{(1)} \cdot f^{(2)}, \quad (18)$$

and so on. N -soliton solutions of the KdV equation are found by assuming that $f^{(1)}$ has the form

$$f^{(1)} = \sum_{j=1}^N \exp(\eta_j), \quad \eta_j = k_j x - \omega_j t + x_{j0}, \quad (19)$$

and k_j , ω_j and x_{j0} are constants, provided that the series (15) truncates.

For $N = 1$, we take

$$f^{(1)} = \exp(\eta_1),$$

and by solving (16) we find that

$$f^{(n)} = 0, \quad \text{for } n = 2, 3, \dots$$

Therefore we have

$$f_1 = 1 + \exp(\eta_1), \quad \omega_1 = -k_1^3,$$

and

$$u(x, t) = \frac{k_1^2}{2} \operatorname{sech}^2 \frac{1}{2}(k_1 x - k_1^3 t + x_{10}). \quad (20)$$

For $N = 2$, the two-soliton solution for the KdV equation is similarly obtained from

$$u(x, t) = 2(\ln f_2)_{xx},$$

where

$$f_2 = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12}). \quad (21)$$

Frequently the N -soliton solutions are obtained as follows

$$f_N = \sum_{\mu=0,1} \exp \left(\sum_{j=1}^N \mu_j \eta_j + \sum_{1 \leq j < l}^N \mu_j \mu_l A_{jl} \right), \quad (22)$$

where the first $\sum_{\mu=0,1}$ means a summation over all possible combinations of $\mu_j = 0, 1$ and $\sum_{1 \leq j < l}^N$ means a summation over all possible pairs (j, l) chosen from the set $\{1, 2, \dots, N\}$, with the condition that $j < l$.

Further Properties of the KdV Equation

Conservation Laws

It was this numerical evidence which prompted Kruskal and his co-workers in Princeton to analytically investigate the KdV equation. Initially, Miura investigated local conservation laws:

$$\partial_t T + \partial_x F = 0, \quad (23)$$

where T and F are polynomials in $u(x, t)$ and its x -derivatives and where ∂_t and ∂_x denote total derivatives. With appropriate boundary conditions this leads to a conserved quantity (constant of the motion). Integrating (23) with respect to x we get:

$$\partial_t \int_A^B T + [F]_A^B = 0. \quad (24)$$

Under periodic (in x) boundary conditions with $A - B$ an integer multiple of the period or with $u(x, t)$ rapidly decreasing as $x \rightarrow \pm \infty$ and $(A, B) = (-\infty, +\infty)$, the square bracket in (24) vanishes and we have the constant of motion $\partial_t \int_A^B T dx$. The quantities T and F are respectively called conserved density and flux. Each T is, in fact, only determined up to an exact x -derivative, so defines an equivalence class of conserved densities:

$$T \sim T + \partial_x S, \quad (25)$$

since this only adds $\partial_t S$ to F and leaves the value of $\int T dx$ unchanged. For the KdV equation the first three are:

$$\begin{aligned} T_0 &= u, & F_0 &= -u_{xx} - 3u^2, \\ T_1 &= \frac{1}{2}u^2, & F_1 &= -uu_{xx} + \frac{1}{2}u_x^2 - 2u^3, \\ T_2 &= u^3 - \frac{1}{2}u_x^2, \\ F_2 &= u_x u_{xxx} - \frac{1}{2}u_{xx}^2 - 3u^2 u_{xx} + 6uu_x^2 - \frac{9}{2}u^4. \end{aligned} \quad (26)$$

The first of these is just the equation itself. These three conservation laws have same physical interpretation, so it was no surprise that they exist. However, Miura discovered several more by direct calculation and was led to the conjecture that there should be *infinitely many*.

In order to ascertain whether the KdV equation was the only such equation with so many conservation laws Miura investigated equations of the form:

$$u_t = u_{xxx} + 6u^n u_x, \quad (27)$$

and found that for $n = 1$ and $n = 2$ (and *only* these values) there existed many conservation laws. With a slight change in notation the second of these, called the modified KdV (MKdV) equation, can be written:

$$v_t = v_{xxx} - 6v^2 v_x = (v_{xx} - 2v^3)_x. \quad (28)$$

The first few conservation laws correspond to conserved densities and fluxes:

$$\begin{aligned} \tilde{T}_{-1} &= v, & \tilde{F}_{-1} &= -v_{xxx} + 2v^3, \\ \tilde{T}_0 &= v^2, & \tilde{F}_0 &= -2vv_{xx} + v_x^2 + 3v^4, \\ \tilde{T}_1 &= \frac{1}{2}(v_x^2 + v^4), & \tilde{F}_1 &= -v_x v_{xxx} + \frac{1}{2}v_{xx}^2 \\ & & & - 2v^3 v_{xx} + 6v^2 v_x^2 + 2v^6. \end{aligned} \quad (29)$$

The Lax Hierarchy

In (Lax 1968) Lax reformulated GGKM's discovery (Gardner et al. 1967, 1974) of the iso-spectral nature of the KdV equation in algebraic form:

$$\left. \begin{aligned} L\psi &\equiv (\partial^2 + u)\psi = \lambda\psi \\ \psi_t &= P\psi \equiv -(4\partial^3 + 6u\partial + 3u_x)\psi \\ \lambda_t &= 0 \end{aligned} \right\} \quad (30)$$

$$\Rightarrow L_t = [P, L] \equiv PL - LP.$$

This led Lax to an interesting generalization:

$$\left. \begin{aligned} L\psi &\equiv (\partial^2 + u)\psi = \lambda\psi \\ \psi_{t_m} &= P_{(m)}\psi \equiv \partial^{2m+1} + \sum_{i=0}^{2m-1} b_i \partial^i \\ \lambda_t &= 0 \end{aligned} \right\} \quad (31)$$

$$\Rightarrow L_{t_m} = [P_{(m)}, L].$$

The integrability condition $L_{t_m} = [P_{(m)}, L]$ is explicitly written as:

$$\begin{aligned} u_{t_m} &= (2m+1)u_x - 2b_{2m-1} \partial^{2m} + \dots \\ &+ (P_{(m)}u - Lb_0). \end{aligned} \quad (32)$$

Equating coefficients of $\partial^i, i = 0, \dots, 2m$, gives us $2m+1$ equations, from which we deduce the $2m$ coefficients $b_0, \dots, b_{2m-1}$:

$$\begin{aligned} b_{2m-1} &= \frac{1}{2}(2m+1)u_x, \\ b_{2m-2} &= \frac{1}{4}(2m+1)(2m-1)u_x, \dots \end{aligned} \quad (33)$$

together with the isospectral flow:

$$u_{t_m} = P(m)u - Lb_0. \quad (34)$$

Nontrivial flows only exist for odd-order equations, since the adjoint of the integrability condition implies $P^\dagger = -P$. There exists an infinite hierarchy of such isospectral flows, the first three of which are:

$$u_{t_0} = u_x, \quad (35)$$

$$u_{t_1} = \frac{1}{4}(u_{xxx} + 6uu_x), \quad (36)$$

$$u_{t_2} = \frac{1}{16} \times (u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x), \quad (37)$$

corresponding respectively to operators:

$$P_{(0)} = \partial, \quad (38)$$

$$P_{(1)n} = \partial^3 + \frac{3}{4}(u\partial + \partial u), \quad (39)$$

$$P_{(2)} = \partial^5 + \frac{5}{4}(u\partial^3 + \partial^3 u) + \frac{5}{16}((3u^2 - u_{xx})\partial + \partial(3u^2 - u_{xx})). \quad (40)$$

Future Directions

In mathematics, the KdV equation is a mathematical model of waves on shallow water surfaces. It is particularly famous as the prototypical example of an exactly solvable model, that is, a nonlinear partial differential equation whose solutions can be exactly and precisely specified. The solutions in turn are the prototypical examples of solitons; these may be found by means of the inverse scattering transform. The mathematical theory behind the KdV equation is rich and interesting, and, in the broad sense, is a topic of active mathematical research.

The KdV equation has several connections to physical problems. In addition to being the

governing equation of the string in the Fermi–Pasta–Ulam problem in the continuum limit, it approximately describes the evolution of long, one-dimensional waves in many physical settings, including shallow-water waves with weakly nonlinear restoring forces, long internal waves in a density-stratified ocean, ion-acoustic waves in a plasma, acoustic waves on a crystal lattice, and more.

The scientists' interest for analytical solutions of the KdV equation stems from the fact that in applying numerical methods to nonlinear partial differential equations, the KdV equation is well suited as a test object, since having an analytical solution statements can be made on the quality of the numerical solution in comparing the numerical result to the exact result. More significantly, the existence of several analytical methods reflects a rich algebraic structure of the KdV equation. It is hoped that the study of the KdV equation could further assist in understanding, identifying and classifying nonlinear integrable differential equations and their exact solutions.

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Semi-analytical Methods for Solving the KdV and mKdV Equations

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Article Outline

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Glossary

Adomian decomposition method, homotopy analysis method, homotopy perturbation method, and variational perturbation method These are some of the semi-analytical/numerical methods for solving Ordinary Differential Equation (in short ODE) or Partial Differential Equation (in short PDE) in literature.

Exact solution A solution to a problem that contains the entire physics and mathematics of a problem, as opposed to one that is approximate, perturbative, closed, etc.

Korteweg-de Vries equation The classical nonlinear equations of interest usually admit for the existence of a special type of the traveling wave solutions, which are either solitary waves or solitons.

Modified Korteweg-de Vries This equation is a modified form of the classical KdV equation in the nonlinear term.

Soliton This concept can be regarded as solutions of nonlinear partial differential equations.

Definition of the Subject

In this study, some semi-analytical/numerical methods are applied to solve the Korteweg-de Vries (KdV) equation and the modified Korteweg-de Vries (mKdV) equation, which are characterized by the solitary wave solutions of the classical nonlinear equations that lead to solitons. Here, the classical nonlinear equations of interest usually admit for the existence of a special type of the traveling wave solutions which are either solitary waves or solitons. These approaches are based on the choice of a suitable differential operator which may be ordinary or partial, linear or nonlinear, and deterministic or stochastic. It does not require discretization and consequently massive computation.

In this scheme, the solution is performed in the form of a convergent power series with easily computable components. This section is particularly concerned with the Adomian decomposition method (ADM), and the results obtained are compared to those obtained by the variational iteration method (VIM), homotopy analysis method (HAM), and homotopy perturbation method (HPM). Some numerical results of these particular equations are also obtained for the purpose of numerical comparisons of those considered approximate methods. The numerical results demonstrate that the ADM is relatively accurate and easily implemented.

Introduction

In this part, a certain nonlinear partial differential equation is introduced which is characterized by the solitary wave solutions of the classical

nonlinear equations that lead to solitons (Debnath 1997, 2007; Drazin and Johnson 1989; Wazwaz 2002; Whitham 1974). The classical nonlinear equations of interest usually admit for the existence of a special type of the traveling wave solutions which are either solitary waves or solitons. In this study, a few solutions arising from the semi-analytical work on the Korteweg-de Vries (KdV) equation and modified Korteweg-de Vries (mKdV) equation will be reviewed.

In this work, a brief history of the above mentioned nonlinear KdV equation is given, and how this type of equation leads to the soliton solutions is then presented as an introduction to the theory of solitons. This theory is an important branch of applied mathematics and mathematical physics. In the last decade, this topic has become an active and productive area of research, and applications of the soliton equations in physical cases have been considered. These have important applications in fluid mechanics, nonlinear optics, ion plasma, classical and quantum field theories, etc.

The best introduction for this study may be the Scottish naval engineer J. Scott Russell's seminal 1844 report to the Royal Society. In the time of the eighteenth century, Scott Russell's report was called "Report on Waves" (Russell 1844). Around 40 years later, starting from Scott Russell's experimental observations in 1934, the theoretical scientific work of Lord Rayleigh and Joseph Boussinesq around 1870 (Rayleigh 1876) independently confirmed Russell's prediction and derived formula 1 from the equation of motion for an inviscid incompressible liquid. They also gave the solitary wave profile $z = u(x, t)$ as

$$u(x, t) = a \operatorname{sech}^2[\beta(x - Ut)] \quad (1)$$

where $\beta^2 = 3a \div \{4 h^2(h + a)\}$ any $a > 0$. They drew the solitary wave profile as in the following (Fig. 1).

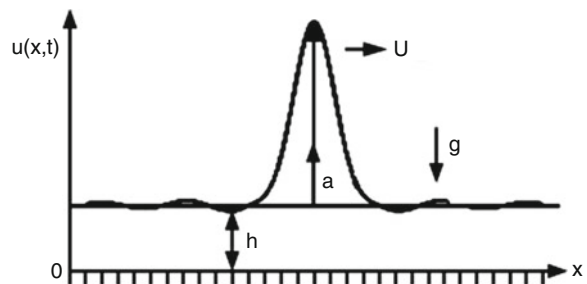
Finally, two Dutch scientists, Korteweg and de Vries, developed a nonlinear partial differential equation to model the propagation of shallow water waves applicable to situation in 1895 (Korteweg and de Vries 1895). This work was really what Scott Russell fortuitously witnessed. This famous classical equation is known simply as the KdV equation. Korteweg and de Vries published a theory of shallow water waves which reduced Russell's observations to its essential features. The nonlinear classical dispersive equation was formulated by Korteweg and de Vries in the form

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} + \varepsilon \frac{\partial^3 u}{\partial x^3} + \gamma u \frac{\partial u}{\partial x} = 0, \quad (2)$$

where c , ε , and γ are physical parameters. This equation now plays a key role in soliton theory.

After setting the theory and applications of the KdV solitary waves, scientists should know a numerical model for such nonlinear equations. This model was constructed and published as a Los Alamos Scientific Laboratory Report in 1955 by Fermi, Pasta, and Ulam (FPU). This was a numerical model of a discrete nonlinear mass-spring system (Fermi et al. 1974). The natural question arises of how the energy would be distributed among all modes in this nonlinear discrete system. The energy would be distributed uniformly among all modes in accordance with the principle of the equipartition of energy stated

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Fig. 1 A solitary wave**



by Fermi, Pasta, and Ulam. This model follows the form of a coupled nonlinear ordinary differential equation system, where w_i is a function of t , and this system is written as

$$\frac{m}{k} \frac{d^2 w_i}{dt^2} = (w_{i+1} + w_{i-1} - 2w_i) + \alpha [(w_{i+1} - w_i)^2 - (w_i - w_{i-1})^2] \quad (3)$$

where w_i is the displacement of the i th mass from the equilibrium position, $i = 1, 2, \dots, n$, $w_0 = w_n = 0$, k is the linear spring constant, and $\alpha (>0)$ measures the strength of nonlinearity.

In the late 1960s, Zabusky and Kruskal numerically studied, and then analytically solved, the KdV equation (Zabusky and Kruskal 1965) from the result of the FPU experiment inspiration. They came to the result that the stable pulse-like waves could exist in a problem described by the KdV equation from their numerical study. A remarkable aspect of the discovered solitary waves is that they retain their shapes and speeds after a collision. After the observation of Zabusky and Kruskal, they called these waves “solitons,” because the character of the waves had a particle-like nature. In fact, they considered the initial value problem for the KdV equation in the form

$$v_t + vv_x + \delta v_{xxx} = 0, \quad (4)$$

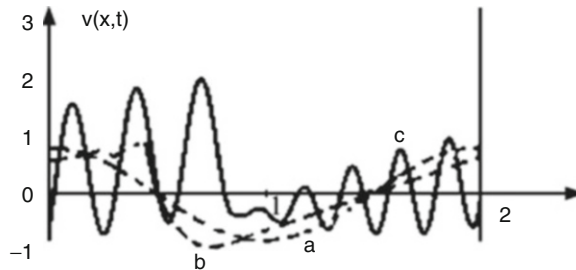
where $\delta = (h/\ell)^2$ and ℓ is a typical horizontal length scale, with the initial condition

$$v(x, 0) = \cos \pi x, \quad 0 \leq x \leq 2 \quad (5)$$

and the periodic boundary conditions with period 2, so that $v(x, t) = v(x + 2, t)$ for all t . Their numerical study with $\sqrt{\delta} = 0.022$ produced a lot of new interesting results, which are shown as follows (Fig. 2).

In 1967, this numerical development was placed on a settled mathematical basis with C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura’s discovery of the inverse scattering transform method (Gardner et al. 1967, 1974). This way of calculation was an ingenious method for finding the exact solution of the KdV equation. Almost simultaneously, Miura (1968) and Miura et al. (1968) formulated another ingenious method to derive an infinite set of conservation laws for the KdV equations by introducing the so-called Miura transformation. Subsequently, Hirota (1971, 1973, 1976) constructed analytical solutions of the KdV equation which provide the description of the interaction among N solitons for any positive integral N .

The soliton concept can be regarded with solutions of nonlinear partial differential equations. The soliton solution of a nonlinear equation is usually used as single wave. If there are several soliton solutions, these solutions are called “solitons.” On the other hand, if a soliton separates infinitely from other solitons, this soliton is a single wave. Besides, a single wave solution cannot be a sech^2 function for equations different from nonlinear equations, such as the KdV equation. But this solution can be sech or $\tan^{-1}(e^{zx})$.



Semi-analytical Methods for Solving the KdV and mKdV Equations, Fig. 2 Development of solitary waves: a initial profile at $t = 0$, b profile at $t = \pi^{-1}$, and c wave profile at $t = (3.6)\pi^{-1}$ (Fermi et al. 1974)

At this stage, one could ask what the definition of a soliton solution is. It is not easy to define the soliton concept. Wazwaz (2002) describes solitons as solutions of nonlinear differential equations such that:

1. A long and shallow water wave should not lose its permanent forms.
2. A long and shallow water wave of the solution is localized, which means that either the solutions decay exponentially to zero such as the solitons admitted by the KdV equation or approach a constant at infinity such as the solitons provided by the sine-Gordon equation.
3. A long and shallow water wave of the solution can interact with other solitons and still preserve its character.

There is also a more formal definition of this concept, but these definitions require substantial mathematics (Edmundson and Enns 1992). On the other hand, the phenomena of the solitons use not quite the same, as have stated above, three properties in all expressions of the solutions. For example, the concept referred to as “light bullets” in nonlinear optics is often called solitons despite losing energy during interaction. This idea can be found at an Internet Web page of the Simon Fraser University, British Columbia, Canada ([Light Bullet Home Page](#)).

Nonlinear phenomena play a crucial role in applied mathematics and physics. Calculating exact and numerical solutions, particularly traveling wave solutions, of nonlinear equations in mathematical physics plays an important role in soliton theory (Debnath 1997; Whitham 1974). It has recently become more interesting to obtain exact solutions of nonlinear partial differential equations utilizing symbolical computer programs (such as Maple, Matlab, and Mathematica) which facilitate complex and tedious algebraical computations. It is also important to find exact solutions of nonlinear partial differential equations. These equations exist because of the mathematical models of complex physical phenomenon that arise in engineering, chemistry, biology, mechanics, and physics. Various

effective methods have been developed to understand the mechanisms of these physical models, to help physicians and engineers, and to ensure knowledge for physical problems and its applications.

Many explicit exact methods have been introduced in literature (Debnath 1997, 2007; Drazin and Johnson 1989; Inan and Kaya 2006, 2007; Khater et al. 1997; Ugurlu and Kaya 2008; Wazwaz 2002, 2007a, b, c; Wazwaz and Helal 2004; Whitham 1974). Some of them are Bäcklund transformation, Cole-Hopf transformation, generalized Miura transformation, inverse scattering method, Darboux transformation, Painleve method, similarity reduction method, tanh (and its variations) method, homogeneous balance method, Exp-function method, sine-cosine method, and so on. There are also many numerical methods implemented for nonlinear equations (Geyikli and Kaya 2005a, b; Helal and Mehanna 2007; Kaya 2003, 2006; Kaya and Al-Khaled 2007; Kaya and El-Sayed 2003a, b, c; Polat et al. 2006; Shawagfeh and Kaya 2004). Some of them are finite difference methods, finite element method, sinc-Galerkin method, and some approximate or semi-analytical/numerical methods such as Adomian decomposition method, variational analysis method, homotopy analysis method, homotopy perturbation method, and so on.

An Analysis of the Semi-Analytical Methods and their Applications

The recent years have seen a significant development in the use of various methods to find the numerical and semi-analytical/numerical solution of a linear or nonlinear or deterministic or stochastic ODE or PDE. In this work, we will only discuss the nonlinear PDE case of the problem.

Before giving the semi-analytical and numerical implementations of the considered methods, some conclusions may be drawn from Liao's book (Liao 2003) and the recent papers of Abbasbandy (2006), Chowdhury et al. (2009), and Domairry and Nadim (2008) for comparisons of the ADM, HAM, HPM, and VIM. ADM can be applied to

solve linear or nonlinear ordinary and partial differential equations, no matter whether they contain small/large parameters, and thus is rather general. In some cases, the Adomian decomposition series converge rapidly. However, this method has some restrictions, e.g., if you are solving nonlinear equation, you have to sort out Adomian polynomials. In the same case, this needs to take more calculations. Liao (2003) pointed out that “in general, convergence regions of power series are small, thus acceleration techniques are often needed to enlarge convergence regions. This is mainly due to the fact that a power series is often not an efficient set of base functions to approximate a nonlinear problem, but unfortunately ADM does not provide us with freedom to use different base functions. Like the artificial small parameter method and the ϵ -expansion method, ADM itself also does not provide us with a convenient way to adjust the convergence region and the rate of approximation solutions.”

Many authors (Abbasbandy 2006; Chowdhury et al. 2009; Domairry and Nadim 2008) made their own studies in which they gave implementations of equations by using HAM and HPM and then found numerical solutions. Their results show that (i) Liao’s HAM can produce much better approximations than the previous solutions for nonlinear differential equations. (ii) They compared the approximations of these two methods, and they observe that although HAM is faster than HPM (see Figs. 1–6 in Domairry and Nadim (2008)), both of the methods converge to the exact solution quite fast. HAM and HPM are in some cases similar to each other; e.g., the obtained solution function of the equation is shorter if HPM is used instead of HAM. The corresponding results converge more rapidly if HAM is used. (iii) The other advantage of the HAM is that it gives the flexibility to choose an auxiliary parameter h to adjust and control the convergence and its rate for the solution series and the defined different functions which originate from the nature of the considered problems (Abbasbandy 2006). (iv) If one compares the approximated numerical solution with the corresponding exact solution by using HAM and HPM, one can see that when the small parameter

h is increased, the error of the first method is less than the second one.

Chowdhury et al. (2009) show that their obtained numerical results by the 5-term HAM are exactly the same as the ADM solutions and HPM solutions for the special case of the auxiliary parameter $h = -1$ and auxiliary function $H(x) = 1$. Because of this conclusion, they admitted that the HPM and ADM are special cases of HAM.

Adomian Decomposition Method

The aim of the present section is to give an outline and implementation of the Adomian decomposition method (ADM) for nonlinear wave equations and to obtain analytic and approximate solutions which are obtained in a rapidly convergent series with elegantly computable components by this method. The approach is based on the choice of a suitable differential operator which may be ordinary or partial, linear or nonlinear, and deterministic or stochastic (Adomian 1988; Adomian and Rach 1992; Adomian 1994; Wazwaz 1997, 2002). It allows one to obtain a decomposition series solution of the equation which is calculated in the form of a convergent power series with easily computable components.

The inhomogeneous problem is quickly solved by observing the self-canceling “noise” terms, where the sum of the components vanishes in the limit. Many tests which model problems from mathematical physics, linear and nonlinear, are discussed to illustrate the effectiveness and the performance of the ADM. Adomian and Rach (1992) and Wazwaz (1997) have investigated the phenomena of the self-canceling “noise” terms where the sum of the components vanishes in the limit. An important observation was made that the “noise” terms appear for nonhomogeneous cases only. Further, it was formally justified that if terms in u_0 are canceled by terms in u_1 , even though u_1 includes further terms, then the remaining non-canceled terms in u_1 constitute the exact solution of the equation.

ADM is valid for ordinary and partial differential equations, whether or not they contain small/large parameters, and thus is rather general. Moreover, the Adomian approximation series converge quickly. However, this method has some restrictions. Approximate solutions given by ADM often contain polynomials. In general, convergence regions of power series are small; thus acceleration techniques are often needed to enlarge convergence regions. This is mainly due to the fact that power series is often not an efficient set of base functions to approximate a nonlinear problem, but unfortunately ADM does not provide us with freedom to use different base functions.

An outline of the method will be given here in order to obtain analytic and approximate solutions by using the ADM. Considering a generalized KdV equation (Kaya and Inan 2005):

$$u_t + \alpha u^m u_x + u_{xxx} = 0, \quad u(x, 0) = g(x), \quad (6)$$

where α and $m > 0$ are constants. An operator form of this equation can be written as

$$L_t(u) + \alpha Nu + L_{xxx}(u) = 0, \quad (7)$$

where $L_t \equiv \partial/(\partial t)$, $Nu = u^m u_x$, and $L_{xxx} \equiv \partial^3/(\partial x^3)$. It is assumed that L_t^{-1} is an integral operator given by $L_t^{-1} \equiv \int_0^t (\cdot) dt$. Operating with the integral operator L_t^{-1} on both sides of Eq. 7 with

$$L_t^{-1} L_t(u) = -\alpha L_t^{-1}(Nu) - L_t^{-1} L_{xxx}(u). \quad (8)$$

Therefore, it follows that

$$u(x, t) = u(x, 0) - \alpha L_t^{-1}(Nu) - L_t^{-1} L_{xxx}(u).$$

We find that the zeroth component is obtained by

$$u_0 = u(x, 0), \quad (9)$$

which is defined by all terms that arise from the initial condition and from integrating the source term and decomposing the unknown function $u(x, t)$ as a sum of components defined by the decomposition series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \quad (10)$$

The nonlinear term $u^m u_x$ can be decomposed into the infinite series of polynomials given by

$$Nu = u^m u_x = \sum_{n=0}^{\infty} A_n,$$

where the components $u_i(x, t)$ will be determined recurrently and the A_n polynomials are the so-called Adomian polynomials (Wazwaz 2002) of $u_0, u_1, u_2, \dots, u_i$ defined by

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \Phi \left(\sum_{k=1}^{\infty} \lambda^k u_k \right) \right] \bigg|_{\lambda=0}, \quad n \geq 0. \quad (11)$$

Substituting Eqs. (11) and (9) into Eq. (8) gives rise to

$$u_{n+1} = -L_t^{-1}(A_n) - L_t^{-1} L_{xxx}(u_n), \quad n \geq 0 \quad (12)$$

where L_t^{-1} is the previously given integration operator. The solution $u(x, t)$ must satisfy the requirements imposed by the initial conditions. The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques. To give a clear overview of the methodology, the following examples will be discussed.

Example 1

Let us consider the mKdV Eq. (6) for the value of $m = 1$, which is called the classical KdV equation, and take this equation with the following initial value condition (Kaya 2007):

$$u(x, 0) = \frac{3\lambda}{\alpha} - \frac{3\lambda}{\alpha} \tanh^2 \left(\frac{\sqrt{\lambda}}{2} x \right), \quad (13)$$

and then basically Eq. (6) is taken in an operator form exactly in the same manner as the form of Eq. (6) and using Eq. (9) to find the zeroth component of $u_0 = u(x, 0)$. Taking into consideration Eq. (12) with Eq. (11), one can calculate the rest of

the terms of the series with the aid of Mathematica and then write these terms to Eq. (10), yielding.

$$\begin{aligned}
 u(x, t) = & \frac{1}{4\alpha} \left\{ 3t^2 \lambda^4 \left(-2 + \cosh(x\sqrt{\lambda}) \right) \sec h^4 \left(\frac{x\sqrt{\lambda}}{2} \right) \right\} \\
 & + \frac{1}{64\alpha} \left\{ t^4 \lambda^7 \left(-33 - 26 \cosh(x\sqrt{\lambda}) \right) \right. \\
 & \left. + \cosh(2x\sqrt{\lambda}) \right\} \sec h^6 \left(\frac{x\sqrt{\lambda}}{2} \right) \left\{ \right. \\
 & + \frac{1}{8\alpha} \left\{ t^3 \lambda^{\frac{11}{2}} \sec h^5 \left(\frac{x\sqrt{\lambda}}{2} \right) \left(-11 \sin h \left(\frac{x\sqrt{\lambda}}{2} \right) \right. \right. \\
 & \left. \left. + \sin h \left(\frac{3x\sqrt{\lambda}}{2} \right) \right) \right\} \\
 & + \frac{1}{640\alpha} \left\{ t^5 \lambda^{\frac{7}{2}} \sec h^7 \left(\frac{x\sqrt{\lambda}}{2} \right) \left(302 \sin h \left(\frac{x\sqrt{\lambda}}{2} \right) \right. \right. \\
 & \left. \left. - 57 \sin h \left(\frac{3x\sqrt{\lambda}}{2} \right) + \sin h \left(\frac{5x\sqrt{\lambda}}{2} \right) \right) \right\} \\
 & + \frac{1}{\alpha} \left\{ 9t \lambda^{\frac{5}{2}} \sec h^2 \left(\frac{x\sqrt{\lambda}}{2} \right) \tan h \left(\frac{x\sqrt{\lambda}}{2} \right) \right\} \\
 & - \frac{1}{\alpha} \left\{ 6t \lambda^{\frac{5}{2}} \sec h^4 \left(\frac{x\sqrt{\lambda}}{2} \right) \tan h \left(\frac{x\sqrt{\lambda}}{2} \right) \right\} \\
 & - \frac{1}{\alpha} \left\{ 6t \lambda^{\frac{5}{2}} \sec h^2 \left(\frac{x\sqrt{\lambda}}{2} \right) \tan h^3 \left(\frac{x\sqrt{\lambda}}{2} \right) \right\} \\
 & + \frac{1}{\alpha} \left\{ 3\lambda \left(1 - \tan h^2 \left(\frac{x\sqrt{\lambda}}{2} \right) \right) \right\} + \dots
 \end{aligned}$$

Example 2

Consider the mKdV Eq. (6) with the values of $m = 2$ and $\alpha = 6$; it has the traveling wave solution which can be obtained subject to the initial condition (Kaya and Inan 2005)

$$u(x, 0) = \sqrt{c} \operatorname{sech}[k + \sqrt{c}x] \quad (14)$$

for all $c \geq 0$, where k is an arbitrary constant.

Again, to find the solution of this equation, simply take the equation in an operator form exactly in the same manner as the form of Eq. 6 and use Eq. 12 to find the zeroth component of

$u_0 = u(x, 0)$ and obtain sequential terms by using Eq. 12 with Eq. 11 to determine other individual terms of the decomposition series with the aid of Mathematica to get

$$\begin{aligned}
 u(x, t) = & \sqrt{c} \sec h(k + \sqrt{c}x) \\
 & + \frac{1}{4} c^{\frac{7}{2}} t^2 (-3 + \cosh(2k + 2\sqrt{c}x)) \sec h^3(k + \sqrt{c}x) \\
 & + \frac{1}{192} c^{\frac{13}{2}} t^4 (115 - 76 \cosh(2k + 2\sqrt{c}x) \\
 & + \cosh(4k + 4\sqrt{c}x)) \sec h^5(k + \sqrt{c}x) \\
 & + \frac{1}{23040} \left\{ c^{\frac{19}{2}} t^6 (-11774 + 10543 \cosh(2k + 2\sqrt{c}x) \right. \\
 & - 722 \cosh(4k + 4\sqrt{c}x) + \cosh(6k + 6\sqrt{c}x)) \\
 & \times \sec h^7(k + \sqrt{c}x) \left. \right\} \\
 & + \frac{1}{4} c^5 t^3 \sec h^4(k + \sqrt{c}x) (-23 \sin h(k + \sqrt{c}x) \\
 & + \sin h(3k + 3\sqrt{c}x) \\
 & + \frac{1}{1920} \{ c^8 t^5 \sec h^6(k + \sqrt{c}x) (1682 \sin h(k + \sqrt{c}x) \\
 & - 237 \sin h(3k + 3\sqrt{c}x) + \sin h(5k + 5\sqrt{c}x)) \} \\
 & + \frac{1}{322560} \{ c^{11} t^7 \sec h^8(k + \sqrt{c}x) \\
 & (-259723 \sin h(k + \sqrt{c}x) + 60657 \sin h(3k + 3\sqrt{c}x) \\
 & - 2179 \sin h(5k + 5\sqrt{c}x) + \sin h(7k + 7\sqrt{c}x)) \} \\
 & + c^2 t \sec h^3(k + \sqrt{c}x) \tan h(k + \sqrt{c}x) \\
 & + c^2 t \sec h(k + \sqrt{c}x) \tan h^3(k + \sqrt{c}x) + \dots
 \end{aligned}$$

Homotopy Analysis Method

In this section, the homotopy analysis method (HAM) (Liao 1992, 2003) is considered. HAM has been constructed and successfully implemented as an approximate and numerical solution for many types of nonlinear problems (Hayat et al. 2006; Hayat and Sajid 2007; Liao 1995, 2003, 2004a, b, 2005; Sajid and Hayat 2008a, b; Sajid et al. 2007) and the references cited therein. A very nice explanation of the basic ideas of the HAM, its relationships with other analytic techniques, and some of its applications in science and engineering is given in Liao's book (Liao 2003). There are many groups of methods similar to HAM in the literature which are implemented for nonlinear problems. Most of these groups of methods are in principle based on a Taylor series in an embedding parameter. If one could guess the initial function and the auxiliary linear operator well, then one can get very good approximations in a few terms, especially for a small value of the variable of the series.

For the purpose of the illustration of the HAM (Liao 1992), the mKdV equation is written in the operator form as

$$L(u) + \alpha u^m u_x + u_{xxx} = 0, \quad (15)$$

where L is a linear operator: $L \equiv \partial/(\partial t)$. Equation 15 can be written in a nonlinear operator form as

$$\begin{aligned} N(\varphi(x, t; q)) &= \frac{\partial \varphi(x, t; q)}{\partial t} \\ &+ \alpha \varphi^m(x, t; q) \frac{\partial \varphi(x, t; q)}{\partial x} \\ &+ \frac{\partial^3 \varphi(x, t; q)}{\partial x^3}, \end{aligned} \quad (16)$$

where $q \in [0, 1]$ is an embedding parameter and $\varphi(x, t; q)$ is a function.

From $u(x, 0) = U_0(x)$, $-\infty < x < \infty$, it is straightforward to express the solution u by a set of base functions

$$\{e_n(x)t^n, n \geq 0\},$$

where $e_n(x)$ as a coefficient is a function with respect to x . This provides us with the so-called rule of solution expression.

Following Liao's method (Liao 1992, 2003), let $u(x, 0) = U_0(x)$ indicate an initial guess of the exact solution u ; $h \neq 0$, an auxiliary parameter; and $H(x, t) \neq 0$, an auxiliary function. A zero-order deformation equation is constructed as

$$\begin{aligned} (1 - q)L[\varphi(x, t; q) - u_0(x, t)] \\ = qhH(x, t)N[\varphi(x, t; q)], \end{aligned} \quad (17)$$

with the initial condition

$$\varphi(x, 0; q) = U_0(x). \quad (18)$$

When $q = 0$ and 1, the above equation has the solution

$$\varphi(x, t; 0) = u_0(x, t) \quad (19)$$

and

$$\varphi(x, t; 1) = u(x, t), \quad (20)$$

respectively.

Assume the auxiliary function $H(x, t)$ and the auxiliary parameter h are properly chosen so that $\varphi(x, t; q)$ can be expressed by the Taylor series

$$\varphi(x, t; q) = u_0(x, t) + \sum_{n=1}^{\infty} u_n(x, t)q^n, \quad (21)$$

where

$$u_n(x, t) = \frac{1}{n!} \left. \frac{\partial^n \varphi(x, t; q)}{\partial q^n} \right|_{q=0} \quad (22)$$

and that the above series is convergent at $q = 1$. Equations 19 and 20 then yield

$$u(x, t) = u_0(x, t) + \sum_{n=1}^{\infty} u_n(x, t). \quad (23)$$

For the sake of simplicity, define the vectors

$$\vec{u}_n(x, t) = \{u_0(x, t), u_1(x, t), \dots, u_n(x, t)\} \quad (24)$$

differentiating the zero-order deformation Eq. 17 n times with respect to the embedding parameter q , then setting $q = 0$, and finally dividing by $n!$, the n th-order deformation equation is written as

$$\begin{aligned} L[u_n(x, t) - \chi_n u_{n-1}(x, t)] \\ = hH(x, t)R_n \left[\vec{u}_{n-1}(x, t) \right], \end{aligned} \quad (25)$$

where

$$R_n \left[\vec{u}_{n-1}(x, t) \right] = \frac{1}{(n-1)!} \left\{ \frac{\partial^{n-1}}{\partial q^{n-1}} N \left[\sum_{m=0}^{\infty} u_m(x, t) q^m \right] \right\} \Big|_{q=0} \quad (26)$$

and

$$\chi_n = \begin{cases} 0, & n \leq 1 \\ 1, & n > 1 \end{cases} \quad (27)$$

with the initial condition

$$u_n(x, 0) = 0, \quad n \geq 1. \quad (28)$$

Therefore, the n th-order approximation of $u(x, t)$ is given by

$$u(x, t) \approx u_0(x, t) + \sum_{m=1}^N u_m(x, t). \quad (29)$$

Example 3

Let us consider the mKdV Eq. (6) for the value of $m = 1$, which is called the classical KdV equation, and take this equation with the following initial value condition:

$$u(x, 0) = u_0 = \frac{3\lambda}{\alpha} \left(1 - \tanh^2 \left(\frac{\sqrt{\lambda}}{2} x \right) \right). \quad (30)$$

All related formulae are the same as those given from Eq. 25:

$$R_n[u_{n-1}] = \frac{\partial u_{n-1}}{\partial t} + \alpha \sum_{i=0}^{n-1} u_i \frac{\partial u_{n-1-i}}{\partial x} + \frac{\partial^3 u_{n-1}}{\partial(31)x^3}. \quad (31)$$

Using Eqs. 24 and 25 with Eq. 31 and the initial function (Eq. 30), three terms of the series (Eq. 29) can be calculated as

$$\begin{aligned} u(x, t) = & \frac{3\lambda}{\alpha} \left(1 - \tanh^2 \left(\frac{\sqrt{\lambda}}{2} x \right) \right) \\ & - \frac{3ht\lambda^{\frac{5}{2}} \operatorname{sech}^2 \left(\frac{x\sqrt{\lambda}}{2} \right) \tanh \left(\frac{x\sqrt{\lambda}}{2} \right)}{\alpha} \\ & + \frac{1}{4\alpha} \left\{ 3ht\lambda^{\frac{5}{2}} \operatorname{sech}^4 \left(\frac{x\sqrt{\lambda}}{2} \right) \right. \\ & \times \left(-2 + \cosh(x\sqrt{\lambda}) \right) - 2(1+h) \sinh(x\sqrt{\lambda}) \Big\} \\ & + \frac{1}{8\alpha} \left\{ ht\lambda^{\frac{5}{2}} \operatorname{sech}^2 \left(\frac{x\sqrt{\lambda}}{2} \right) \left\{ 12h(1+h)t\lambda^{\frac{3}{2}} \right. \right. \\ & \times \left(-2 + \cosh(x\sqrt{\lambda}) \right) \operatorname{sech}^2 \left(\frac{x\sqrt{\lambda}}{2} \right) \\ & - h^2 t^2 \lambda^3 \operatorname{sech}^3 \left(\frac{x\sqrt{\lambda}}{2} \right) \times \left(-11 \sinh \left(\frac{x\sqrt{\lambda}}{2} \right) \right. \\ & \left. \left. + \sinh \left(\frac{3x\sqrt{\lambda}}{2} \right) \right) - 24(1+h)^2 \tanh \left(\frac{x\sqrt{\lambda}}{2} \right) \right\} \Big\} \\ & + \frac{1}{64\alpha} \left\{ ht\lambda^{\frac{5}{2}} \operatorname{sech}^2 \left(\frac{x\sqrt{\lambda}}{2} \right) \times \left\{ 144h(1+h)^2 t\lambda^{\frac{3}{2}} \right. \right. \\ & \times \left(-2 + \cosh(x\sqrt{\lambda}) \right) \times \operatorname{sech}^2 \left(\frac{x\sqrt{\lambda}}{2} \right) \\ & + h^3 t^3 \lambda^{\frac{9}{2}} \left(33 - 26 \cosh(x\sqrt{\lambda}) + \cosh(2x\sqrt{\lambda}) \right) \\ & \times \operatorname{sech}^4 \left(\frac{x\sqrt{\lambda}}{2} \right) - 24h^2(1+h)t^2 \lambda^3 \times \operatorname{sech}^2 \left(\frac{x\sqrt{\lambda}}{2} \right) \\ & \times \left(-11 \sinh \left(\frac{x\sqrt{\lambda}}{2} \right) + \sinh \left(\frac{3x\sqrt{\lambda}}{2} \right) \right) \\ & \left. \left. - 192(1+h)^3 \tanh \left(\frac{x\sqrt{\lambda}}{2} \right) \right\} \right\} + \dots \end{aligned}$$

Example 4

In this example, the mKdV Eq. (6) is considered for the values of $m = 2$ and $\alpha = 6$. It has the traveling wave solution which can be obtained subject to the initial condition

$$u(x, 0) = \sqrt{c} \operatorname{sech}[k + \sqrt{c}x]. \quad (32)$$

For implementation of the HAM for this particular equation with the initial function (Eq. 32), all related formulae are the same as those given from Eqs. 15 and 25 with the following $R_n[u_n - 1]$:

$$R_n[u_{n-1}] = \frac{\partial u_{n-1}}{\partial t} + \alpha \sum_{i=0}^{n-1} u_i u_{i-j} \frac{\partial u_{n-1-i}}{\partial x} + \frac{\partial^3 u_{n-1}}{\partial x^3}. \quad (33)$$

Using Eqs. 24 and 25 with Eq. 33 by with the initial function (Eq. 32), we have

$$\begin{aligned} u(x, t) = & \sqrt{c} \operatorname{sech}[k + \sqrt{cx}] \\ & - c^2(33)ht \operatorname{sech}[k + \sqrt{cx}] \tanh[k + \sqrt{cx}] \\ & + \frac{1}{4} c^2 ht \operatorname{sech}[k + \sqrt{cx}] \\ & \times \left(2c^{\frac{3}{2}} h t (1 - 2 \operatorname{sech}^2(k + \sqrt{cx})) \right. \\ & \left. - 4(1 + h) \tanh(k + \sqrt{cx}) \right) \\ & + \frac{1}{24} c^2 ht \operatorname{sech}(k + \sqrt{cx}) \\ & \times \left(12c^{\frac{3}{2}} h (1 + h) t (-3 + \cosh(2(k + \sqrt{cx}))) \right) \\ & \times \sec h^2(k + \sqrt{cx}) + c^3 h^2 t^2 \operatorname{sech}^3(k + \sqrt{cx}) \\ & \times (23 \sin h(k + \sqrt{cx}) - \sin h(3(k + \sqrt{cx}))) \\ & - 24(1 + h)^2 \tanh(k + \sqrt{cx}) + \dots \end{aligned}$$

which are the three terms of the approximate series solution (29).

Homotopy Perturbation Method

In 1998, Liao's early idea to construct the one-parameter family of equations was introduced by He's so-called homotopy perturbation method (He 1997, 1999a, 2000, 2003), which is only a special case of the homotopy analysis method (Liao 1995). But this homotopy perturbation method is almost the same as Liao's early one-parameter family equation which can be found in Liao (1992). There are some minor differences, and

detailed discussions and proofs can be seen in literature (He 1997, 1999a, 2000, 2003). The author (He 1997, 1999a, 2000, 2003) has proved that these two methods lead to the same solutions of the considered equation for high-order approximations. This is also mainly because of the Taylor series properties. In conclusion of this, nothing is new in He's idea, except the new name "homotopy perturbation method" (Sajid and Hayat 2008a).

To illustrate HPM, consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \quad (34)$$

with the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \quad (35)$$

where $A(u)$ is written as follows:

$$A(u) = L(u) + N(u), \quad (36)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a given analytic right-hand side function, and Γ is the boundary of the domain Ω . The operator A can be generally divided into two parts L and N , where L is linear and N is the nonlinear term. So, Eq. 34 can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \quad (37)$$

By the homotopy technique (Liao 1995), a homotopy $v(r, p)$: $\Omega \times [0, 1] \rightarrow \Re$ is obtained which satisfies

$$\begin{aligned} H(v, p) = & (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] \\ = & 0, p \in [0, 1], \quad r \in \Omega, \end{aligned} \quad (38)$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation of Eq. 34 which satisfies the boundary conditions. Obviously, from Eq. 38 the result will be

$$\begin{aligned} H(v, 0) &= L(v) - L(u_0) = 0, \\ H(v, 1) &= A(v) - f(r) = 0, \end{aligned}$$

changing the process of p from zero to unity just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy.

We consider v as the following:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots \quad (39)$$

According to HPM, the best approximation solution of Eq. 37 can be explained as a series of the power of p :

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + v_3 + \dots \quad (40)$$

The above convergence is given in He (1999a). Some results have been discussed in He (1997, 2000, 2003).

Example 5

Here, the mKdV Eq. (6) is again considered for the value of $m = 1$, which is called classical KdV equation; first, we construct a homotopy as follows:

$$(1 - p) [\dot{Y} - \dot{u}_0] + p [\dot{Y} + \alpha YY' + Y'''] = 0, \quad (41)$$

where

$$\dot{Y} = \frac{\partial Y}{\partial t}, Y' = \frac{\partial Y}{\partial x}, Y''' = \frac{\partial^3 Y}{\partial x^3} \quad \text{and } p \in [0, 1].$$

With the initial approximation

$$Y_0 = u_0 = \frac{3\lambda}{\alpha} - \frac{3\lambda}{\alpha} \tanh^2\left(\frac{\sqrt{\lambda}}{2}x\right),$$

suppose the solution of Eq. 41 has the form

$$\begin{aligned} Y &= Y_0 + pY_1 + p^2Y_2 + p^3Y_3 + \dots \\ &= \sum_{n=0}^{\infty} p^n Y_n(x, t). \end{aligned} \quad (42)$$

Substituting Eq. 42 into Eq. 41 and arranging the coefficients of “ p ” powers, we have

$$p^0 : \dot{Y}_0 - \dot{u}_0 = 0, \quad (43)$$

$$p^1 : \dot{Y}_1 + \dot{u}_0 + \alpha Y_0 Y_0' + Y_0''' = 0, \quad (44)$$

$$p^2 : \dot{Y}_2 + \alpha Y_0 Y_1' + \alpha Y_1 Y_0' + Y_1''' = 0, \quad (45)$$

$$\begin{aligned} p^3 : \dot{Y}_3 + \alpha Y_0 Y_2' + \alpha Y_1 Y_1' + \alpha Y_2 Y_0' + Y_2''' \\ = 0, \end{aligned} \quad (46)$$

$\vdots$

and finally using Mathematica, the solutions of the equation can be obtained as follows:

$$Y_0 = \frac{3\lambda}{\alpha} - \frac{3\lambda}{\alpha} \tanh^2\left(\frac{\sqrt{\lambda}}{2}x\right), \quad (47)$$

$$Y_1 = \frac{3t\lambda^{\frac{5}{2}}}{\alpha} \sec^2 h^2\left(\frac{x\sqrt{\lambda}}{2}\right) \tanh\left(\frac{x\sqrt{\lambda}}{2}\right) \quad (48)$$

$$Y_2 = \frac{3t^2\lambda^4}{4\alpha} \left(-2 + \cosh(x\sqrt{\lambda})\right) \operatorname{sech}^4\left(\frac{x\sqrt{\lambda}}{2}\right), \quad (49)$$

$$\begin{aligned} Y_3 &= \frac{3t^3\lambda^{\frac{11}{2}}}{8\alpha} \operatorname{sech}^5\left(\frac{x\sqrt{\lambda}}{2}\right) \\ &\times \left(-11 \sinh\left(\frac{x\sqrt{\lambda}}{2}\right) + \sinh\left(\frac{3x\sqrt{\lambda}}{2}\right)\right) \end{aligned} \quad (50)$$

$\vdots$

The above terms of the series (Eq. 42) could be calculated. When the series (Eq. 42) is considered

with the terms (Eqs. 43, 44, 45, and 46) and supposing $p = 1$, an approximate series solution of the considered KdV equation is obtained as follows:

$$\begin{aligned} u(x, t) = & \frac{3\lambda}{\alpha} - \frac{3\lambda}{\alpha} \tanh^2\left(\frac{\sqrt{\lambda}}{2}x\right) \\ & + \frac{3t\lambda^{\frac{5}{2}}}{\alpha} \operatorname{sech}^2\left(\frac{x\sqrt{\lambda}}{2}\right) \tanh\left(\frac{x\sqrt{\lambda}}{2}\right) \\ & + \frac{3t^2\lambda^4}{4\alpha} \left(-2 + \cosh(x\sqrt{\lambda})\right) \operatorname{sech}^4\left(\frac{x\sqrt{\lambda}}{2}\right) \\ & + \frac{3t^3\lambda^{\frac{11}{2}}}{8\alpha} \operatorname{sech}^5\left(\frac{x\sqrt{\lambda}}{2}\right) \\ & \left(-11\sinh\left(\frac{x\sqrt{\lambda}}{2}\right) + \sinh\left(\frac{3x\sqrt{\lambda}}{2}\right)\right) + \dots \end{aligned}$$

Example 6

Let us consider the mKdV Eq. (6) with the values of $m = 2$ and $\alpha = 6$. For this method's implementation, a homotopy is constructed as follows:

$$(1-p)\left[\dot{Y} - \dot{u}_0\right] + p\left[\dot{Y} + 6Y^2Y' + Y'''\right] = 0. \quad (51)$$

With the initial approximation

$$Y_0 = u_0 = \sqrt{c} \operatorname{sech}[k + \sqrt{c}x],$$

suppose the solution of Eq. 41 has the form

$$\begin{aligned} Y = & Y_0 + pY_1 + p^2Y_2 + p^3Y_3 + \dots \\ = & \sum_{n=0}^{\infty} p^n Y_n(x, t). \end{aligned} \quad (52)$$

Substituting Eq. 52 into Eq. 51 and arranging the coefficients of " p " powers, we have

$$p^0 : \dot{Y}_0 - \dot{u}_0 = 0, \quad (53)$$

$$p^1 : \dot{Y}_1 + \dot{u}_0 + 6Y_0^2Y_0' + Y_0''' = 0, \quad (54)$$

$$p^2 : \dot{Y}_2 + 12Y_0Y_1Y_0' + 6Y_0^2Y_0' + Y_1''' = 0, \quad (55)$$

$$\begin{aligned} p^3 : & \dot{Y}_3 + 12Y_0Y_2Y_0' + 6Y_1^2Y_0' + 12Y_0Y_1Y_1' \\ & + Y_2''' = 0 \end{aligned} \quad (56)$$

⋮

Finally, using Mathematica, the few terms of the series solution of Eq. 51 can be obtained as follows:

$$\begin{aligned} u(x, t) = & \sqrt{c} \operatorname{sech}(k + \sqrt{c}x) \\ & + \frac{1}{4}c^{\frac{7}{2}}t^2(-3 + \cosh(2(k + \sqrt{c}x))) \\ & \times \operatorname{sech}^3(k + \sqrt{c}x) \\ & + \frac{1}{24}c^5t^3 \operatorname{sech}^4(k + \sqrt{c}x) \\ & \times (-23\sinh(k + \sqrt{c}x) + \sinh(3(k + \sqrt{c}x))) \\ & + c^2t \operatorname{sech}(k + \sqrt{c}x) \tanh(k + \sqrt{c}x) + \dots \end{aligned}$$

Variational Iteration Method

In this section, a kind of semi-analytical technique is considered for a nonlinear problem called the variational iteration method (VIM) (He 1997, 1999b, 2006). It is proposed by He (1997), and the method is based on the use of restricted variations and correction functionals. In this technique, a correction functional is defined by a general Lagrange multiplier using the ingenuity of variational theory. This method is implemented to get approximate solutions for many linear and nonlinear problems in physics and mathematics (Abdou and Soliman 2005a, b; He and Wu 2006; Momani and Abuasad 2006; Ramos 2008; Wazwaz 2007d). When the method is implemented on the differential equation, it does not require the presence of small parameters, and the solution of the equation is obtained as a sequence of iterates. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives (Abdou and Soliman 2005a, b; He and Wu 2006; Momani and Abuasad 2006; Ramos 2008; Wazwaz 2007d).

To illustrate the VIM, let us consider the nonlinear KdV Eq. (6):

$$Lu + Nu = 0, \quad (57)$$

where L and N are linear and nonlinear operators, respectively. In Abdou and Soliman (2005a, b), He and Wu (2006), Momani and Abuasad (2006), Ramos (2008), and Wazwaz (2007d), He proposed the VIM where a correction functional for Eq. 57 can be written as

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda \{Lu_n(x, \tau) + \tilde{N}u_n(x, \tau)\} d\tau, \quad n \geq 0, \quad (58)$$

where λ is a general Lagrange multiplier (He 1997), which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation, which means $\delta\tilde{u}_n = 0$. It is required first to determine the Lagrangian multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x, t)$, $n \geq 0$, of the solution $u(x, t)$, will be readily obtained upon using the Lagrangian multiplier obtained and by using any selective function u_0 . The initial values $u(x, 0)$ and $u_t(x, 0)$ are usually used for the selective zeroth approximation u_0 . Having λ determined, then several approximations $u_j(x, t)$, $j \geq 0$ can be determined. Consequently, the solution is given by

$$u = \lim_{n \rightarrow \infty} u_n. \quad (59)$$

In what follows, the VIM will apply to two nonlinear KdV equations with similar initial functions to illustrate the strength of the method and to compare the given above methods.

Example 7

Let us consider the mKdV Eq. (6) for the value of $m = 1$ with the following initial function:

$$u(x, 0) = \frac{3\lambda}{\alpha} - \frac{3\lambda}{\alpha} \tanh^2\left(\frac{\sqrt{\lambda}}{2}x\right), \quad (60)$$

where λ and α are arbitrary constants. The correction functional for this equation is

$$\begin{aligned} u_{n+1}(x, t) = & u_n(x, t) \\ & + \int_0^t \lambda \left\{ (u_n(x, \tau))_\tau + \frac{\alpha}{2} (\tilde{u}_n^2(x, \tau))_x \right. \\ & \left. + (u_n(x, \tau))_{xxx} \right\} d\tau, \quad n \geq 0, \end{aligned} \quad (61)$$

where λ is the general Lagrange multiplier (He 1997) whose optimal value is found using variational theory and u_0 is an initial solution with or without unknown parameters. In the case of no unknown parameters, u_0 should satisfy initial boundary conditions, and $\tilde{u}_n^2$ is the restricted variation (He 1999b), i.e., $\tilde{u}_n^2 = 0$. To find the optimal value of λ , we have

$$\begin{aligned} \delta u_{n+1}(x, t) = & \delta u_n(x, t) \\ & + \delta \int_0^t \lambda \left\{ (u_n(x, \tau))_\tau \right. \\ & + \frac{\alpha}{2} (\tilde{u}_n^2(x, \tau))_x \\ & \left. + (u_n(x, \tau))_{xxx} \right\} d\tau, \end{aligned} \quad (62)$$

and this yields the stationary conditions

$$\lambda'(\tau) = 0, \quad 1 + \lambda(\tau)|_{\tau=t} = 0.$$

This in turn gives $\lambda = -1$; therefore, Eq. 51 can be written as follows:

$$\begin{aligned} u_{n+1}(x, t) = & u_n(x, t) \\ & - \int_0^t \lambda \left\{ (u_n(x, \tau))_\tau + \frac{\alpha}{2} (\tilde{u}_n^2(x, \tau))_x \right. \\ & \left. + (u_n(x, \tau))_{xxx} \right\} d\tau. \end{aligned} \quad (63)$$

Substituting the value of $m = 1$ into the mKdV Eq. (6) with an initial value (Eq. 60) and then substituting this into Eq. 63 and using Mathematica, the solutions of Eq. 6 with initial value (Eq. 60) can be obtained as follows:

$$\begin{aligned}
u(x, t) = & \frac{1}{560\alpha \left(1 + \cosh\left(\frac{x\sqrt{\lambda}}{2}\right)\right)} \\
& \times \left\{ 105\lambda \left\{ 32 + t^4 \lambda^6 \operatorname{sech}^2\left(\frac{x\sqrt{\lambda}}{2}\right) \right. \right. \\
& \times \left\{ 88 + 4(-149 + 6t^2 \lambda^3) \operatorname{sech}^2\left(\frac{x\sqrt{\lambda}}{2}\right) \right. \\
& + (1095 - 124t^2 \lambda^3) \operatorname{sech}^4\left(\frac{x\sqrt{\lambda}}{2}\right) \\
& + 198(-3 + t^2 \lambda^3) \operatorname{sech}^6\left(\frac{x\sqrt{\lambda}}{2}\right) \\
& \left. \left. - 99t^2 \lambda^3 \operatorname{sech}^8\left(\frac{x\sqrt{\lambda}}{2}\right) \right\} \right\} - 4t\lambda^{\frac{5}{2}} \{-140(6 + t^2 \lambda^3) \\
& + 9t^2 \lambda^3 \operatorname{sech}^2\left(\frac{x\sqrt{\lambda}}{2}\right) \{140 - 141t^2 \lambda^3 \\
& - 35(4 + 3t^2 \lambda^3) \operatorname{sech}^2\left(\frac{x\sqrt{\lambda}}{2}\right) \\
& + (497t^2 \lambda^3 - 80t^4 \lambda^6) \operatorname{sech}^4\left(\frac{x\sqrt{\lambda}}{2}\right) \\
& + 20t^2 \lambda^3 (-21 + 20t^2 \lambda^3) \operatorname{sech}^6\left(\frac{x\sqrt{\lambda}}{2}\right) \\
& - 630t^4 \lambda^6 \operatorname{sech}^8\left(\frac{x\sqrt{\lambda}}{2}\right) \\
& + 315t^4 \lambda^6 \operatorname{sech}^{10}\left(\frac{x\sqrt{\lambda}}{2}\right) \} \} \\
& \times \tanh\left(\frac{x\sqrt{\lambda}}{2}\right) \} + \dots
\end{aligned}$$

Example 8

Again the mKdV Eq. (6) is considered with the values of $m = 2$ and $\alpha = 6$ and the following initial function:

$$u(x, 0) = \sqrt{c} \operatorname{sech}[k + \sqrt{c}x]. \quad (64)$$

The correction functional for this equation is

$$\begin{aligned}
u_{n+1}(x, t) = & u_n(x, t) \\
& + \int_0^t \lambda \left\{ (u_n(x, \tau))_\tau + \frac{\alpha}{2} (\tilde{u}_n^2(x, \tau))_x \right. \\
& \left. + (u_n(x, \tau))_{xxx} \right\} d\tau, n \geq 0. \quad (65)
\end{aligned}$$

Substituting the value of $m = 2$ into the mKdV Eq. (6) with initial value (Eq. 64) and then substituting this into Eq. 65 and using Mathematica, the solutions of the considered equation with initial value (Eq. 64) can be obtained as follows:

$$\begin{aligned}
u(x, t) = & \frac{1}{4} \operatorname{sech}(k + \sqrt{c}x) \\
& \times \left\{ 2c^{\frac{3}{2}}t^2(1 - 2\operatorname{sech}^2(k + \sqrt{c}x)) \right. \\
& + 2c^5t^3 \operatorname{sech}^5(k + \sqrt{c}x) \\
& \times (-17\sinh(k + \sqrt{c}x) + 3\sinh(3(k + \sqrt{c}x))) \\
& + 3c^{\frac{13}{2}}t^4(-3 + \cosh(2(k + \sqrt{c}x))) \\
& \times \operatorname{sech}^4(k + \sqrt{c}x) \tanh^2(k + \sqrt{c}x) \\
& \left. + 4(\sqrt{c} + c^2t \tanh(k + \sqrt{c}x)) \right\} + \dots
\end{aligned}$$

We have just written the series solution for the mKdV as three terms rather than four terms.

Numerical Experiments

For numerical comparison purposes, two KdV equations are considered. The first one is the classical KdV, and the second one is the modified KdV equation. The formula of numerical results for ADM, HAM, and HPM is given as follows:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \phi_n = & u(x, t) \\
\text{where } \phi_n(x, t) = & \sum_{k=0}^n u_k(x, t), n \geq 0. \quad (66)
\end{aligned}$$

The formula of the numerical results for VIM is given as Eq. 59. The recurrence relations of the methods are given in Eqs. 12, 25, 43, 44, 45, 46, and 58, respectively.

Moreover, the decomposition series of the KdV equation's solutions generally converge very rapidly in real physical problems (Adomian and Rach 1992; Adomian 1994; Wazwaz 2002). The results were obtained about the speed of convergence of this method, providing us methods to solve linear and nonlinear functional equations. The convergence of the HAM and VIM is numerically shown in some works (Abdou and Soliman 2005b; Ramos 2008) and references therein. Here, how all these methods are

converged to their corresponding exact solutions has been proved.

Numerical approximations show a high degree of accuracy, and in most cases of ϕ_n , the n -term approximation is accurate for quite low values of n . The solutions are very rapidly convergent by utilizing the ADM, VIM, HPM, and HAM. The obtained numerical results justify the advantage of these methodologies, and even with few terms, the approximation is accurate. Furthermore, as all these methods do not require discretization of the variables, i.e., time and space, it is not affected by computation roundoff errors, and one is not faced with the necessity of large computer memory and time.

In order to verify numerically whether the proposed methodologies lead to higher accuracy, the numerical solutions can be evaluated using the n -term approximation (Eqs. 66 and 59). The numerical results of Eq. 6 (for KdV $m = 1$ and for mKdV $m = 2$) for the various values of x and t are illustrated in Tables 1 and 3, respectively.

These tabulated results show the differences of the absolute errors between the exact solution and the numerical solution for the ADM, VIM, and HAM. One can conclude that the values on the interval $0 \leq x \leq 15$ and for some small values of t all the considered methods give very close numerical solutions to a corresponding KdV equation. If one closely looks at Tables 1 and 3, then it is evident that the numerical results are almost the same for a few terms of the series solutions, such as the first three terms. The calculation for the later terms of the series has stopped because the scheme VIM does not work very well in the sense of computer time. This problem needs to be approached in a completely different way. Moreover, it has also been numerically proved that HPM is a special case of the HAM which is tabulated in Table 2 for the various values of h for interval $-1.4 \leq h \leq 0.6$ and similar constant values as in Table 1. It is numerically shown that when we take the value of $h = -1$ in the formula HAM, then (numerically) $\text{HAM} \equiv \text{HPM}$ in

Semi-analytical Methods for Solving the KdV and mKdV Equations, Table 1 Comparison between the absolute error of the solution of KdV Eq. (6) ($m = 1$) at

various values of t and $x = 0.5$ for $\alpha = 1, \gamma = 0.01$ in ADM, VIM, and HAM ($h = -1$).

	ADM	HAM	VIM	ADM	HAM	VIM
t	$x = 0$			$x = 5$		
0.1	2.9925×10^{-8}	2.9925×10^{-8}	$3. \times 10^{-8}$	0.0000130999	0.0000145274	0.0000145274
0.2	1.197×10^{-7}	1.197×10^{-7}	1.2×10^{-7}	0.0000261535	0.0000291008	0.000029101
0.3	2.69323×10^{-7}	2.69323×10^{-7}	2.69998×10^{-7}	0.0000391608	0.0000437201	0.0000437206
0.4	4.78795×10^{-7}	4.78795×10^{-7}	4.79995×10^{-7}	0.0000521216	0.0000583853	0.0000583862
0.5	7.48113×10^{-7}	7.48113×10^{-7}	7.49987×10^{-7}	0.000065036	0.0000730962	0.0000730976
0.6	1.07727×10^{-6}	1.07727×10^{-6}	1.07997×10^{-6}	0.0000779036	0.0000878527	0.0000878548
0.7	1.46628×10^{-6}	1.46628×10^{-6}	1.46995×10^{-6}	0.0000907246	0.000102655	0.000102658
0.8	1.91512×10^{-6}	1.91512×10^{-6}	1.91992×10^{-6}	0.000103499	0.000117502	0.000117506
0.9	2.42379×10^{-6}	2.42379×10^{-6}	2.42987×10^{-6}	0.000116226	0.000132395	0.0001324
1.	2.9923×10^{-6}	2.9923×10^{-6}	2.9998×10^{-6}	0.000128906	0.000147333	0.000147339
t	$x = 10$			$x = 15$		
0.1	0.0000207071	0.0000229046	0.0000229046	0.0000216022	0.0000238682	0.0000238682
0.2	0.0000413971	0.000045826	0.0000458261	0.0000432119	0.0000477289	0.0000477289
0.3	0.0000620701	0.000068764	0.0000687642	0.0000648289	0.0000715819	0.0000715819
0.4	0.0000827257	0.0000917187	0.000091719	0.0000864532	0.0000954272	0.0000954271
0.5	0.000103364	0.00011469	0.00011469	0.000108085	0.000119265	0.000119265
0.6	0.000123985	0.000137677	0.000137678	0.000129723	0.000143094	0.000143094
0.7	0.000144588	0.000160681	0.000160682	0.000151369	0.000166916	0.000166916
0.8	0.000165173	0.0001837	0.000183702	0.000173022	0.00019073	0.000190729
0.9	0.000185741	0.000206736	0.000206738	0.000194682	0.000214535	0.000214535
1.	0.00020629	0.000229787	0.00022979	0.000216348	0.000238333	0.000238332

Semi-analytical Methods for Solving the KdV and mKdV Equations, Table 2 Comparison between the absolute error of the solution for Eq. 6 ($m = 1$) by HAM

and HPM at various values t and $x = 0.5$ with different values of h for HAM; in fact HPM is the value of $h = -1$ for HAM.

t	HPM	HAM ($h = -1$)	HAM ($h = -1.4$)	HAM ($h = -0.9$)	HAM ($h = -0.6$)
0.1	1.60354×10^{-6}	1.60354×10^{-6}	1.60838×10^{-6}	1.60346×10^{-6}	1.59877×10^{-6}
0.2	3.26676×10^{-6}	3.26676×10^{-6}	3.27654×10^{-6}	3.26662×10^{-6}	3.25727×10^{-6}
0.3	4.98966×10^{-6}	4.98966×10^{-6}	5.00446×10^{-6}	4.98946×10^{-6}	4.97551×10^{-6}
0.4	6.77222×10^{-6}	6.77222×10^{-6}	6.79214×10^{-6}	6.77196×10^{-6}	6.75346×10^{-6}
0.5	8.61444×10^{-6}	8.61444×10^{-6}	8.63955×10^{-6}	8.61411×10^{-6}	8.59111×10^{-6}
0.6	0.0000105163	0.0000105163	0.0000105467	0.0000105159	0.0000104885
0.7	0.0000124777	0.0000124777	0.0000125135	0.0000124773	0.0000124455
0.8	0.0000144988	0.0000144988	0.0000145401	0.0000144984	0.0000144621
0.9	0.0000165795	0.0000165795	0.0000166263	0.000016579	0.0000165384
1.	0.0000187197	0.0000187197	0.0000187722	0.0000187192	0.0000186744

Semi-analytical Methods for Solving the KdV and mKdV Equations, Table 3 Comparison between the absolute error of the solution of Eq. 6 ($m = 2$) for $c = 1$ and $k = -7$.

$t x$	0.1	0.2	0.3	0.4	0.5
$u \text{ exact} - \varphi_3$ for ADM					
0.1	8.23236×10^{-9}	1.29161×10^{-7}	6.41349×10^{-7}	1.98865×10^{-6}	4.76447×10^{-6}
0.2	9.098×10^{-9}	1.42742×10^{-7}	7.08789×10^{-7}	2.19776×10^{-6}	5.26547×10^{-6}
0.3	1.00546×10^{-8}	1.57752×10^{-7}	7.83317×10^{-7}	2.42885×10^{-6}	5.81914×10^{-6}
0.4	1.11118×10^{-8}	1.74338×10^{-7}	8.65678×10^{-7}	2.68424×10^{-6}	6.43099×10^{-6}
0.5	1.228×10^{-8}	1.92667×10^{-7}	9.56694×10^{-7}	2.96645×10^{-6}	7.10715×10^{-6}
$u \text{ exact} - u_3$ for VIM					
0.1	8.19516×10^{-9}	1.2858×10^{-7}	6.38476×10^{-7}	1.97978×10^{-6}	4.74334×10^{-6}
0.2	9.04779×10^{-9}	1.41958×10^{-7}	7.0491×10^{-7}	2.18579×10^{-6}	5.23696×10^{-6}
0.3	9.98686×10^{-9}	1.56692×10^{-7}	7.78082×10^{-7}	2.4127×10^{-6}	5.78065×10^{-6}
0.4	1.10203×10^{-8}	1.72908×10^{-7}	8.58611×10^{-7}	2.66243×10^{-6}	6.37904×10^{-6}
0.5	1.21566×10^{-8}	1.90738×10^{-7}	9.47155×10^{-7}	2.93702×10^{-6}	7.03703×10^{-6}
$u \text{ exact} - \varphi_3$ for HPM $\equiv$ HAM with value of $h = -1$					
0.1	8.19516×10^{-9}	1.2858×10^{-7}	6.38476×10^{-7}	1.97978×10^{-6}	4.74334×10^{-6}
0.2	9.04779×10^{-9}	1.41958×10^{-7}	7.0491×10^{-7}	2.18579×10^{-6}	5.23696×10^{-6}
0.3	9.98686×10^{-9}	1.56692×10^{-7}	7.78082×10^{-7}	2.4127×10^{-6}	5.78065×10^{-6}
0.4	1.10203×10^{-8}	1.72908×10^{-7}	8.58611×10^{-7}	2.66243×10^{-6}	6.37904×10^{-6}
0.5	1.21566×10^{-8}	1.90738×10^{-7}	9.47155×10^{-7}	2.93702×10^{-6}	7.03703×10^{-6}
$u \text{ exact} - \varphi_3$ for HAM with value of $h = -0.9$					
0.1	2.1809×10^{-9}	1.26484×10^{-7}	1.18355×10^{-7}	1.28924×10^{-7}	5.68147×10^{-7}
0.2	2.41029×10^{-9}	1.39787×10^{-7}	1.30802×10^{-7}	1.42489×10^{-7}	6.27919×10^{-7}
0.3	2.66382×10^{-9}	1.54489×10^{-7}	1.44558×10^{-7}	1.57481×10^{-7}	6.93984×10^{-7}
0.4	2.94403×10^{-9}	1.70737×10^{-7}	1.59761×10^{-7}	1.74053×10^{-7}	7.67005×10^{-7}
0.5	3.25373×10^{-9}	1.88694×10^{-7}	1.76562×10^{-7}	1.9237×10^{-7}	8.47719×10^{-7}

Table 2. It is also our experience that the choice of the h value depends on the chosen equation and the solution of the equation, too. This can be seen in Tables 2 and 3.

Overall, one can see that the closeness of the approximate solutions and exact solutions of all methods gives almost equal absolute values of error, except for the approximate solutions from

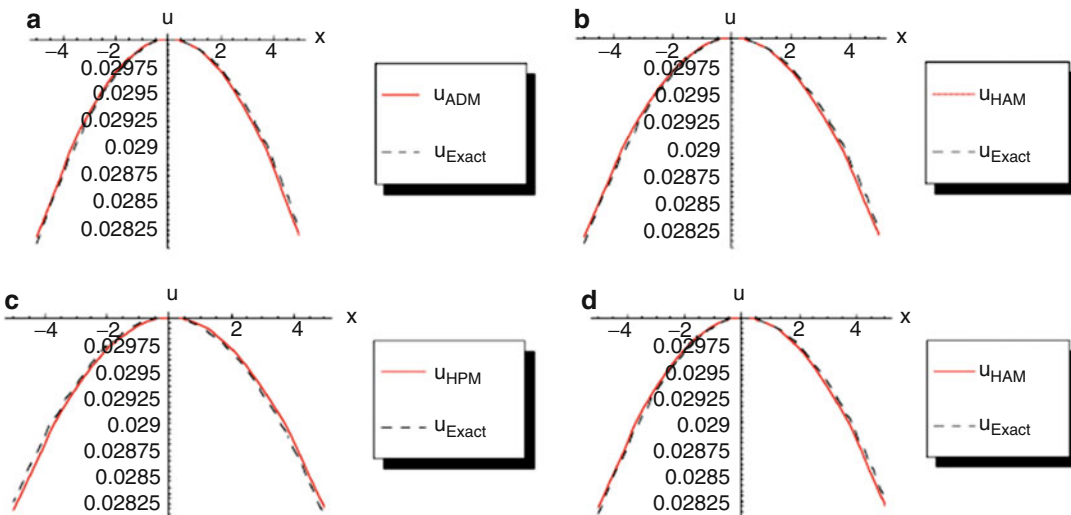
HAM. The numerical solutions with a special value of $h = -0.9$ for the HAM perform very well, as one can see in Table 3. It is also a valuable opportunity for us to use HAM among the considered methods because it gives the users flexibility to choose different values of h , depending on which value is necessary for the problem.

The graphs of the numerical values of the exact and approximate solutions of the classical KdV equation are depicted in Figs. 3, 4, 5, and 6. In Figs. 3, 4, and 5, numerical results are given for the KdV equation and corresponding exact solution by using the considered methods. It is to be noted that only four terms were used in evaluating the approximate solutions with different values of the constant $\lambda = 0.01, 0.1, 1$ in order to get how close the approximate solutions were to the exact solution. A very good approximation to the actual solution of the equation was achieved by only using the four terms of the series derived above (Eqs. 59 and 66) for all considered methods with the value of $\lambda = 0.01$ which are depicted in Fig. 3. This is because of the nature of the series methods. The closeness of the numerical results for approximate solutions and the exact solution diverges for the other value of $\lambda = 0.1$ in all considered methods which are depicted in Fig. 4. This

divergence is very clear in Fig. 5 for the results of all the methods. But a sharp divergence can be seen in the numerical results of VIM especially at the value of $x = 0.5$, in Fig. 5b. In fact, these are illustrated in Tables 1 and 2 and Figs. 3 and 4. It is evident that the overall errors can be made smaller by adding new terms of the series (Eqs. 59 and 66).

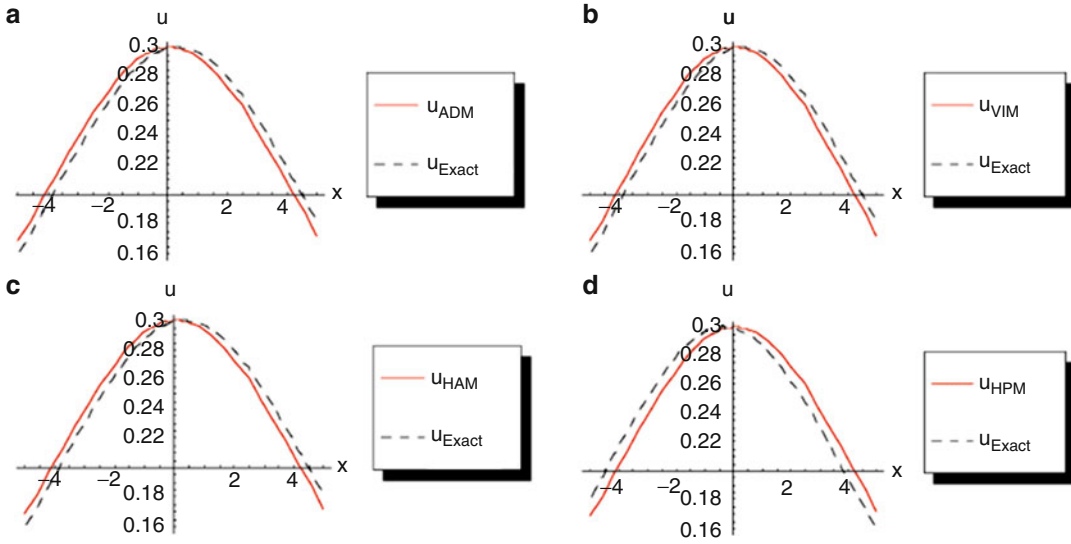
One can see from Fig. 6a that the absolute values of the numerical results for approximate solutions and exact solutions are very small at the value of $0 \leq x \leq 1$, but other than these values of x , neither positive nor negative values of x give small errors for all considered methods with the value of $\lambda = 0.01$. However, in all considered methods with the value of $\lambda = 1$, a relatively small absolute error is given at the value of $0 \leq x \leq 1$, but all the methods are given biggest absolute error at a value of x around ± 2 . The VIM is poorly performed at the value of x around -2 , but besides this all other methods are performed relatively well at the other values of x , which can be seen in Fig. 4b.

Lastly, the clear conclusion can be drawn from the numerical results that the ADM and HAM algorithms provide highly accurate numerical solutions without spatial discretizations for nonlinear partial differential equations. It is also worth noting



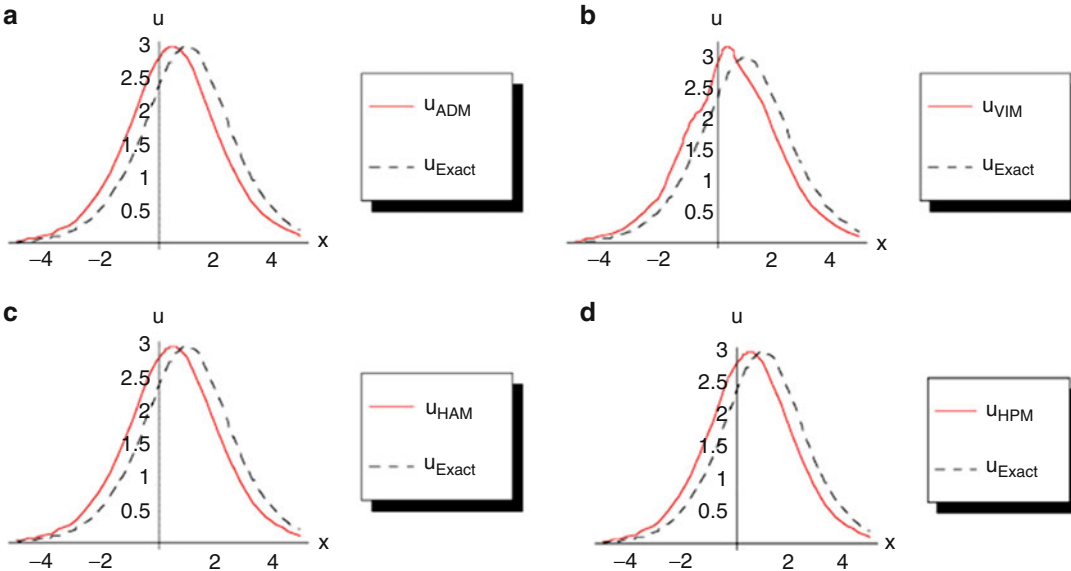
Semi-analytical Methods for Solving the KdV and mKdV Equations, Fig. 3 A comparison by the ADM, VIM, HPM, and HAM with exact solution of Eq. 6 with

$m = 1$ at $t = 0.5$ for $\alpha = 1$ and $\lambda = 0.01$ in (a–c) and $\alpha = 1$, $\lambda = 0.01$, and $h = -1$ in (d), respectively.



Semi-analytical Methods for Solving the KdV and mKdV Equations, Fig. 4 A comparison by the ADM, VIM, HAM, and HPM with exact solution of Eq. 6 with

$m = 1$ at $t = 0.5$ for $\alpha = 1$ and $\lambda = 0.1$ in (a–c) and $\alpha = 1$, $\lambda = 0.1$, and $h = -1$ in (d), respectively.



Semi-analytical Methods for Solving the KdV and mKdV Equations, Fig. 5 A comparison by the ADM, VIM, HAM, and HPM with exact solution of Eq. 6 with

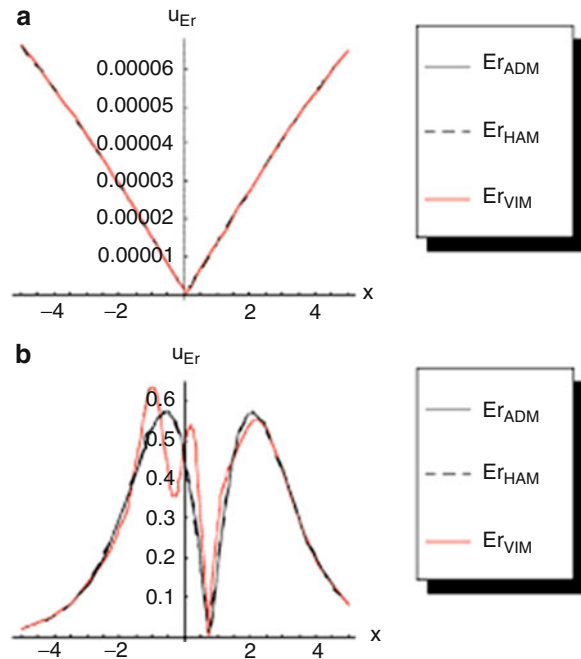
$m = 1$ at $t = 0.5$ for $\alpha = 1$ and $\lambda = 1$ in (a–c) and $\alpha = 1$, $\lambda = 1$, and $h = -1$ in (d), respectively.

that the advantage of the approximation of the series methodologies displays a fast convergence of the solutions. The illustrations show the dependence of the rapid convergence depends on the character and behavior of the solutions just as in

closed form solutions. Finally, it can be pointed out that for given equations with initial values $u(x, 0)$, the corresponding analytical and numerical solutions are obtained according to the recurrence relations (12), (25), (43, 44, 45, 46), and (58) using

Semi-analytical Methods for Solving the KdV and mKdV Equations,

Fig. 6 These two graphs are a comparison of $u_{Er} = |u_{Exact} - u_{Approx}|$ for ADM, HAM, and VIM of the solutions of Eq. 6 with $m = 1$ at $t = 0.5$ for $\alpha = 1$, $\lambda = 0.01$, and $h = -1$ in (a) and $\alpha = 1$, $\lambda = 1$, and $h = -1$ in (b), respectively



Mathematica package version of Mathematica 4 in PC computer.

Future Directions

Nonlinear phenomena play a crucial role in applied mathematics and physics. Furthermore, when an original nonlinear equation is directly calculated, the solution will preserve the actual physical characters of the solutions. Explicit solutions to the nonlinear equations are of fundamental importance. Various effective methods have been developed to understand the mechanisms of these physical models, to help physicists and engineers, and to ensure knowledge for physical problems and their applications.

In the future, the scientist will be very busy doing many works via applications of the nonlinear evolution equations in order to obtain exact and numerical solutions. One has to find out more efficient and effective methods to solve the nonlinear problems in applied mathematical areas of constructing either an exact solution or a numerical solution. Scientists have a long way to go in this nonlinear study.

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Books and Reviews

The following, referenced by the end of the entry, is intended to give some useful for further reading

For another obtaining of the KdV equation for water waves, see Kevorkian and Cole (1981); one can see the work of the Johnson (1972) for a different water-wave application with variable depth, for waves on arbitrary shears in the work of Freeman and Johnson (1970) and Johnson (1980) for a review of one and two-dimensional KdV equations. In addition to these; one can see the book of Drazin and Johnson (1989) for some numerical solutions of nonlinear evolution equations. In the work of the Zabusky, Kruskal and Deam (F1965) and Eilbeck (F1981), one can see the motion pictures of soliton interactions. See a comparison of the KdV equation with water wave experiments in Hammack and Segur (1974)

For further reading of the classical exact solutions of the nonlinear equations can be seen in the works: the Lax approach is described in Lax (1968); Calogero and Degasperis (1982, A.20), the Hirota's bilinear approach is developed in Matsuno (1984), the Bäcklund transformations are described in Rogers and Shadwick (1982); Lamb (1980, Chap. 8), the Painleve properties is discussed by Ablowitz and Segur (1981, Sect. 3.8), In the book of Dodd, Eilbeck, Gibbon and Morris (1982, Chap. 10) can found review of the many numerical methods to solve nonlinear evolution equations and shown many of their solutions

Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV)

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Article Outline

Glossary
Definition of the Subject
Introduction
Some Numerical Methods for Solving the
Korteweg–de Vries (KdV) Equation
The Adomian Decomposition Method (ADM)
The Homotopy Analysis Method (HAM)
The Variational Iteration Method (VIM)
The Homotopy Perturbation Method (HPM)
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Glossary

Absolute error When a real number x is approximated by another number x^* , the error is $x - x^*$. The absolute error is $|x - x^*|$.

Adomian decomposition method The method was introduced and developed by George Adomian. The method proved to be powerful, effective, and can easily handle a wide class of linear or nonlinear differential equations.

Adomian polynomials It is well known now that the ADM suggests that the unknown linear function u may be presented by the

decomposition series $u = \sum_{n=0}^{\infty} u_n$, the nonlinear term $F(u)$, such as $u^2, u^3, \sin u, e^u, u_x^2$, etc. can be expressed by an infinite series of the so-called Adomian polynomials A_n .

Finite difference method The method handles the differential equation by replacing the derivatives in the equation with difference quotients.

Homotopy analysis method The HAM was introduced by Shi-Jun Liao in 1992. For details see section “The Variational Iteration Method (VIM)”.

Homotopy perturbation method The HPM was introduced by Ji-Huan He in 1999. For details see section “Numerical Applications and Comparisons”.

Korteweg–de Vries equation This is one of the simplest and most useful nonlinear model equations for solitary waves.

Lagrange multiplier The multiplier in the functional should be chosen such that its correction solution is superior to its initial approximation (trial function) and is the best within the flexibility of trial function, accordingly we can identify the multiplier by variational theory.

Initial conditions The PDEs mostly arise to govern physics phenomena such as heat distribution, wave propagation phenomena. Most of the PDEs depend on the time t . The initial values of the dependent variable u at the starting time $t = 0$ should be prescribed.

Solitons It is interesting to point out that there is no precise definition of a soliton. However, a soliton can be defined as a solution of a nonlinear partial differential equation.

Variational iteration method The VIM was first proposed by Ji-Huan He. This method has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions.

Definition of the Subject

In this work, the Adomian decomposition method (ADM), the homotopy analysis method (HAM), the variational iteration method (VIM), the homotopy perturbation method (HPM) and the explicit finite difference method (EFDM) are implemented to investigate numerical solutions of the Korteweg–de Vries (KdV) equation with initial condition. The five methods are compared and it is shown that the HAM and EFDM are more efficient and effective than the ADM, the VIM and the HPM, and also converges to its exact solution more rapidly. The HAM contains the auxiliary parameter $\hbar$ which provides us with a simple way to adjust and control the convergence region of solution series.

Introduction

John Scott Russell first experimentally observed the solitary wave, a long water wave without change in shape, on the Edinburgh-Glasgow Canal in 1834. He called it the great wave of translation and, then, reported his observations at the British Association in his 1844 paper “Report on waves” (Russell 1844).

After 60 years from this discovery, the two scientists D.J. Korteweg and G. de Vries formulated a mathematical model equation to provide an explanation of the phenomenon observed by Russell (Korteweg and de Vries 1895). They derived the famous equation, the Korteweg–de Vries (KdV) equation, for the propagation of waves in one dimension on the surface of water. This is one of the simplest and most useful nonlinear model equations for solitary waves, and it represents the longtime evolution of wave phenomena in which the steepening effect of the nonlinear term is counterbalanced by dispersion. The KdV equation is a nonlinear partial differential equation of third order.

Modern developments in the theory and applications of the KdV equation began with the seminal work published as a Los Alamos Scientific Laboratory Report in 1955 by Fermi, Pasta and Ulam (FPU) on a numerical model of a discrete nonlinear mass-springs system (Fermi et al. 1955). This curious result of the FPU experiment inspired

Zabusky and Kruskal (1965) marked the birth of the new concept of the soliton, a name intended to signify particle like quantities. They found that stable pulse like waves could exist in a system described by the KdV equation. A remarkable quality of these solitary waves was that they could collide with each other and yet preserve their shapes and speeds after the collision. This means that a collision between KdV solitary waves is elastic. Subsequently, Zabusky (1967) confirmed, numerically, the actual physical interaction of two solitons, and Lax (1968) gave a rigorous analytical proof that the identities of two distinct solitons are preserved through the nonlinear interaction governed by the KdV equation. Subsequently, a paper by Gardner et al. (Gardner et al. 1967) demonstrated that it was possible to write many exact solutions to the equation by using ideas from scattering theory. They discovered the first integrable nonlinear partial differential equation. Gardner et al. (1974) and Hirota (1971, 1973) constructed analytical solutions of the KdV equation that provide the description of the interaction among N solitons for any positive integer N . Experimental confirmation of solitons and their interactions has been provided successfully by Zabusky and Galvin (1971), Hammack and Segur (1974), and Weidman and Maxworthy (1978). During the past 15 years a rather complete mathematical description of solitons has been developed.

The nondispersive nature of the soliton solutions to the KdV equation arises not because the effects of dispersion are absent, but because they are balanced by nonlinearities in the system (Dodd et al. 1982; Lombdahl 1984; Debnath 1997; Helal 2002). The KdV equation (Korteweg and de Vries 1895; Gardner et al. 1967; Davidson 1972) arises in several areas of nonlinear physics such as hydrodynamics, plasma physics, etc.

During the last five decades or so an exciting and extremely active area of research has been devoted to the construction of exact and numerical solutions for a wide class of nonlinear equations. This includes the most famous nonlinear one of Korteweg and de Vries.

Exact solutions of the KdV equation have been used for many powerful methods such as the application of the Jacobian elliptic function (Lamb 1980; Drazin and Johnson 1989; Drazin 1992), the

powerful inverse scattering transform (Khater et al. 1998), the Painlevé analysis (Khater and El-Sabbagh 1989), the Backlund transformations method (Wahlquist and Estabrook 1973), the Lie group theoretical methods (Olver 1986), the direct algebraic method (Hereman et al. 1985), the tanh-method (Malfliet 1992), the sine-cosine method (Yan 1996), the homogeneous balance method (Wang 1996), the Riccati expansion method with constant coefficient (Yan and Zhang 2001) and the mapping method (Peng 2003). In addition, Sawada and Kotera (1974), Rosales (1978), Whitham (1927), Wadati and Sawada (1980a, b) have all employed perturbation techniques. Recently, Helal and El-Eissa (1996), Khater et al. (Khater and El-Kalaawy 1997; Khater et al. 2000), Helal (2001) have studied analytically and numerically some physical problems that lead to the KdV equation and the soliton solutions have been obtained (Helal 2002).

The KdV equation is a generic equation for the study of weakly nonlinear long waves. The KdV type of equation has been an important class of nonlinear evolution equations with numerous applications in physical sciences and engineering fields. For example, in plasma physics, these equations give rise to the ion acoustic solitons (Das and Sarma 1999); in geophysical fluid dynamics, they describe a long wave in shallow seas and deep oceans (Osborne 1995; Ostrovsky and Stepanyants 1989). Their strong presence is exhibited in cluster physics, super deformed nuclei, fission, thin film, radar and rheology (Ludu and Draayer 1998; Reatto and Galli 1999), optical-fiber communications (Turitsyn et al. 1998) and superconductors (Coffey 1996).

Thus many analytical solutions of the KdV equation are found, and their existence and uniqueness have been studied for a certain class of initial functions (Dodd et al. 1982). The numerical solutions of the KdV equation are essential because of solutions which are not analytically available. Many methods have been proposed for numerical treatment of the KdV equation for the various boundary and initial conditions. For appropriate initial conditions, Gradner et al. (1967) have shown the existence and uniqueness of solutions of the KdV equation. For the KdV equation, several quite successful

numerical methods have been available such as spectral/pseudo spectral methods (Gou and Shen 2001; Huang and Sloan 1992), finite-difference methods and Fourier spectral methods developed by many authors from both the theoretical and computational points of view (Man and Sun 2000; Man and Sun 2001). The exponential finite-difference method is used to solve the KdV equation with the initial and boundary conditions by Bahadir (2005). Jain et al. (1997) developed a numerical method for solving the KdV equation by using splitting method and quintic spline approximation technique. Soliman (2004) worked numerical solutions for the KdV equation based on the collocation method using septic splines as element shape functions were set up. Frauendiener and Klein (2006) presented the hyperelliptic theta-functions with spectral methods for numerical solutions of the KP and KdV equations. Bhatta and Bhatti (2006) studied numerical solution of the KdV equation by using modified Bernstein polynomials. Helal and Mehanna (2007) presented a comparative study between the Adomian decomposition method and the finite-difference method for solving the general KdV equation. Kutluay et al. (2000) used the heat balance integral method to the KdV equation prescribed by appropriate homogeneous boundary conditions and a set of initial conditions to obtain its approximate analytical solutions at small times. An analytical-numerical method was applied to the KdV equation with a variant of boundary and initial conditions to obtain its numerical solutions at small times by Özer and Kutluay (2005). Recently, Dehghan and Shokri (2007) proposed a numerical scheme to solve the third-order nonlinear KdV equation using collocation points and approximating the solution using multi-quadratic radial basis function. Dag and Dereli (2008) presented numerical solution of the KdV equation by using the meshless method based on the collocation with radial basis functions.

Some Numerical Methods for Solving the Korteweg–de Vries (KdV) Equation

The Korteweg–de Vries (KdV) equation has the following form:

$$u_t - 6uu_x + u_{xxx} = 0, \quad (1)$$

subject to initial condition

$$u(x, 0) = f(x), \quad (2)$$

where $u(x, t)$ is a differentiable function and $f(x)$ is bounded. We shall assume that the solution $u(x, t)$, along with its derivatives, tends to zero as $|x| \rightarrow \infty$. The second and third terms uu_x and u_{xxx} represent the nonlinear convection and dispersion effects, respectively. The solitons of the KdV equation arise as a balance between nonlinear convection and dispersion terms. The nonlinear effect causes the steepening of the waveform (Dag and Dereli 2008), while the dispersion effect makes the waveform spread. Due to the competition of these two effects, a stationary waveform (solitary wave) exists. The solution is obtained among the nonlinear and dispersion by stability. The soliton solutions of nonlinear wave equations shown by Wadati (1973, 2001) have the following properties:

- Some certain waves propagate which does not change its special behavior.
- Reginal waves do not lose their properties and also they are stable towards the collisions.

The basic purpose of this work is to approach the KdV Eq. (1) with initial condition (2) differently, by using four semi inverse methods and the EFDm. In this work, we will use the Adomian decomposition method (ADM), the homotopy analysis method (HAM), the variational iteration method (VIM), the homotopy perturbation method (HPM) and the explicit finite difference method (EFDm). The results are compared with those obtained using the five methods. More details for the five methods can be found in next sections.

The Adomian Decomposition Method (ADM)

The ADM was first introduced by Adomian in the beginning of (Adomian 1989, 1994, 1998). The method is useful for obtaining both a closed

form and the explicit solution and numerical approximations of linear or nonlinear differential equations and it is also quite straight forward for writing computer codes. This method has been applied to obtain a formal solution to a wide class of stochastic and deterministic problems in science and engineering involving algebraic, differential, integro-differential, differential delay, integral and partial differential equations (Lesnic and Elliott 1999; Wazwaz 2002; Wazwaz 2003; Cherruault 1998; Dehghan 2004a; Dehghan 2004b; Babolian et al. 2004; El-Sayed 2002; Inc 2006a, 2007a). The convergence of ADM for partial differential equations was presented by Cherruault (1990). Application and convergence of this method for nonlinear partial differential equations are found in (Ngarhasta et al. 2002; Mavoungou and Cherruault 1992; Inc 2006b; Hashim et al. 2006).

In general, it is necessary to construct the solution of the problems in the form of a decomposition series solution. In the simplest case, the solution can be developed as a Taylor series expansion about the function, not the point at which the initial condition and integration of the right-hand side function of the problem are determined (the first term u_0 of the decomposition series for $n \geq 0$). The sum of the $u_0, u_1, u_2, \dots$ terms are simply the decomposition series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \quad (3)$$

Suppose that the differential equation operator, including both linear and nonlinear terms, can be formed as

$$Lu + Ru + Nu = F(x, t), \quad (4)$$

with initial condition

$$u(x, 0) = g(x), \quad (5)$$

where L is the higher-order derivative which is assumed to be invertible, R is a linear differential operator of order less than L , N is the nonlinear term and $F(x, t)$ is a source term. We next apply

the inverse operator L^{-1} to both sides of Eq. (4) and using the given condition (5) to obtain

$$u(x, t) = g(x) + f(x, t) - L^{-1}(Ru) - L^{-1}(Nu), \quad (6)$$

where the function $f(x, t)$ represents the terms arising from integrating the source term $F(x, t)$ and from using the given conditions, all are assumed to be prescribed. The nonlinear term can be written as

$$Nu = \sum_{n=0}^{\infty} A_n, \quad (7)$$

where A_n is the Adomian polynomials. These polynomials are defined as

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{k=0}^{\infty} \lambda^k u_k(x, t) \right) \right]_{\lambda=0}, \quad (8)$$

$$n = 0, 1, 2, \dots$$

for example

$$\begin{aligned} A_0 &= N(u_0), \\ A_1 &= u_1 N'(u_0), \\ A_2 &= u_2 N'(u_0) + \frac{1}{2} u_1^2 N''(u_0), \\ A_3 &= u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{6} u_1^3 N'''(u_0), \end{aligned} \quad (9)$$

and so on, the other polynomials can be constructed in a similar way (Adomian 1998). As indicated before, Adomian method defines the solution u by an infinite series of components given by Eq. (4) and the components $u_0, u_1, u_2, \dots$ are usually recurrently determined. Thus, the formal recursive relation is defined by

$$\begin{aligned} \{u_0(x, t) &= g(x) + f(x, t), u_{n+1}(x, t) \\ &= -L^{-1}(Ru_n) - L^{-1}(Nu_n), \quad n \geq 0, \end{aligned} \quad (10)$$

which are obtained for all components of u . As a result, the terms of the series $u_0, u_1, u_2, \dots$ are identified and the exact solution may be entirely determined by using the approximation

$$u(x, t) = \lim_{n \rightarrow \infty} \varphi_n(x, t), \quad (11)$$

where

$$\varphi_n(x, t) = \sum_{k=0}^{n-1} u_k(x, t), \quad (12)$$

or

$$\begin{cases} \varphi_0 = u_0, \\ \varphi_1 = u_0 + u_1, \\ \varphi_2 = u_0 + u_1 + u_2, \\ \vdots \\ \varphi_n = u_0 + u_1 + u_2 + \dots + u_{n-1}, \quad n \geq 0. \end{cases} \quad (13)$$

The Homotopy Analysis Method(HAM)

The HAM was developed in 1992 by Liao in (Liao 2002, 2003a, b, 2005). This method has been successfully applied by many authors (Sajid et al. 2006; Hayat et al. 2006; Hayat and Khan 2005; Abbasbandy 2006, 2007; Inc 2007b). The HAM contains the auxiliary parameter $\hbar$ which provides us with a simple way to adjust and control the convergence region of solution series for large or small values of x and t .

We consider the following differential equation

$$N[u(x, t)] = 0, \quad (14)$$

where N is a nonlinear differential operator, x and t denote independent variables, $u(x, t)$ is an unknown function. By means of the HAM, one first constructs a zero-order deformation equation

$$\begin{aligned} (1-p)\mathcal{L}[\phi(x, t; p) - u_0(x, t)] \\ = p \hbar N[\phi(x, t; p)], \end{aligned} \quad (15)$$

where $p \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is a non-zero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $u_0(x, t)$ is an initial guess of $u(x, t)$ and $\phi(x, t; p)$ is a unknown function. It is important that one has great freedom to choose

auxiliary things in the HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\begin{aligned}\phi(x, t; 0) &= u_0(x, t), \quad \phi(x, t; 1) \\ &= u(x, t),\end{aligned}\quad (16)$$

respectively. The solution $\phi(x, t; p)$ varies from the initial guess $u_0(x, t)$ to the solution $u(x, t)$. Expanding $\phi(x, t; p)$ in Taylor series about the embedding parameter

$$\phi(x, t; p) = u_0(x, t) + \sum_{m=1}^{+\infty} u_m(x, t) p^m, \quad (17)$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; p)}{\partial p^m} \right|_{p=0}. \quad (18)$$

The convergence of the series (17) depends upon the auxiliary parameter $\hbar$. If it is convergent at $p = 1$, one has

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{+\infty} u_m(x, t). \quad (19)$$

According to the definition (19), the governing equation can be deduced from the zero-order deformation Eq. (15). Define the vector

$$\vec{u}_n = \{u_0(x, t), u_1(x, t), \dots, u_n(x, t)\}.$$

Differentiating Eq. (15) m -times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called m th-order deformation equation

$$\mathfrak{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar \mathfrak{R}_m(\vec{u}_{m-1}), \quad (20)$$

where

$$\mathfrak{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(x, t; p)]}{\partial p^{m-1}} \right|_{p=0}, \quad (21)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (22)$$

It should be emphasized that $u_m(x, t)$ for $m \geq 1$ is governed by the nonlinear Eq. (20) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

The Variational Iteration Method(VIM)

The VIM was first proposed by He (1997a, b, 1999a, 2000, 2006a). This method has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions. The idea of VIM is to construct a correction functional by a general Lagrange multiplier. The multiplier in the functional should be chosen such that its correction solution is superior to its initial approximation (trial function) and is the best within the flexibility of the trial function, accordingly we can identify the multiplier by variational theory. The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing the boundary/initial conditions.

We consider the following general differential equation:

$$Lu + Nu = g, \quad (23)$$

where L and N are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term.

According to the VIM, the terms of a sequence $\{u_n\}$ are constructed such that this sequence converges to the exact solution. u_n s are calculated by a correction functional as follows:

$$\begin{aligned}u_{n+1}(x, t) &= u_n(x, t) \\ &+ \int_0^t \lambda \{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\} d\xi, \\ &\geq 0,\end{aligned}\quad (24)$$

where λ is the general Lagrange multiplier, which can be identified optimally by the variational theory, the subscript n denotes the n th approximation and $\tilde{u}_n$ considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$. For linear problems, the exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. In nonlinear problems, in order to determine the Lagrange multiplier in a simple manner, the nonlinear terms have to be considered as restricted variations. Consequently, the exact solution may be obtained by using

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t). \quad (25)$$

This method has been employed to solve a large variety of linear and nonlinear problems with approximations converging to accurate solutions. This technique was used to find the solutions of partial differential equations, ordinary differential equations, boundary-value problems and integral equations by many authors (Abdou and Soliman 2005a, b; Bildik and Konuralp 2006; Draganescu 2006; Tatari and Dehghan 2007a, b; Wazwaz 2007; Inc 2007c, d).

The Homotopy Perturbation Method (HPM)

The HPM was developed by He (1999b, 2005a, b, 2006b). In recent years, the applications of HPM in nonlinear problems has been devoted to scientists and engineers (Ganji and Rajabi 2006; Siddiqui et al. 2006; Ganji and Rafei 2006; Chowdhury and Hashim 2007).

We consider the following general nonlinear differential equation:

$$A(y) - f(r) = 0, \quad r \in \Omega, \quad (26)$$

with boundary conditions

$$B(y, \partial y / \partial n) = 0, \quad r \in \Gamma, \quad (27)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω . The

operator A can be generally divided into two parts L and N , where L is linear and N is nonlinear. Therefore, Eq. (26) can be written as follows:

$$L(y) + N(y) - f(r) = 0. \quad (28)$$

We construct a homotopy of Eq. (26) $y(r, p) : \Omega \times [0, 1] \rightarrow R$ which satisfies

$$\begin{aligned} H(y, p) &= (1 - p)[L(y) - L(y_0)] \\ &\quad + p[A(y) - f(r)] \\ &= 0, \quad r \in R, \end{aligned} \quad (29)$$

which is equivalent to

$$\begin{aligned} H(y, p) &= L(y) - L(y_0) + pL(y_0) \\ &\quad + p[A(y) - f(r)] = 0, \end{aligned} \quad (30)$$

where $p \in [0, 1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the boundary conditions. It follows from Eqs. (29) and (30) that

$$\begin{aligned} H(y, 0) &= L(y) - L(y_0) = 0 \text{ and} \\ H(y, 1) &= A(y) - f(r) = 0. \end{aligned} \quad (31)$$

Thus, the changing process of p from 0 to 1 is just that of $y(r, p)$ from $y_0(r)$ to $y(r)$. In topology this is called deformation and $L(y) - L(y_0)$ and $A(y) - f(r)$ are called homotopic. Here the embedding parameter is introduced much more naturally, unaffected by artificial factors; further it can be considered as a small parameter for $0 \leq p \leq 1$. So it is very natural to assume that the solution of (29) and (30) can be expressed as

$$y(x) = u_0(x) + pu_1(x) + p^2u_2(x) + \cdots. \quad (32)$$

According to HPM, the approximate solution of (26) can be expressed as a series of the power of p , i.e.,

$$\begin{aligned} y &= \lim_{p \rightarrow 1} y(x) \\ &= u_0(x) + u_1(x) + u_2(x) + \cdots. \end{aligned} \quad (33)$$

The convergence of series (33) has been proved by He in (He 1999b).

Numerical Applications and Comparisons

We now consider the initial value problem associated with the KdV equation:

$$\begin{cases} u_t - 6uu_x + u_{xxx} = 0, \\ u(x, 0) = -\frac{c}{2} \sec h^2\left(\frac{\sqrt{c}}{2}x\right), \end{cases} \quad (34)$$

with $u(x, t)$ is a sufficiently smooth function and c is the velocity of the wavefront.

In this section, we apply the above described four methods and the explicit finite difference method on the KdV Eq. (34) so that the comparisons are made numerically.

The ADM for Eq. (14)

According to the ADM (Adomian 1989, 1994, 1998), Eq. (34) can be written in an operator form

$$\begin{cases} Lu = 6uu_x - u_{xxx}, \\ u(x, 0) = f(x) = -\frac{c}{2} \sec h^2\left(\frac{\sqrt{c}}{2}x\right), \end{cases} \quad (35)$$

where the differential operator L is

$$L \equiv \frac{\partial}{\partial t}, \quad (36)$$

and the inverse operator L^{-1} is an integral operator given by

$$L^{-1}(\circ) = \int_0^t (\circ) dt. \quad (37)$$

The ADM (Adomian 1989, 1994, 1998) assumes a series solution for the unknown function $u(x, t)$ given by Eq. (3) and the nonlinear term $Nu = uu_x$ can be decomposed into a infinite series of polynomials given by Eq. (5). Operating with the integral operator (37) on both sides of (35) and using the initial condition, we get

$$u(x, t) = f(x) + L^{-1}(6uu_x - u_{xxx}). \quad (38)$$

Substituting (3) and (5) into the functional Eq. (38) yields

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x, t) &= f(x) \\ &+ L^{-1} \left[6 \left(\sum_{n=0}^{\infty} A_n \right) - \left(\sum_{n=0}^{\infty} u_n \right)_{xxx} \right]. \end{aligned} \quad (39)$$

The ADM admits that the zeroth component $u_0(x, t)$ be identified by all terms that arise from the initial conditions and from the source terms if exist, and as a result, the remaining components $u_n(x, t)$, $n \geq 1$ can be determined by using the recurrence relation:

$$\begin{cases} u_0(x, t) = f(x) = -\frac{c}{2} \sec h^2\left(\frac{\sqrt{c}}{2}x\right), \\ u_{k+1}(x, t) = L^{-1}(6A_k - (u_k)_{xxx}), \quad k \geq 0, \end{cases} \quad (40)$$

where A_k , $k \geq 0$ are Adomianpolynomials that represent the nonlinear term (uu_x) and given by

$$\begin{cases} A_0 = u_0 u_{0x}, \\ A_1 = u_{0x} u_1 + u_0 u_{1x}, \\ A_2 = u_1 u_{1x} + u_{0x} u_2 + u_0 u_{2x}, \\ A_3 = u_1 u_{2x} + u_{1x} u_2 + u_{0x} u_3 + u_0 u_{3x}. \end{cases} \quad (41)$$

Thus, some of the symbolically computed components are found as:

$$\begin{aligned} u_0(x, t) &= -\frac{c}{2} \sec h^2\left(\frac{\sqrt{c}}{2}x\right), \\ u_1(x, t) &= -\frac{c^{5/2}}{2} \sec h^2\left(\frac{\sqrt{c}}{2}x\right) \tanh\left(\frac{\sqrt{c}}{2}x\right) t, \\ u_2(x, t) &= -\frac{c^4}{8} (\cosh[\sqrt{c}x] - 2) \sec h^4\left(\frac{\sqrt{c}}{2}x\right) t^2, \\ u_3(x, t) &= -\frac{c^{11/2}}{48} \sec h^5\left(\frac{\sqrt{c}}{2}x\right) \\ &\quad \cdot \left(-11 \sinh\left(\frac{\sqrt{c}}{2}x\right) + \sinh\left(\frac{3\sqrt{c}}{2}x\right) \right) t^3, \end{aligned}$$

$$u_4(x, t) = -\frac{c^7}{384} \sec h^6 \left(\frac{\sqrt{c}}{2} x \right) \cdot [33 - 26 \cosh [\sqrt{c}x] + \cosh [2\sqrt{c}x]] t^4, \quad \mathbb{E}[k] = 0, \quad (45)$$

where k is a constant. We now define a nonlinear operator

and so on. In this manner the other components of the decomposition series can be easily obtained using any symbolic program. The solution in a series for is given by

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\ &= u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots, \\ &= -\frac{c}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} x \right) \\ &\quad - \frac{c^{5/2}}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} x \right) \tanh \left(\frac{\sqrt{c}}{2} x \right) t \\ &\quad - \frac{c^4}{8} (\cosh [\sqrt{c}x] - 2) \sec h^4 \left(\frac{\sqrt{c}}{2} x \right) t^2 \\ &\quad - \frac{c^{11/2}}{48} \sec h^5 \left(\frac{\sqrt{c}}{2} x \right) \\ &\quad \cdot \left(-11 \sinh \left(\frac{\sqrt{c}}{2} x \right) + \sinh \left(\frac{3\sqrt{c}}{2} x \right) \right) t^3 \\ &\quad + \dots \end{aligned} \quad (42)$$

so that the exact solution

$$u(x, t) = -\frac{c}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} (x - ct) \right), \quad (43)$$

is readily obtained.

The convergence analysis of the ADM applied to the KdV Eq. (35) without the initial condition has been conducted by Helal and Mehanna (Frauendiener and Klein 2006).

The HAM for Eq. (34)

We will apply the HAM to the Eq. (34) to illustrate the strength of the method and to establish exact solution for this Eq. (34). We choose the linear operator

$$\mathbb{E}[\phi(x, t; p)] = \frac{\partial \phi(x, t; p)}{\partial t}, \quad (44)$$

with property

$$\begin{aligned} N[\phi(x, t; p)] &= \frac{\partial \phi(x, t; p)}{\partial t} \\ &\quad - 6\phi(x, t; p) \frac{\partial \phi(x, t; p)}{\partial x} + \frac{\partial^3 \phi(x, t; p)}{\partial x^3}. \end{aligned} \quad (46)$$

We construct the zeroth-order deformation equation

$$\begin{aligned} (1-p)\mathbb{E}[\phi(x, t; p) - u_0(x, t)] \\ = p \hbar N[\phi(x, t; p)]. \end{aligned} \quad (47)$$

For $p = 0$ and $p = 1$, we can write

$$\begin{cases} \phi(x, t; 0) = u(x, 0) = u_0(x, t), \\ \phi(x, t; 1) = u(x, t). \end{cases} \quad (48)$$

So we obtain m th-order deformation equation

$$\mathbb{E}[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar \mathfrak{R}_m(\vec{u}_{m-1}), \quad (49)$$

where

$$\begin{aligned} \mathfrak{R}_m(\vec{u}_{m-1}) &= \frac{\partial \phi_{m-1}(x, t; p)}{\partial t} \\ &\quad - 6 \sum_{n=0}^{m-1} \phi_n(x, t; p) \frac{\partial \phi_{m-1-n}(x, t; p)}{\partial x} \\ &\quad + \frac{\partial^3 \phi_{m-1}(x, t; p)}{\partial x^3}. \end{aligned} \quad (50)$$

Now the solution of the m th-order deformation Eq. (49) for $m \geq 1$ become

$$\begin{aligned} u_m(x, t) &= \chi_m u_{m-1}(x, t) \\ &\quad + \hbar \mathbb{E}^{-1} [\mathfrak{R}_m(\vec{u}_{m-1})]. \end{aligned} \quad (51)$$

Thus we get the following terms by using (51) for Eq. (34):

$$\begin{aligned}
u_0(x, t) &= -\frac{c}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} x \right), \\
u_1(x, t) &= \frac{c^{5/2}}{2} \hbar \operatorname{sech}^2 \left(\frac{\sqrt{c}}{2} x \right) \tanh \left(\frac{\sqrt{c}}{2} x \right) t, \\
u_2(x, t) &= -\frac{c^{5/2}}{2} \hbar t \operatorname{sech}^4 \left(\frac{\sqrt{c}}{2} x \right) \left[c^{3/2} \hbar t \cosh(\sqrt{c} x) \right. \\
&\quad \left. - 2(c^{3/2} \hbar t + (1 + \hbar) \sinh(\sqrt{c} x)) \right], \\
u_3(x, t) &= \frac{c^{5/2}}{48} \hbar t \sec h^2 \left(\frac{\sqrt{c}}{2} x \right) \left[c^3 \hbar^2 t^2 \sec h^3 \left(\frac{\sqrt{c}}{2} x \right) \right. \\
&\quad \cdot \sinh \left(\frac{3\sqrt{c}}{2} x \right) + 24\hbar(1 + \hbar) \tanh \left(\frac{\sqrt{c}}{2} x \right) \\
&\quad + \operatorname{sech}^2 \left(\frac{\sqrt{c}}{2} x \right) (-12c^{3/2} \hbar(1 + \hbar) t \cosh(\sqrt{c} x) \\
&\quad + 12(1 + \hbar) \sinh(\sqrt{c} x) \\
&\quad \left. \left. + c^{3/2} \hbar t \left(24 + 24\hbar - 11c^{3/2} \hbar t \tanh \left(\frac{\sqrt{c}}{2} x \right) \right) \right) \right],
\end{aligned}$$

and so on. In this manner the other components of the decomposition series can be easily obtained using any symbolic program.

Liao (2002, 2003a, b, 2005) showed that whatever a solution series converges to it will be one of the solutions of considered problem. Liao (2002, 2003a, b, 2005) showed the approximate solutions obtained by the HAM to be controlled by the auxiliary parameter and the rate of convergence of $\hbar$.

It is noted that the approximate solutions converge at $-2.1 \leq \hbar \leq -1.1$ for 5th-order approximation and at $-4 \leq \hbar \leq 2$ for 10th-order approximation for the KdV Eq. (34) when $x = 10$, $t = 0.5$, $c = 1$ and $c = 5$ (see Figs. 2 and 3).

The VIM for Eq. (34)

To solve the KdV Eq. (34) by means of the VIM, we construct a correction functional which reads as

$$\begin{aligned}
u_{n+1}(x, t) &= u_n(x, t) \\
&+ \int_0^t \lambda \left(\frac{\partial u_n(x, \tau)}{\partial \tau} - N\tilde{u}_n(x, \tau) + \frac{\partial^3 \tilde{u}_n(x, \tau)}{\partial x^3} \right) d\tau,
\end{aligned} \quad (52)$$

where λ is the general Lagrange multiplier (He 2006b) whose optimal value is found using variational theory, $u_0(x, t)$ is an initial approximation which must be chosen suitable and $N\tilde{u}_n$ is the restricted variation, i.e., $\delta\tilde{u}_n = 0$ (He 2005a). To find the optimal values of λ we have

$$\begin{aligned}
\delta u_{n+1}(x, t) &= \delta u_n(x, t) \\
&+ \delta \int_0^t \lambda \left(\frac{\partial u_n(x, \tau)}{\partial \tau} \right) d\tau,
\end{aligned} \quad (53)$$

that results

$$\begin{aligned}
\delta u_{n+1}(x, t) &= \delta u_n(x, t)(1 + \lambda) \\
&- \int_0^t \delta u_n(x, t) \lambda' d\tau,
\end{aligned} \quad (54)$$

which yields

$$\lambda'(\tau) = 0|_{\tau=t}, 1 + \lambda(\tau) = 0|_{\tau=t}. \quad (55)$$

Thus we have

$$\lambda(t) = -1, \quad (56)$$

and we obtain the following iteration formula

$$\begin{aligned}
u_{n+1}(x, t) &= u_n(x, t) - \int_0^t \left(\frac{\partial u_n(x, \tau)}{\partial \tau} \right. \\
&\quad \left. - 6u_n(x, \tau) \frac{\partial u_n(x, \tau)}{\partial x} + \frac{\partial^3 u_n(x, \tau)}{\partial x^3} \right) d\tau.
\end{aligned} \quad (57)$$

Now using (57) we can find the solution of the KdV Eq. (34) as a convergent sequence. We get the following components:

$$u_0(x, t) = -\frac{c}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} x \right),$$

$$\begin{aligned}
u_1(x, t) &= -\frac{c}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} x \right) \\
&\quad \left[\cosh \left(\frac{\sqrt{c}}{2} x \right) + c^{3/2} t \sinh \left(\frac{\sqrt{c}}{2} x \right) \right],
\end{aligned}$$

$$\begin{aligned}
u_2(x, t) = & -\frac{c}{32} \sec h^7 \left(\frac{\sqrt{c}}{2} x \right) \left[-2(4c^3 t^2 - 5) \right. \\
& \cdot \cosh \left(\frac{\sqrt{c}}{2} x \right) + (5 - c^3 t^2) \cosh \left(\frac{3\sqrt{c}}{2} x \right) \\
& + \cosh \left(\frac{5\sqrt{c}}{2} x \right) + c^3 t^2 \cosh \left(\frac{5\sqrt{c}}{2} x \right) \\
& + 4c^{3/2} t \sinh \left(\frac{\sqrt{c}}{2} x \right) - 60c^{9/2} t^3 \sinh \left(\frac{\sqrt{c}}{2} x \right) \\
& + 6c^{3/2} t \sinh \left(\frac{3\sqrt{c}}{2} x \right) + 12c^{9/2} t^3 \sinh \left(\frac{3\sqrt{c}}{2} x \right) \\
& \left. + 2c^{3/2} t \sinh \left(\frac{5\sqrt{c}}{2} x \right) \right]
\end{aligned}$$

and so on. In the same manner the rest of components of the iteration formulae (57) can be obtained using any symbolic packages.

So we obtain

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) = -\frac{c}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} (x - ct) \right). \quad (58)$$

The HPM for Eq. (34)

To investigate the approximate solution of Eq. (34), we first construct a homotopy as follows:

$$(1 - p)(\dot{Y} - u_0) + p(\dot{Y} - 6YY' + Y''') = 0, \quad (59)$$

where “primes” denote differentiation with respect to x , and “dot” denotes differentiation with respect to t . In order to obtain the unknowns of $Y_i(x, t)$, $i = 1, 2, 3, 4$, we must construct and solve the following system which includes four equations with three unknowns, considering the initial condition of $Y(x, 0) = u(x, 0)$ and having the initial approximation of Eq. (34):

$$\begin{aligned}
p^0 : \dot{Y}_0 - u_0 &= 0 \\
p^1 : \dot{Y}_1 + u_0 - 6Y_0Y'_0 + Y'''_0 &= 0 \\
p^2 : \dot{Y}_2 - 6Y_0Y'_1 - 6Y_1Y'_0 + Y'''_1 &= 0 \\
p^3 : \dot{Y}_3 - 6Y_0Y'_2 - 6Y_1Y'_1 - 6Y_2Y'_0 + Y'''_2 &= 0
\end{aligned} \quad (60)$$

Thus, we will obtain:

$$u(x, t) = \lim_{p \rightarrow 1} Y_k(x, t), \quad k = 0, 1, 2, 3, \dots \quad (61)$$

To calculate the terms of the homotopy series (61) for $u(x, t)$, we substitute the initial condition into the systems (60) and finally using Mathematica, the solutions of the equation can be obtained as follows:

$$u_0(x, t) = -\frac{c}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} x \right),$$

$$u_1(x, t) = -\frac{c^{5/2}}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} x \right) \tanh \left(\frac{\sqrt{c}}{2} x \right) t, \quad (62)$$

$$\begin{aligned}
u_2(x, t) = & \frac{c^{5/2}}{2} t^2 \sec h^4 \left(\frac{\sqrt{c}}{2} x \right) \\
& \left[-c^{3/2} \cosh(\sqrt{c}x) + 2c^{3/2} \right],
\end{aligned}$$

and so on. In this manner the other components can be easily obtained. Substituting Eq. (62) into Eq. (61):

$$\begin{aligned}
u(x, t) = & Y_0(x, t) + Y_1(x, t) + Y_2(x, t) + \dots \\
= & -\frac{c}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} x \right) \\
& -\frac{c^{5/2}}{2} \sec h^2 \left(\frac{\sqrt{c}}{2} x \right) \tanh \left(\frac{\sqrt{c}}{2} x \right) t \\
& -\frac{c^4}{2} t^2 \sec h^4 \left(\frac{\sqrt{c}}{2} x \right) [\cosh(\sqrt{c}x) - 2] + \dots
\end{aligned} \quad (63)$$

The EFDM for Eq. (34)

The finite difference methods are the most frequently used and universally applicable. These methods are approximate in the sense that the derivatives at a point are approximated by difference quotients over a small interval (Smith 1987). In order to obtain a finite difference replacement of the KdV Eq. (34) the region to be examined is divided into equal rectangular meshes with sides

Δx and Δt parallel to the x - and t -axes respectively. The function $u(x, t)$ is approximated by $u_i^n = u(i\Delta x, n\Delta t)$ where i and n are integers and $i = n = 0$ is the origin. Let us define

$$U_t|_{(i, n)} = \frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t}, \quad (64)$$

$$U_x|_{(i, n)} = \frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}, \quad (65)$$

$$U_{xxx}|_{(i, n)} = \frac{1}{2(\Delta x)^3} [U_{i+2}^n - 2U_{i+1}^n + 2U_{i-1}^n - U_{i-2}^n]. \quad (66)$$

The explicit scheme computes the value of the numerical solution at the forward time step in terms of known values at the previous time step. An explicit scheme for solving the KdV equation produced originally by Zabusky and Galvin (1971) is centered in time and space. Substituting (64)–(66) into the KdV Eq. (34) with

$$U|_{(i, n)} = \frac{1}{3} (U_{i-1}^n + U_i^n + U_{i+1}^n), \quad (67)$$

leads to

$$U_i^{n+1} = U_i^{n-1} + 2\Delta t (U_{i-1}^n + U_i^n + U_{i+1}^n) (U_{i+1}^n - U_{i-1}^n) - \frac{\Delta t}{(\Delta x)^3} (U_{i+2}^n - 2U_{i+1}^n + 2U_{i-1}^n - U_{i-2}^n). \quad (68)$$

For the initial step, we use a scheme which is forward in time and centered in space

$$U_i^1 = U_i^0 + \Delta t (U_{i-1}^0 + U_i^0 + U_{i+1}^0) (U_{i+1}^0 - U_{i-1}^0) - \frac{\Delta t}{(\Delta x)^3} (U_{i+2}^0 - 2U_{i+1}^0 + 2U_{i-1}^0 - U_{i-2}^0) \quad (69)$$

It is clear that Eq. (68) is a three-level scheme of time, i.e. in order to obtain U_i at the time level $n + 1$, we need the following values of U_{i-2} , U_{i-1} , U_{i+1} and U_{i+2} at the previous time level n in addition to the value of U_i at the time level $n - 1$. The explicit difference scheme (68) has

second order accuracy in Δt and Δx as the truncation error is $O(\Delta t)^2 + O(\Delta x)^2$ and is also consistent with KdV equation. A stability analysis of the nonlinear numerical scheme (68) using Fourier mode method is not easy to handle unless it is assumed that U , in the nonlinear term, is locally constant. This is equivalent to replacing the term (67) in Eq. (68) by U^* . This linearized scheme for the KdV Eq. (34) has stability condition (Vilegenthart 1971):

$$\frac{\Delta t}{\Delta x} \left[-6|U^*| + \frac{4}{(\Delta x)^2} \right] \leq 1. \quad (70)$$

In Table 7, h is x -direction mesh size and k is t -direction time step.

Conclusions and Discussions

In this paper, the ADM, HAM, VIM, HPM and EFDM were used for the KdV Eq. (34) with initial conditions. We made comparisons of the numerical results obtained by using ADM, HAM, VIM, HPM and EFDM. The approximate solutions to the equation has been also calculated by using the ADM, HAM, VIM, HPM and EFDM without any need to use transformation techniques and linearization of the equation. The ADM, HAM and HPM avoids the difficulties and massive computational work by determining analytic solutions of the nonlinear equations. The main advantage of HAM, VIM and HPM are to overcome the difficulty arising in calculating Adomian's polynomials in the ADM. But the main disadvantage of VIM and EFDM, in general, very big terms and the consuming time to compute it is big, so we need a large computer memory and time. In addition, it is difficult and takes a lot of time to calculate the terms after the fifth term in the VIM for the KdV Eq. (14). It is convenient to examine Tables 1, 2, 3, 4, 5, 6, 7 and Figs. 1, 3, 4, 5 for comparing exact solution and numerical solutions which one gets by using these five methods. Tables and figures are calculated for these methods by using five terms and $c = 1$, except Tables 2 and 4. The results which are obtained by the EFDM are better than the results which are

Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Table 1 Error between the ADM (φ_5) and exact solutions for the KdV Eq. (14) at $c = 1$

t/x	10	15	20	25	30
0.1	2.00679×10^{-4}	1.35233×10^{-6}	9.11171×10^{-9}	6.13942×10^{-11}	4.13671×10^{-13}
0.2	2.22178×10^{-4}	1.49452×10^{-6}	1.00712×10^{-8}	6.78512×10^{-11}	4.57177×10^{-13}
0.3	2.45102×10^{-4}	1.65169×10^{-6}	1.11292×10^{-8}	7.49865×10^{-11}	5.05255×10^{-13}
0.4	2.70872×10^{-4}	1.82535×10^{-6}	1.22991×10^{-8}	8.28713×10^{-11}	5.58381×10^{-13}
0.5	2.99336×10^{-4}	2.01722×10^{-6}	1.35919×10^{-8}	9.15815×10^{-11}	6.17071×10^{-13}

Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Table 2 Error between the ADM (φ_{10}) and exact solutions for the KdV Eq. (34) at $c = 5$

t/x	10	15	20	25	30
0.1	2.00679×10^{-8}	1.35233×10^{-13}	9.11171×10^{-18}	6.13942×10^{-23}	4.1367×10^{-28}
0.2	2.22178×10^{-8}	1.49452×10^{-12}	1.00712×10^{-18}	6.78512×10^{-23}	4.5717×10^{-27}
0.3	2.45102×10^{-7}	1.65169×10^{-12}	1.11292×10^{-17}	7.49865×10^{-22}	5.0525×10^{-27}
0.4	2.70872×10^{-7}	1.82535×10^{-12}	1.22991×10^{-17}	8.28713×10^{-22}	5.5838×10^{-26}
0.5	2.99336×10^{-6}	2.01722×10^{-11}	1.35919×10^{-16}	9.15815×10^{-21}	6.1707×10^{-26}

Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Table 3 Error between the HAM (5th - order approximation) and exact solutions for the KdV Eq. (34) at $c = 1$ and $h = -0.5$

t/x	10	15	20	25	30
0.1	1.94277×10^{-4}	1.30915×10^{-6}	8.82101×10^{-9}	5.94355×10^{-11}	4.00473×10^{-13}
0.2	2.08022×10^{-4}	1.40179×10^{-6}	9.44517×10^{-9}	6.36412×10^{-11}	4.28811×10^{-13}
0.3	2.22925×10^{-4}	1.50222×10^{-6}	1.01219×10^{-8}	6.82009×10^{-11}	4.59534×10^{-13}
0.4	2.39102×10^{-4}	1.61125×10^{-6}	1.08565×10^{-8}	7.31508×10^{-11}	4.92886×10^{-13}
0.5	2.56683×10^{-4}	1.72975×10^{-6}	1.16549×10^{-8}	7.85304×10^{-11}	5.29134×10^{-13}

Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Table 4 Error between the HAM (10th - order approximation) and exact solutions for the KdV Eq. (34) at $c = 5$ and $h = 0.5$

t/x	10	15	20	25	30
0.1	8.12684×10^{-9}	1.13334×10^{-13}	1.58053×10^{-18}	2.20415×10^{-23}	3.0738×10^{-28}
0.2	3.62126×10^{-8}	5.05121×10^{-13}	7.04272×10^{-18}	9.82156×10^{-23}	1.3696×10^{-27}
0.3	8.86781×10^{-7}	1.23668×10^{-12}	1.72463×10^{-17}	2.40512×10^{-22}	3.3541×10^{-27}
0.4	1.87306×10^{-7}	2.61211×10^{-12}	3.64277×10^{-17}	5.08009×10^{-22}	7.0845×10^{-27}
0.5	5.02718×10^{-7}	7.01076×10^{-12}	9.77698×10^{-17}	1.36347×10^{-21}	1.9014×10^{-26}

Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Table 5 Error between the VIM (5 terms) and exact solutions for the KdV Eq. (34) at $c = 1$

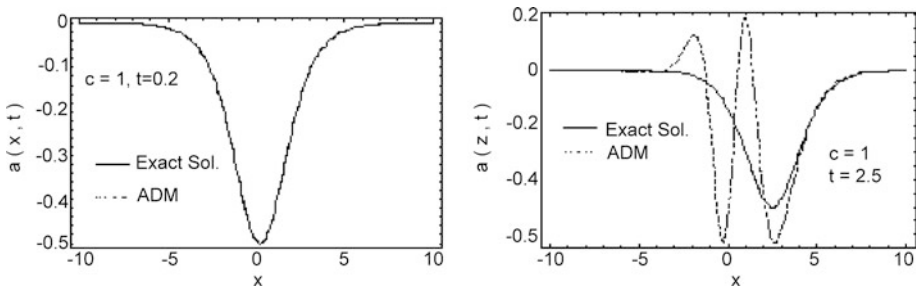
t/x	10	15	20	25	30
0.1	2.33263×10^{-4}	1.57189×10^{-6}	1.05913×10^{-8}	7.13638×10^{-11}	4.24233×10^{-13}
0.2	2.99151×10^{-4}	2.01592×10^{-6}	1.35833×10^{-8}	9.15216×10^{-11}	6.16666×10^{-13}
0.3	3.81873×10^{-4}	2.57322×10^{-6}	1.73383×10^{-8}	1.16824×10^{-10}	7.87156×10^{-13}
0.4	4.84308×10^{-4}	3.26302×10^{-6}	2.19864×10^{-8}	1.48141×10^{-10}	9.98164×10^{-13}
0.5	6.09593×10^{-4}	4.10597×10^{-6}	2.76657×10^{-8}	1.86417×10^{-10}	1.25602×10^{-12}

Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Table 6 Error between the HPM (five iterations) and exact solutions for the KdV Eq. (34) at $c = 1$

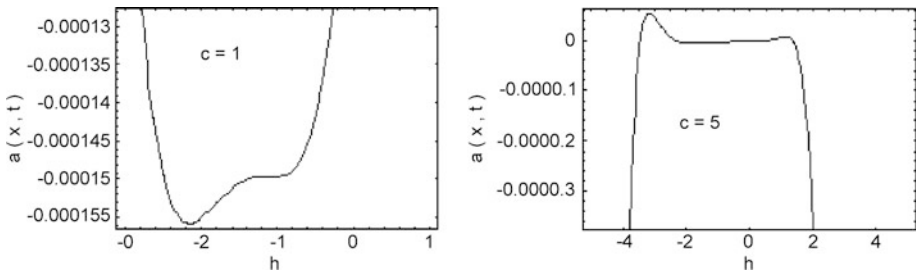
t/x	10	15	20	25	30
0.1	2.00679×10^{-4}	1.35232×10^{-6}	9.11171×10^{-9}	6.13942×10^{-11}	4.13671×10^{-13}
0.2	2.21782×10^{-4}	1.49452×10^{-6}	1.00734×10^{-8}	6.78513×10^{-11}	4.57177×10^{-13}
0.3	2.45102×10^{-4}	1.65169×10^{-6}	1.11296×10^{-8}	7.49865×10^{-11}	5.05255×10^{-13}
0.4	2.70871×10^{-4}	1.82535×10^{-6}	1.22991×10^{-8}	8.28716×10^{-11}	5.58381×10^{-13}
0.5	2.99337×10^{-4}	2.01722×10^{-6}	1.35919×10^{-8}	9.15815×10^{-11}	6.17071×10^{-13}

Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Table 7 Error between the EFDm and exact solutions for the KdV Eq. (34) at $c = 1$, $h = 0.05$ and $k = 0.025$

t/x	10	15	20	25	30
0.1	6.15156×10^{-6}	4.14842×10^{-8}	2.79520×10^{-10}	1.88339×10^{-12}	1.26902×10^{-14}
0.2	7.54177×10^{-6}	5.09160×10^{-8}	3.43074×10^{-10}	2.31161×10^{-12}	1.55755×10^{-14}
0.3	9.14755×10^{-6}	6.19126×10^{-8}	4.17173×10^{-10}	1.70489×10^{-12}	1.89396×10^{-14}
0.4	1.10086×10^{-5}	7.46977×10^{-8}	5.03358×10^{-10}	3.39162×10^{-12}	1.38607×10^{-14}
0.5	1.35102×10^{-5}	8.94343×10^{-8}	6.03380×10^{-10}	4.06554×10^{-12}	2.73934×10^{-14}



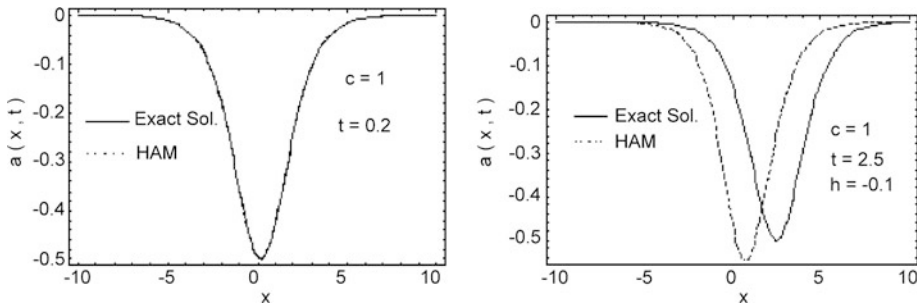
Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 1 The comparison of the ADM (φ_5) and the exact solutions for Eq. (34) at $c = 1$



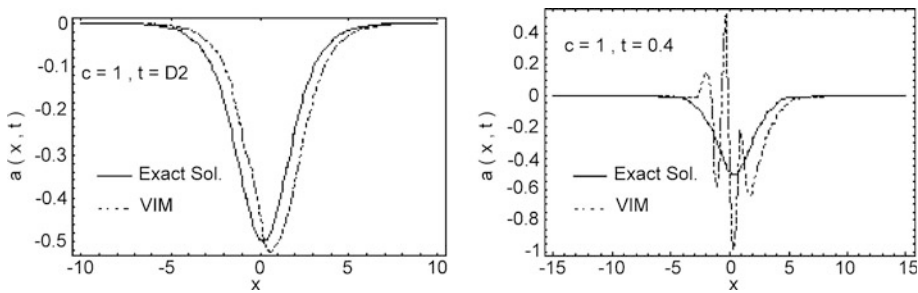
Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 2 The h -curves for $x = 10$, $t = 0.5$, $c = 1$ for 5th-order approximation and $x = 10$, $t = 0.5$, $c = 5$ for 10th-order approximation of $u(x, t)$, respectively

obtained by the other four (semi-inverse) methods for five terms (see Tables 1, 3, 5, 6, 7. If you increase the numbers of terms in the semi-inverse methods, we can see that the results are better (see

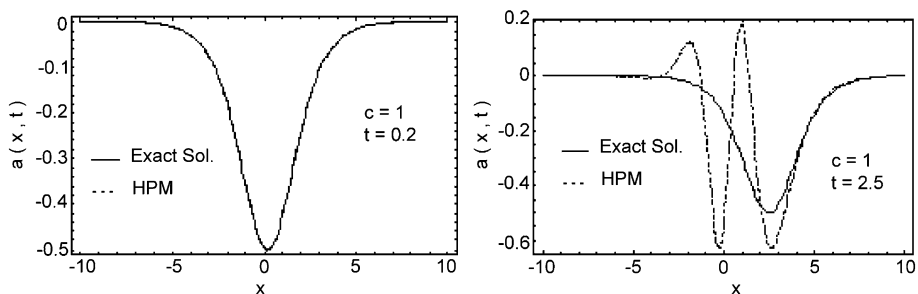
Tables 2 and 4. Obtaining serial solutions with the semi-inverse methods are easier and more efficient than pure numerical methods (the finite difference methods, the finite element methods,



Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 3 The comparison of the HAM (5th - order approximation) and the exact solutions for Eq. (34) at $c = 1$



Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 4 The comparison of the VIM (five iterations) and the exact solutions for Eq. (34) at $c = 1$



Some Numerical Methods for Solving the Korteweg-de Vries Equation (KdV), Fig. 5 The comparison of the HPM (five iterations) and the exact solutions for Eq. (34) at $c = 1$

B-spline methods, etc.). Finding numerical results by the EFDm are difficult and take time. The present methods give nearly the same results for $c = 1$ and large values of x but the HAM is the more sensitive method than the others. Because it is possible to obtain a rapidly convergent solution for large and small values of x and t by choosing convenient helper the auxiliary parameter h . For this it is enough to consider Figs. 1, 3, 4, 5. Finally, we point out that, for given equations

with initial value, the corresponding analytical and numerical solutions are obtained according to the recurrence equations by using Mathematica.

Future Directions

The Korteweg-de Vries equation, which is the most important equation of mathematical physics, has solitons and solitary wave solutions. It

wassolved by using analytical and numerical methods by many authors. We have mentioned these analytical and numerical methods in the introduction. The theoretical studies on the KdV equation have been nearly completed, but we may find new soliton solutions by producing new analytical methods or expanding the old methods. It is also possible to perform same operations on numerical methods. We may show the interaction of KdV solitons with each other by numerical.

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Nonlinear Internal Waves

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Glossary

Nonlinear waves Nonlinear waves are such waves which arise as solutions of the nonlinear mathematical models simulating physical phenomena in fluids.

Shallow water Shallow water means water waves for which the ratio between the amplitude and the wave length is relatively small. The linear theory of motion is inadequate for the description of shallow water waves.

Internal waves Internal waves are gravity waves that oscillate within a fluid medium. They arise from perturbations to hydrostatic equilibrium, where balance is maintained between the force

of gravity and the buoyant restoring force. A simple example is a wave propagating on the interface between two fluids of different densities, such as oil and water. Internal waves typically have much lower frequencies and higher amplitudes than surface gravity waves because the density differences (and therefore the restoring forces) within a fluid are usually much smaller than the density of the fluid itself.

Pycnocline Pycnocline is a rapid change in water density with depth. In freshwater environments, such as lakes, this density change is primarily caused by water temperature, while in seawater environments such as oceans, the density change may be caused by changes in water temperature and/or salinity.

Solitary waves Solitary waves are localized traveling waves, which asymptotically tend to zero at large distances.

Solitons Solitons are waves which appear as a result of a balance between a weakly nonlinear convection and a linear dispersion. The solitons are localized highly stable waves that retain its identity (shape and speed) upon interaction and resemble particle like behavior. In the case of a collision, solitons undergo a phase shift.

Baroclinic fluid Baroclinic fluid is such a fluid for which the density depends on both the temperature and the pressure. In atmospheric terms, the baroclinic areas are generally found in the mid-latitude/polar regions.

Barotropic fluid Barotropic fluid is such a fluid for which the density depends only on the pressure. In atmospheric terms, the barotropic zones of the Earth are generally found in the central latitudes or tropics.

Introduction

The work on internal gravity waves has been started in the middle of the twentieth century by Keulegan (1953) followed by Long (1956), where they applied Boussinesq's and Rayleigh's techniques, respectively, on free surface waves to the problem of internal waves.

In physical oceanography, fluid density varies with the depth due to the salinity and the temperature of the fluid (Le Blond and Mysak 1978). Several works, dealing with variable density, have been worked out (e.g., Peters and Stoker 1960; Robinson 1969; Garret and Munk 1979; Helal and Moline 1981; Kabbaj 1985; Abou-Dina and Helal 1990, 1995, ...).

The geophysical problem of the propagation waves in stratified fluid is very important in physical oceanography. The physical problem is simulated by a suitable model of a free-surface and interface fluid flow over a horizontal bottom or over a certain topography.

The fluid in ocean may be considered as a stratified one according to the variation of the salinity and temperature, due to the weather, with respect to the vertical coordinate normal to the surface. The mathematical model of a two-fluid system (stratified fluid) is a good description to simulate wave motions in physical applications.

Based on results of many field observations, it is found that the fluid's density changes rapidly within the regions of pycnocline.

Nonlinear theoretical models for waves in a two-fluid system have been established with various restrictions on length scales. Among these, Benjamin (1966) derived the KdV equation for thin layers in comparison with the wave length. Later, Benjamin (1967), Davis and Acrivos (1967), and Ono (1975) constructed the BO equation under the assumptions of a thin upper layer and an infinitely deep lower layer. Kubota et al. (1978) and Choi and Camassa (1996, 1999) carried out a series of investigations to derive model equations for weakly and strongly nonlinear wave propagation in stratified fluids. The upper boundary is allowed to be either free or rigid. The assumptions they stated are as follows:

1. The wavelength of interfacial wave is long.
2. The upper layer is thin.
3. No depth restriction is made on the lower layer.

The effect of a submerged obstacle on an incident wave inside an ideal two-layer stratified shallow water has been studied in two dimensions by

Abou-Dina and Helal (1992) by applying the shallow water approximation theory.

In 1995, Pinettes, Renouard, and Germain presented a detailed analytical and experimental study of the oblique passing of a solitary wave over a shelf in a two-layer fluid.

Later, Barthelemy et al. (2000) applied the WKBJ technique to investigate theoretically the scattering of a surface long wave by a step bottom in a two-layer fluid. This method enabled to overcome the main inconsistency of the shallow water theory in the presence of obstacles.

Matsuno (1993) attempted to unify the KdV, BO, and ILW models using the assumption of small wave steepness. Lynett and Liu (2002) assumed a small density difference and derived a set of model equations.

Recently, Craig et al. (2005) presented a Hamiltonian perturbation theory for the long-wave limit and provided a uniform treatment of various long-wave models (KdV, BO, and ILW models). Their formulation is shown to be very effective for perturbation calculations and represents a basis for numerical simulations. For a single-fluid system, in the last two decades, the traditional Boussinesq equations have been improved to extend their applicability from shallow water to deep water (see, e.g., Nwogu 1993; Chen and Liu 1995; Wei et al. 1995; Madsen and Schaffer 1998; Gobbi et al. 2000).

Most recently, Liu et al. (2008) published an excellent study on the essential properties of Boussinesq equations for internal and surface waves in a two-fluid system.

Most studies applied the technique of the Padé approximant to allow a much higher-order accuracy without increasing the order of the derivatives.

Although theoretical works for surface waves are excessive, studies of internal waves based on Boussinesq equations are still lacking.

In the present work, and for a better description of the physical problem, we shall investigate, in the frame of the shallow water theory, the effect of the bounded motion of a vertical plane wave maker on the generation and propagation of waves inside a stratified fluid. The wave maker is situated at the finite end of a semi-infinite

channel of constant depth occupied by two non-immiscible fluid layers of different constant densities. The problem considered here is supposed to be a two-dimensional one with an irrotational motion, the fluid layers are taken ideal, and the surface tension is neglected.

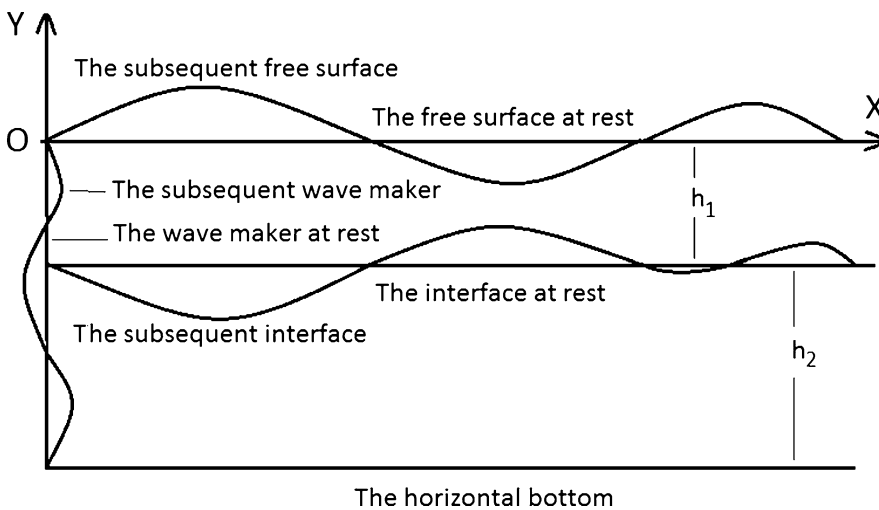
Double series representation for the unknown functions, in terms of a certain small parameter characterizing the motion of the wave maker, is used. The first few orders of approximations obtained following this technique indicate the presence of a secular term increasing indefinitely in going away from the wave maker and vanishing at the position of this later. To overcome this physically unacceptable result, a multiple-scale transformation of variable is carried out, and a single-series representation, for the unknown functions, is proposed. This technique leads to solutions up to the fifth order of approximations free from secular terms and shows that the generation and propagation processes of internal waves are governed by the well-known KdV equation. The initial conditions needed for the solution of this nonlinear partial differential equation are obtained from the results of the preceding method at the position of the wave maker where the solution is free from secular terms. On the light of this result, it is expected to obtain an internal solitonic waves.

Problem and Frame of Reference

A vertical plane wave maker is situated at the finite extremity of a semi-infinite channel with a constant depth. The channel is occupied by two layers of immiscible and inviscid liquids of constant densities. The wave maker is set in motion generating waves which propagate, inside both fluid layers, toward the downstream extremity of the channel. The required is to calculate the produced waves and also to determine the motion that should be assigned to the wave maker in order to generate internal waves only in the channel.

In the mathematical model, the problem is assumed to be a two-dimensional one, the flow is considered irrotational, and the free surface and the interface are assumed to remain always near their positions at rest. Furthermore, the motion of the wave maker is considered slow and bounded and the problem is studied in the frame of the nonlinear shallow water theory of motion.

A fixed rectangular system of reference is used for the description of the motion of the fluid. The origin O is taken in the free surface at rest with horizontal x -axis pointing along the direction of propagation of waves. The y -axis is directed vertically upward (Fig. 1).



Nonlinear Internal Waves, Fig. 1 Problem and frame of reference

Notation

The following notation is used throughout this entry:

g	The acceleration of gravity
$h_1, h_2 (h = h_1 + h_2)$	The constant depths of the upper and lower layers of the fluid, respectively
$P(x, y)$	The pressure
S	The stocks parameter
t	The time
$D_x = \varepsilon f(y, \varepsilon t)$	The horizontal displacement of the wave maker ($= \varepsilon f^{(1)}(y, \varepsilon t)$ in the upper layer and $\varepsilon f^{(2)}(y, \varepsilon t)$ in the lower layer)
$\Phi^{(j)}(x, y, t)$	The velocity potentials: $j = 1, 2$ for the upper and lower layers, respectively
$y = \eta^{(1)}(x, t)$	The equation of the free surface
$y = -h_1 + \eta^{(2)}(x, t)$	The equation of the interface
(x, y)	Cartesian coordinates of a point
$\delta_{n, m}$	The Kronecker delta = 1 if $n = m$; 0 if $n \neq m$
ε	A small parameter
ρ_1, ρ_2	The constant densities of the upper and lower layers, respectively ($\rho_1 < \rho_2$)
D_z	An operator of total differentiation w.r.t. z
$\partial_{x, y, \dots, z}$	An operator of partial differentiation w.r.t. $x, y, \dots, \text{znd } z$

Superscripts

- 1 upper layer
- 2 lower layer
- ’, ’’ first and second derivatives w.r.t. the argument of the superscripted function

Subscripts

- e The barotropic (external) mode
- i The baroclinic (internal) mode

Equations of Motion

The general system of equations and simplifying conditions describing water wave motion are expressed in terms of the velocity potential in the case of irrotational motion for both homogeneous and stratified fluids (Boussinesq 1871; Stoker 1957; Wehausen and Laitone 1960).

As in Abou-Dina and Helal (1990, 1992, 1995), we start with the set of distorted variables: $\hat{x}, \hat{y}, \hat{t}$ defined in terms of x, y, t as:

$$\hat{x} = \varepsilon x, \quad \hat{y} = y, \quad \hat{t} = \varepsilon t, \quad (1)$$

where ε is a small parameter.

Let us denote by $\hat{\Phi}^{(j)}, \hat{P}^{(j)}$ and $\hat{\eta}^{(j)}$ the functions $\Phi^{(j)}(\hat{x}/\varepsilon, \hat{y}, \hat{t}/\varepsilon)$, $P^{(j)}(\hat{x}/\varepsilon, \hat{y}, \hat{t}/\varepsilon)$, and $\eta^{(j)}(\hat{x}/\varepsilon, \hat{t}/\varepsilon)$, respectively.

According to the physical and simplifying conditions adopted here, the system of equations and conditions governing the problem is written in terms of the distorted variables $\hat{x}, \hat{y}, \hat{t}$ as follows Abou-Dina and Helal (1990, 1992, 1995) and Abou-Dina and Hassan (2006):

- (I) In the fluid layers with constant densities:

$$\varepsilon^2 \partial_{\hat{x}\hat{x}} \hat{\Phi}^{(j)} + \partial_{\hat{y}\hat{y}} \hat{\Phi}^{(j)} = 0, \quad (2)$$

$$\begin{aligned} & \hat{P}^{(j)}(\hat{x}, \hat{y}, \hat{t}) \\ &= \rho_j \left[\varepsilon \partial_{\hat{t}} \hat{\Phi}^{(j)} + \frac{1}{2} \left(\varepsilon^2 \left(\partial_{\hat{x}} \hat{\Phi}^{(j)} \right)^2 + \left(\partial_{\hat{y}} \hat{\Phi}^{(j)} \right)^2 \right) + g \hat{y} \right], \end{aligned} \quad (3)$$

where j hereafter stands for both 1 and 2.

- (II) On the free surface which is impermeable and isobaric:

$$\begin{aligned}\partial_{\hat{y}}\hat{\Phi}^{(1)} &= \varepsilon^2 \partial_{\hat{x}}\hat{\eta}^{(1)}\partial_{\hat{x}}\hat{\Phi}^{(1)} + \varepsilon \partial_{\hat{t}}\hat{\eta}^{(1)} \quad \text{at } \hat{y} \\ &= \hat{\eta}^{(1)}(\hat{x}, \hat{t}),\end{aligned}\quad (4)$$

$$\begin{aligned}\varepsilon \partial_{\hat{t}}\hat{\Phi}^{(1)} + \frac{1}{2} \left(\varepsilon^2 \left(\partial_{\hat{x}}\hat{\Phi}^{(1)} \right)^2 + \left(\partial_{\hat{y}}\hat{\Phi}^{(1)} \right)^2 \right) \\ + g\hat{y} = 0 \quad \text{at } \hat{y} = \hat{\eta}^{(1)}(\hat{x}, \hat{t}).\end{aligned}\quad (5)$$

(III) At the interface, the impermeability gives:

$$\begin{aligned}\partial_{\hat{y}}\hat{\Phi}^{(j)} &= \varepsilon^2 \partial_{\hat{x}}\hat{\eta}^{(2)}\partial_{\hat{x}}\hat{\Phi}^{(j)} + \varepsilon \partial_{\hat{t}}\hat{\eta}^{(2)} \quad \text{at } \hat{y} \\ &= -h_1 + \hat{\eta}^{(2)}(\hat{x}, \hat{t}),\end{aligned}\quad (6)$$

Also the compatibility condition for the pressure across this boundary implies:

$$\begin{aligned}\rho_1 \left[g \left(h_1 + \hat{\eta}^{(2)} \right) + \varepsilon \partial_{\hat{t}}\hat{\Phi}^{(1)} + \frac{1}{2} \left(\varepsilon^2 \left(\partial_{\hat{x}}\hat{\Phi}^{(1)} \right)^2 + \left(\partial_{\hat{y}}\hat{\Phi}^{(1)} \right)^2 \right) \right] = \\ \rho_2 \left[g \left(h_1 + \hat{\eta}^{(2)} \right) + \varepsilon \partial_{\hat{t}}\hat{\Phi}^{(2)} + \frac{1}{2} \left(\varepsilon^2 \left(\partial_{\hat{x}}\hat{\Phi}^{(2)} \right)^2 + \left(\partial_{\hat{y}}\hat{\Phi}^{(2)} \right)^2 \right) \right] \\ \text{at } \hat{y} = -h_1 + \hat{\eta}^{(2)}(\hat{x}, \hat{t}).\end{aligned}\quad (7)$$

(IV) On the horizontal bottom:

$$\partial_{\hat{y}}\hat{\Phi}^{(2)} = 0 \quad \text{at } \hat{y} = -h. \quad (8)$$

(V) The radiation condition implies that no wave comes from infinity.

(VI) The initial conditions: At the initial instant of time ($\hat{t} = 0$), the fluid is at rest with horizontal free surface at $\hat{y} = 0$ and interface at $\hat{y} = -h_1$. This initial condition assumes that the functions $f^{(j)}(\hat{y}, \hat{t})$ have initial vanishing values and implies at $\hat{t} = 0$ that:

- Inside the fluid layers

$$\hat{\Phi}^{(j)}(\hat{x}, \hat{y}, 0) = 0, \quad (9a)$$

$$\partial_{\hat{x}}\hat{\Phi}^{(j)}(\hat{x}, \hat{y}, 0) = 0, \quad (9b)$$

$$\partial_{\hat{y}}\hat{\Phi}^{(j)}(\hat{x}, \hat{y}, 0) = 0, \quad (9c)$$

- At the free surface

$$\hat{\eta}^{(2)}(\hat{x}, 0) = 0 \quad (9d)$$

- At the interface

$$\hat{\eta}^{(2)}(\hat{x}, 0) = 0 \quad (9e)$$

provided that

$$f^{(j)}(\hat{y}, 0) = 0. \quad \text{for } j = 1, 2 \quad (9f)$$

(VII) On the wave maker:

$$\begin{aligned}\partial_{\hat{x}}\hat{\Phi}^{(j)}(\hat{x}, \hat{y}, \hat{t}) &= \varepsilon \frac{\partial}{\partial \hat{t}} f^{(j)}(\hat{y}, \hat{t}), \quad \text{at } \hat{x} \\ &= 0, \quad -h + h_2\delta_{j,1} \leq \hat{y} \\ &\leq h_1\delta_{j,2}.\end{aligned}\quad (10)$$

To the system of equations and conditions governing the problem in the non-distorted (physical) space, we have to replace ε by unity and omit the hats (^) over the symbols in Eqs. (2)–(10).

The Shallow Water Theory

In the framework of the shallow water theory, the nonlinear system of Eqs. (2)–(10) will be solved using the double series representation:

$$\hat{\Phi}^{(j)}(\hat{x}, \hat{y}, \hat{t}) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varepsilon^n \exp\left(-\frac{m\pi\hat{x}}{h_j\varepsilon}\right) \times \hat{\Phi}_{n,m}^{(j)}(\hat{x}, \hat{y}, \hat{t}), \quad (11a)$$

$$\hat{\eta}^{(j)}(\hat{x}, \hat{t}) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \varepsilon^n \exp\left(-\frac{m\pi\hat{x}}{h_j\varepsilon}\right) \hat{\eta}_{n,m}^{(j)}(\hat{x}, \hat{t}). \quad (11b)$$

The validity of using the above representation has been studied by Germain (1971, 1972). The small parameter ε represents the ratio of the water depth to the wave length. Hence, $S \geq 1$, and we are dealing with the nonlinear acoustic analogy.

According to the theory under consideration, at each order (n,m), the above system of equations and conditions must be verified.

For simplification in the following, the hats ($\hat{}$) over the symbols will be omitted.

Verification of the Homogeneous Equations

The above procedure, when applied to the homogeneous Eqs. (2)–(8) along with the radiation condition (V), can lead after some manipulations to the following expression for the total velocity potential up to the second order of the small parameter ε :

$$\begin{aligned} \Phi^{(j)}(x, y, t) = & \varepsilon [A^{(j)}(x - C_1 t) + B^{(j)}(x - C_2 t)] \\ & + \varepsilon^2 [R^{(j)}(x - C_1 t) + S^{(j)}(x - C_2 t)] \\ & + \varepsilon^2 \sum_{m=1}^{\infty} A_m^{(j)}(t) \exp\left(-\frac{m\pi x}{h_j \varepsilon}\right) \\ & \cos\left(\frac{m\pi}{h_j} \left(y + \delta_{j2} h^2\right)\right) + O(\varepsilon^3), \end{aligned} \quad (12)$$

where:

$$C_j^2 = \frac{g}{2} \left\{ h - (-1)^j \sqrt{(h_1 - h_2)^2 + 4(\rho_1/\rho_2)h_1 h_2} \right\} \quad (13a)$$

The functions $A^{(j)}$, $B^{(j)}$, and $A_m^{(j)}$ are arbitrary functions to be determined, with:

$$A^{(1)}(x - C_1 t) = \gamma_1 A^{(2)}(x - C_1 t) \quad (13b)$$

$$B^{(1)}(x - C_2 t) = \gamma_2 B^{(2)}(x - C_2 t) \quad (13c)$$

and:

$$\gamma_j = \frac{gh_2}{C_j^2 - gh_1}. \quad (13d)$$

The functions $R^{(j)}$ and $S^{(j)}$ are arbitrary functions combined by similar relations to those in Eq. (12) in terms of $A^{(j)}$ and $B^{(j)}$, respectively.

Equations (11)–(13) indicate the existence of two different wave speeds C_1 and C_2 ($C_1 > C_2$). Accordingly, we obtain local oscillations along with progressive waves of two different modes:

- (i) External (or barotropic) modes corresponding to the wave speed C_1
- (ii) Internal (or baroclinic) modes corresponding to the wave speed C_2

The corresponding expressions of the free surface and the interface are given, to the third order, by:

$$\eta^{(1)}(x, t) = -(\varepsilon/g) \partial_t \Phi^{(1)}(x, y, t) \text{ at } y = 0, \quad (14a)$$

$$\begin{aligned} \eta^{(2)}(x, t) = & \left(\frac{\varepsilon}{g(\rho_2 - \rho_1)} \right) \\ & \times \partial_t \{ \rho_1 \Phi^{(1)}(x, y, t) - \rho_2 \Phi^{(2)}(x, y, t) \} \text{ at } \\ & y = -h_1. \end{aligned} \quad (14b)$$

Complete Determination of the Solution

To complete the determination of the velocity potentials $\Phi^{(j)}(x, y, t)$, we verify the initial conditions (9) and the condition on the wave maker (10) which yield, together with expressions (12) for the velocity potential functions, the following relations:

- In the upper layer:

$$\begin{aligned} A^{(1)'}(-C_1 t) + B^{(1)'}(-C_2 t) \\ - \frac{\pi}{h_1} \sum_{m=1}^{\infty} m A_m^{(1)}(t) \cos\left(\frac{m\pi}{h_1} y\right) = \frac{\partial}{\partial t} f^{(1)}(y, t), \\ -h_1 \leq y \leq 0 \end{aligned} \quad (15a)$$

- In the lower layer:

$$\begin{aligned}
& A^{(2)'}(-C_1 t) + B^{(2)'}(-C_2 t) \\
& - \frac{\pi}{h_2} \sum_{m=1}^{\infty} (-1)^m m A_m^{(2)}(t) \cos\left(\frac{m\pi}{h_2}(y+h_1)\right) \\
& = \frac{\partial}{\partial t} f^{(2)}(y, t), \quad -h \leq y \leq h_1
\end{aligned} \tag{15b}$$

Relations (15a) and (15b) together with the initial conditions (9) lead, after some manipulations, to the following expressions:

$$\begin{aligned}
A^{(2)}(x - C_1 t) = \frac{C_1}{\gamma_1 - \gamma_2} & \left[\frac{\gamma_2}{h_2} \int_{-h}^{-h_1} f^{(2)}\left(y, -\frac{x - C_1 t}{C_1}\right) dy \right. \\
& \left. - \frac{1}{h_1} \int_{-h_1}^0 f^{(1)}\left(y, -\frac{x - C_1 t}{C_1}\right) dy \right]
\end{aligned} \tag{16a}$$

$$\begin{aligned}
B^{(2)}(x - C_2 t) = \frac{C_2}{\gamma_2 - \gamma_1} & \left[\frac{\gamma_1}{h_2} \int_{-h}^{-h_1} f^{(2)}\left(y, -\frac{x - C_2 t}{C_2}\right) dy \right. \\
& \left. - \frac{1}{h_1} \int_{-h_1}^0 f^{(1)}\left(y, -\frac{x - C_2 t}{C_2}\right) dy \right]
\end{aligned} \tag{16b}$$

$$A_m^{(1)}(t) = \frac{-2}{m\pi} \int_{-h_1}^0 \left[\frac{\partial}{\partial t} f^{(1)}(y, t) \right] \cos\left(\frac{m\pi}{h_1} y\right) dy \tag{16c}$$

$$\begin{aligned}
A_m^{(1)}(t) = \frac{-2}{m\pi} \int_{-h}^{-h_1} & \left[\frac{\partial}{\partial t} f^{(2)}(y, t) \right] \cos\left(\frac{m\pi}{h_2}(y+h_1)\right) \\
& \times dy
\end{aligned} \tag{16d}$$

The functions $A^{(1)}(x - C_1 t)$ and $B^{(1)}(x - C_2 t)$ are given in terms of the functions $A^{(1)}(x - C_1 t)$ and $B^{(1)}(x - C_2 t)$ by relations (13b, c).

elevation, $\eta_e^{(1)}(x, t)$ and $\eta_i^{(1)}(x, t)$, are given using (12) and (14a) as:

$$\begin{aligned}
\eta_e^{(1)}(x, t) = \frac{-\varepsilon^2 C_1 \gamma_1}{g(\gamma_1 - \gamma_2)} \frac{\partial}{\partial t} & \left[\frac{\gamma_2}{h_2} \int_{-h}^{-h_1} f^{(2)}\left(y, -\frac{x - C_1 t}{C_1}\right) dy \right. \\
& \left. - \frac{1}{h_1} \int_{-h_1}^0 f^{(1)}\left(y, -\frac{x - C_1 t}{C_1}\right) dy \right]
\end{aligned} \tag{16e}$$

$$\begin{aligned}
\eta_i^{(1)}(x, t) = \frac{-\varepsilon^2 C_2 \gamma_2}{g(\gamma_2 - \gamma_1)} \frac{\partial}{\partial t} & \left[\frac{\gamma_1}{h_2} \int_{-h}^{-h_1} f^{(2)}\left(y, -\frac{x - C_2 t}{C_2}\right) dy \right. \\
& \left. - \frac{1}{h_1} \int_{-h_1}^0 f^{(1)}\left(y, -\frac{x - C_2 t}{C_2}\right) dy \right]
\end{aligned} \tag{16f}$$

The contributions of external and internal modes to the interface elevation, $\eta_e^{(2)}(x, t)$ and $\eta_i^{(2)}(x, t)$, are given in terms of $\eta_e^{(1)}(x, t)$ and $\eta_i^{(1)}(x, t)$ using (12)–(14) as:

$$\eta_e^{(2)}(x, t) = \frac{\rho_2 - \gamma_1 \rho_1}{\gamma_1(\rho_2 - \rho_1)} \eta_e^{(1)}(x, t) \tag{17a}$$

$$\eta_i^{(2)}(x, t) = \frac{\rho_2 - \gamma_1 \rho_1}{\gamma_1(\rho_2 - \rho_1)} \eta_i^{(1)}(x, t) \tag{17b}$$

It is well known, in the studies dealing with stratified fluids, that the external mode is dominant in the neighborhood of the free surface and has a negligible contribution on the interface while the internal mode is dominant in the neighborhood of the interface and has a negligible contribution on the free surface. If the motion of the wave maker is such that the function $\eta_e^{(2)}(x, t)$ given by (16a) vanishes, the major contribution of both modes is localized in the neighborhood of the interface, and in such cases we say that the wave maker generates internal waves only in the channel.

Free Surface and Interface Elevations of Different Modes

The contributions of the progressive waves of external and internal modes to the free surface

Secular Term

A suitable form of the double series representation (11) was used in studying certain problems in the

case of homogeneous fluids (Abou-Dina and Helal 1992; Abou-Dina and Hassan 2006). It has been shown that this procedure leads to a secular term of the third order in ε in the expression of the velocity potential. This secular term, which increases indefinitely with the increase of the variable x , vanishes at $x = 0$. The same result can be shown in the case of stratified fluids, according to the analysis of Abou-Dina and Helal (1990). Hence, although expressions (12)–(17) are valid at $x = 0$, they are not adequate for the description of the propagation of waves far from the wave maker. This unacceptable result is due to certain aspects of the mathematical procedure used (see Abou-Dina and Helal 1992; Abou-Dina and Hassan 2006 for the justification) and needs to be remedied.

Our main interest, in the remaining part of the present paper, is to modify the mathematical procedure used above in order to describe the propagation of waves generated by the wave maker in going toward the downstream extremity of the channel.

Multiple-Scale Transformation of Variables

We use the set of variables u , v , and y defined in terms of the distorted set x , y , t by (cf. Benney and Lin 1960; Temperville 1985):

$$u = x - Ct, \quad v = \varepsilon^2 x, \quad (18)$$

where C is a real constant to be precised.

Equations (2), (4), up to (8) can be written in terms of u , v , and y as:

(i) In the fluid mass:

$$\varepsilon^6 \partial_{vv} \Phi^{(j)} + 2\varepsilon^4 \partial_{uv} \Phi^{(j)} + \varepsilon^2 \partial_{uu} \Phi^{(j)} + \partial_{yy} \Phi^{(j)} = 0. \quad (19)$$

(ii) On the free surface:

$$\begin{aligned} \partial_y \Phi^{(1)} &= \varepsilon^6 \partial_v \eta^{(1)} \partial_v \Phi^{(1)} \\ &+ \varepsilon^4 \{ \partial_u \eta^{(1)} \partial_v \Phi^{(1)} + \partial_v \eta^{(1)} \partial_u \Phi^{(1)} \} \\ &+ \varepsilon^2 \partial_u \eta^{(1)} \partial_u \Phi^{(1)} - \varepsilon C \partial_u \eta^{(1)} \\ &\text{at } y = \eta^{(1)}(u, v), \end{aligned} \quad (20a)$$

$$\begin{aligned} &g \eta^{(1)} - \varepsilon C \partial_u \Phi^{(1)} \\ &+ \frac{1}{2} \left[\varepsilon^2 \{ \partial_u \Phi^{(1)} + \varepsilon^2 \partial_v \Phi^{(1)} \}^2 + \left(\partial_y \Phi^{(1)} \right)^2 \right] = 0 \\ &\text{at } y = \eta^{(1)}(u, v) \end{aligned} \quad (20b)$$

(iii) On the interface:

$$\begin{aligned} \partial_y \Phi^{(j)} &= \varepsilon^6 \partial_v \eta^{(2)} \partial_v \Phi^{(j)} \\ &+ \varepsilon^4 \{ \partial_u \eta^{(2)} \partial_v \Phi^{(j)} + \partial_v \eta^{(2)} \partial_u \Phi^{(j)} \} \\ &+ \varepsilon^2 \partial_u \eta^{(2)} \partial_u \Phi^{(j)} - \varepsilon C \partial_u \eta^{(2)} \\ &\text{at } y = \eta^{(1)}(u, v) \text{ for } j = 1, 2 \end{aligned} \quad (21a)$$

$$\begin{aligned} &\rho_1 \left\{ g(h_1 + \eta^{(2)}) - \varepsilon C \partial_u \Phi^{(1)} + \frac{1}{2} \left[\varepsilon^2 \{ \partial_u \Phi^{(1)} + \varepsilon^2 \partial_v \Phi^{(1)} \}^2 + \left(\partial_y \Phi^{(1)} \right)^2 \right] \right\} = \\ &\rho_2 \left\{ g(h_1 + \eta^{(2)}) - \varepsilon C \partial_u \Phi^{(2)} + \frac{1}{2} \left[\varepsilon^2 \{ \partial_u \Phi^{(2)} + \varepsilon^2 \partial_v \Phi^{(2)} \}^2 + \left(\partial_y \Phi^{(2)} \right)^2 \right] \right\} \\ &\text{at } y = -h_1 + \eta^{(2)}(u, v) \end{aligned} \quad (21b)$$

(iv) On the bottom:

$$\partial_y \Phi^{(2)} = 0 \quad \text{at } y = -h. \quad (22)$$

In the regions where we are interested in, far

from the wave maker, the contribution of the local disturbance on the motion of the fluid layers can be neglected, and we use the following representations for the functions $\Phi^{(j)}(u, v, y)$ and $\eta^{(j)}(u, v)$ (cf. Temperville 1985):

$$\Phi^{(j)}(u, v, y) = \sum_{n=1}^{\infty} \varepsilon^{2n-1} \Phi_{2n-1}^{(j)}(u, v, y), \quad (23a)$$

$$\eta^{(j)}(u, v) = \sum_{n=1}^{\infty} \varepsilon^{2n} \eta_{2n}^{(j)}(u, v). \quad (23b) \quad \text{and:}$$

$$M = 3\gamma + \frac{2(1-\gamma)(\rho_2 - \gamma\rho_1)}{\gamma(\rho_2 - \rho_1)}, \quad (28b)$$

$$N = 2(h_2 + \gamma h_1) \quad (28c)$$

$$\gamma = \frac{gh_2}{(C^2 - gh_1)}. \quad (28d)$$

Derivation of the KdV Equation

It can be shown from the above equations that for $n = 1$, the functions $\Phi_1^{(j)}(u, v, y)$, $j = 1, 2$ are independent of the variable y and that

$$\eta_2^{(1)}(u, v) = \frac{C}{g} \partial_u \Phi_1^{(j)}(u, v, y), \quad \text{at } y = 0, \quad (24)$$

$$\eta_2^{(2)}(u, v) = \frac{C}{g(\rho_2 - \rho_1)} \left[\rho_2 \partial_u \Phi_1^{(2)}(u, v, y) - \rho_1 \partial_u \Phi_1^{(1)}(u, v, y) \right], \quad \text{at } y = -h_1 \quad (25)$$

Also, for $n = 2$, the values of the constant C appearing in (18) can be obtained as:

$$C = \begin{cases} C_1 & \text{for the barotropic (external) mode,} \\ C_2 & \text{for the baroclinic (internal) mode,} \end{cases} \quad (26)$$

where C_j is given by (13a) for $j = 1, 2$.

It can also be shown that the results up to $n = 3$, contain no secular terms and that the function $\eta_2^{(1)}(u, v)$, characterizing the free surface elevation satisfies the following KdV equation:

$$L \partial_{uuu} \eta_2^{(1)} + M \eta_2^{(1)} \partial_u \eta_2^{(1)} + N \partial_v \eta_2^{(1)} = 0, \quad (27)$$

where:

$$L = \frac{1}{6} (2h^3 + h_1^3 - 3hh_1^2) + \frac{\gamma}{2} \left(\frac{C^2}{g} - \frac{h_1}{3} \right), \quad (28a)$$

The initial condition at $v = 0$ (or $x = 0$ and $u = -Ct$) needed for the solution of the KdV equation (27) is obtained from (16) in the form:

- Barotropic mode:

$$\eta_e^{(1)}(u, 0) = \frac{\varepsilon^2 C_1^2 \gamma_1}{g(\gamma_1 - \gamma_2)} \frac{\partial}{\partial u} \left[\frac{\gamma_2}{h_2} \int_{-h}^{-h_1} f^{(2)} \left(y, -\frac{u}{C_1} \right) dy - \frac{1}{h_1} \int_{-h_1}^0 f^{(1)} \left(y, -\frac{u}{C_1} \right) dy \right] \quad (29a)$$

- Baroclinic mode:

$$\eta_i^{(1)}(u, 0) = \frac{\varepsilon^2 C_2^2 \gamma_2}{g(\gamma_2 - \gamma_1)} \frac{\partial}{\partial u} \left[\frac{\gamma_1}{h_2} \int_{-h}^{-h_1} f^{(2)} \left(y, -\frac{u}{C_2} \right) dy - \frac{1}{h_1} \int_{-h_1}^0 f^{(1)} \left(y, -\frac{u}{C_2} \right) dy \right] \quad (29b)$$

The following relation is also obtained between $\eta_2^{(2)}(u, v)$ and $\eta_2^{(1)}(u, v)$:

$$\eta_2^{(2)}(u, v) = \frac{\rho_2 - \gamma\rho_1}{\gamma(\rho_2 - \rho_1)} \eta_2^{(1)}(u, v) \quad (30)$$

Conclusion

A multiple-scale transformation of variables and single-series representations for the velocity potentials and the free surface and the interface

elevations show that these elevations up to the second order satisfy the KdV equation. The initial conditions needed for the solution of this equation are obtained at the position of the wave maker using another technique depending on a double series representation of the unknown functions. The particular choice of the motion of the wave maker can minimize the elevation of the free surface and hence lead to dominant nonlinear internal waves only in the channel.

Future Directions

For a future work following the present one, it is intended to investigate the following items:

- Study the possibility of generating nonlinear internal waves by different types of the motion of the plane wave maker. This will need analytical and numerical work as well.
- Study the inverse problem, precisely the possibility of the recuperation of the water waves' energy, and employ it in producing a controlled solid body motion.
- Study the possibility of generating nonlinear internal waves in model problems with different and more complicated geometry, to simulate the actual physical situations.

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Partial Differential Equations that Lead to Solitons

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Article Outline

Definition of the Subject
Introduction
Some Nonlinear Models that Lead to Solitons
Future Directions
Bibliography

Definition of the Subject

In this part, we introduce the reader to a certain class of nonlinear partial differential equations which are characterized by solitary wave solutions of the classical nonlinear equations that lead to solitons. The classical nonlinear equations of interest show the existence of special types of traveling wave solutions which are either solitary waves or solitons. In this study, we will review a few solutions arising from the analytic work of the Korteweg–de Vries (KdV) equations, the generalized regularized long-wave RLW equation, Kadomtsev–Petviashvili (KP) equation, the Klein–Gordon (KG) equation, the Sine-Gordon (SG) equation, the Boussinesq equation, Pochhammer–Chree (PC) equation and the nonlinear Schrödinger (NLS) equation, the Fisher equation, Burgers equation, the Korteweg–de Vries Burgers’ equation (KdVB), the two-dimensional Korteweg–de Vries Burgers’ (tdKdVB), the potential Kadomtsev–Petviashvili equation, the Kawahara equation, Generalized Zakharov–Kuznetsov (gZK) equation, the Sharma–Tasso–Olver equation, and the Cahn–Hilliard equation.

Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics. Calculating exact and numerical solutions, in particular, traveling wave solutions, of nonlinear PDEs in mathematical physics plays an important role in soliton theory. Moreover, these equations are mathematical models of complex physical occurrences that arise in engineering, chemistry, biology, mechanics, and physics.

In this work, we give a brief history of the above-mentioned nonlinear equations and how this type of equation has led to the soliton solutions; we then present an introduction to the theory of solitons. Soliton theory is an important branch of applied mathematics and mathematical physics. In the last decade this topic has become an active and productive area of research, and applications of the soliton equations in physical cases have been considered. These have important applications in fluid mechanics, nonlinear optics, ion plasma, classical and quantum fields’ theories etc.

The best introduction to the soliton is that contained in J. Scott Russell’s (a Scottish naval engineer) seminal 1844 report to the Royal Society. Scott Russell’s report titled “Report on Waves” (Russell 1844) was presented to the Royal Society in the eighteenth Century and he wrote:

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel . . .”

Scott Russell's experimental observations in 1834 were followed by the theoretical scientific work of Lord Rayleigh and Joseph Boussinesq around 1870 (Rayleigh 1876) and finally, two Dutchmen, Korteweg and de Vries, developed a nonlinear partial differential equation to model the propagation of shallow water waves applicable to the situation in 1895 (Korteweg and de Vries 1895). This work was really what Scott Russell fortuitously witnessed. This famous classical equation is known simply as the KdV equation. Korteweg and de Vries published a theory of shallow water waves which reduced Russell's observations to its essential features. Then a nonlinear classical dispersive equation was formulated by Korteweg and de Vries in the form

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} + \varepsilon \frac{\partial^3 u}{\partial x^3} + \gamma u \frac{\partial u}{\partial x} = 0$$

where c , ε , γ are physical parameters. This equation plays a key role in soliton theory.

In the late 1960s, Zabusky and Kruskal numerically studied and then analytically solved the KdV equation (Zabusky and Kruskal 1965). They came to the result that stable pulse-like waves could exist in a problem described by the KdV equation from their numerical study. A remarkable aspect is that the discovered solitary waves retain their shapes and speeds after collision. Zabusky and Kruskal called these waves solitons as they resemble particles in nature. In 1967, this numerical development was placed on a settled mathematical basis with Gardner et al.'s discovery of the inverse-scattering-transform method (Gardner et al. 1967).

The soliton concept is related with solutions for nonlinear partial differential equations. The soliton solution of a nonlinear equation usually is used as a single wave. If there are several soliton solutions, these solutions are called solitons. On the other hand, if a soliton is separated infinitely from another soliton, this soliton is called a single wave. Besides, a single wave solution can't be sech^2 function for equations other than nonlinear equations, such as the KdV equation. But this solution can be sech or $\tan^{-1}(e^{\alpha x})$.

At this stage, we could ask what the definition of the soliton solutions is? It is not easy to define the

soliton concept. Wazwaz (2002) describes solitons as solutions of nonlinear differential equations as follows:

- (i) A long and shallow water wave should not lose its permanent form;
- (ii) A long and shallow water wave of the solution is localized, meaning either the solutions decay exponentially to zero such as in the solitons admitted by the KdV equation, or approach a constant at infinity such as solitons provided by the SG equation;
- (iii) A long and shallow water wave of the solution can interact with other solitons preserving its character.

There is also a more formal definition of this concept, but these definitions require substantial mathematics (Edmundson and Enns 1992). On the other hand, the phenomena of the solitons doesn't always follow these three properties. For example, there is the concept of "light bullets" in the subject of nonlinear optics which are often called solitons despite losing energy during interaction. This idea can be found at an internet web page of the *Simon Fraser University* British Columbia, Canada (Edmundson and Enns 1996).

Some Nonlinear Models that Lead to Solitons

In this section, we will deal with the fundamental ideas and some nonlinear partial differential equations that lead to solitons. These equations are the result of a huge amount of research work and physical implementations which focus on the most diverse and active areas of applied mathematics and mathematical physics.

Example 1

In various fields of science and engineering, nonlinear evaluation equations, as well as their analytic and numerical solutions, are of fundamental importance. One of the most attractive and surprising wave phenomena is the creation of solitary waves or solitons. It was approximately two centuries ago, that an adequate theory for solitary

waves was developed, in the form of a modified wave equation known as the KdV equation (Debnath 1983, 1997; Drazin and Johnson 1989; Korteweg and de Vries 1895, Wazwaz 2002, Whitham 1974). The third-order KdV equation is a classical nonlinear partial differential equation originally formulated to model shallow water flow (Korteweg and de Vries 1895).

Herein, we are particularly interested in the original third-order KdV equation (Korteweg and de Vries 1895) given by

$$u_t + uu_x + \alpha u_{xxx} = 0,$$

respectively, and the generalized form

$$u_t + u^p u_x + \alpha u_{xxx} = 0,$$

where p models the dispersion and subscripts in t and x denote partial derivatives with respect to these independent variables. The study of long waves, tides, solitary waves, and related phenomena, leads to an equation referred to as the generalized KdV equation. If $p = 0$, $p = 1$, and $p = 2$, this equation becomes the linearized KdV, nonlinear KdV, and modified KdV equations, respectively (Debnath 1983, 1997; Drazin and Johnson 1989; Wazwaz 2002; Whitham 1974). The modified KdV equation has also found applications in many areas of physical situations, for instance the equation describes nonlinear acoustic waves in an anharmonic lattice (Zabusky 1967b).

Example 2

Consider the solution $u(x, t)$ of the generalized RLW equation

$$u_t + u_x + \alpha u_x^v - \beta u_{xxt} = 0,$$

where v is a positive integer, and α and β are positive constants. The generalized RLW equation was first put forward as a model for small-amplitude long waves on the surface of water in a channel by Peregrine (Kakutani and Ona 1969; Peregrine 1967) and later by Benjamin et al. (1972). In physical situations such as unidirectional waves propagating in a water channel,

long-crested waves in near-shore zones, and many others, the generalized RLW equation serves as an alternative model to the KdV equation (Bona et al. 1981, 1983).

Example 3

The nonlinear KG equations in one-dimensional space

$$u_{tt} - u_{xx} + \frac{dV}{du} = 0,$$

where $V = V(u)$ is a general nonlinear function of u , but not its derivatives. This equation is the most natural nonlinear generalization of the wave equation and first arose in a mathematical context, with $V(u) = \exp(u)$, in the theory of surfaces of constant curvature. In addition, this equation appears in many different fields of application. For instance, a polynomial nonlinearity can be used as a model field theory, while a $\cos u$ term yields the sine-Gordon equation and so on (Debnath 1983, 1997; Drazin and Johnson 1989; Wazwaz 2002; Whitham 1974). We will consider a particular case of equation KG, the so-called sine-Gordon nonlinear hyperbolic equation, which has the form

$$u_{tt} - cu_{xx} + \kappa \sin(u) = 0;$$

$$-\infty < x < \infty, t > 0,$$

where c and κ are constants. The sine-Gordon equation is firstly used in the work of differential geometry and for investigation of the propagation of a ‘slip’ dislocation in crystals (Debnath 1997).

The generalized one-dimensional KG equation

$$u_{tt} - ku_{xx} + b_1 u + b_2 u^{m+1} + b_3 u^{2m+1} = 0,$$

is given in (Zhang et al. 2003). The KG equation represents a nonlinear model of longitudinal wave propagation of elastic rods when $m = 1$ (Clarkson et al. 1986).

Example 4

In this example, we consider the generalized Boussinesq-type equation (Boussinesq 1871)

$$u_{tt} - u_{xx} + \delta u_{xxxx} = -(\psi(u))_{xx};$$

$$-\infty < x < \infty, t > 0,$$

where $\delta \geq 0$ is constant, $\psi(u) = |u|^{\alpha-1}u$ and $\alpha > 1$. This equation represents a generalization of the classical Boussinesq equation which arises in the modeling of nonlinear strings. The Boussinesq equation describes in the continuous limit the propagation of waves in a one-dimensional nonlinear lattice and the propagation of waves in shallow water (Bona and Sachs 1988; Boussinesq 1871; Debnath 1997; Zabusky 1967a). A proof of the local well-posedness of the Boussinesq equation has been showed by Bona and Sachs (1988). In (Cole 1951) the authors noticed that the Boussinesq equation admits solitary solutions and the existence of solitary wave solutions illustrates the perfect balance between dispersion and the nonlinearity of the Boussinesq equation. For the α integer, solitary-wave solutions have been shown to be stable under some restrictions on the wave speed by Bona and Sachs (1988).

Scott Russell's study (1844) of solitary water waves motivated the development of nonlinear partial differential equations for the modeling of wave phenomena in fluids, plasmas, elastic bodies, etc. The Boussinesq equation is an important model that approximately describes the propagation of long waves on shallow water like the other Boussinesq equations (with u_{xxtt} , instead of u_{xxxx}). This equation was first deduced by Boussinesq (1871). In the case $\delta > 0$ this equation is linearly stable and governs small nonlinear transverse oscillations of an elastic beam (Debnath 1997). It is called the "good" Boussinesq equation, while the equation with $\delta < 0$ received the name "bad" Boussinesq equation since it possesses linear instability.

Example 5

Consider the PC equation

$$u_{tt} - \frac{1}{p} u_{xx}^\ell - u_{xx} - u_{txx} = 0,$$

where ℓ indicates different material with different values of it. The PC equation is applied as a nonlinear model of longitudinal wave propagation

of elastic rods (Debnath 1997; Drazin and Johnson 1989; Whitham 1974). In the work of Bogolubsky (Whitham 1974), the author obtained exact solitary wave solutions to the PC equation $\ell = 2, 3, 5$, respectively (Debnath 1997).

Example 6

In this paper, we consider the Burgers equation

$$u_t + \varepsilon uu_x - \nu u_{xx} = 0$$

where ε and ν are parameters and the subscripts t and x denote differentiation. The Burgers' equation is a model of flow through a shock wave in a viscous fluid (Cole 1951) and in the Burgers' model of turbulence (Burgers 1948).

Example 7

The Fisher equation

$$u_t - \nu u_{xx} = ku \left(1 - \frac{u}{\kappa}\right),$$

is well known in population dynamics, where $\nu > 0$ is the diffusion constant, $k > 0$ is the linear growth rate, and $\kappa > 0$ is the carrying capacity of the environment. The right-hand side function $ku \left(1 - \frac{u}{\kappa}\right)$ represents a nonlinear growth rate (Debnath 1997). This well-known equation was first proposed by Fisher (1937) for a model of the advancement of a mutant gene in an infinite one-dimensional habitat. In recent years, the equation has been used as a basis for a wide variety of models for the spatial spread of genes in a population and for chemical wave propagation (Fisher 1937).

Example 8

Let us consider the $(1 + 1)$ -dimensional NLS equation with two higher-order nonlinear terms in the form

$$iu_t + \frac{1}{2} u_{xx} + \alpha |u|^\sigma u + \beta |u|^{2\sigma} u = 0,$$

where σ is a positive constant, α and β are constant parameters (Debnath 1997; Drazin and Johnson 1989; Wazwaz 2002). $u = u(x, t)$ is a complex function that represents the complex amplitude of the wave form, the variable t should

be interpreted as the normalized propagation distance, x the normalized transverse coordinate that represents a retarded time. The NLS equation, especially that with lower σ values, appears in various branches of contemporary physics (Fisher 1937; Hayata and Koshiba 1995; Robert et al. 1996; Wang et al. 2007; Zhang et al. 2003; Zhou et al. 1992) and has been extensively investigated for the cases of $\sigma = 1$ and $\sigma = 2$, its various solutions have been obtained. The NLS equation also arises in some form of $\alpha = 0$ and $\sigma = 1$ in some other physical systems such as nonlinear optic (Bespalov and Talanov 1966; Hasegawa and Tappert 1973; Karpman and Krushkal 1969), hydro magnetic and plasma waves (Ichikawa 1979; Schimizu and Ichikawa 1972) and the propagation of solitary waves in piezoelectric semiconductors (Pawlik and Rowlands 1975).

Example 9

In applied mathematics, the KP equation (Kadomtsev and Petviashvili 1970) is a nonlinear partial differential equation. It is also sometimes called the Kadomtsev–Petviashvili–Boussinesq equation. The KP equation is usually written as:

$$(u_t - 6uu_{xx} + u_{xxx})_x + 3\lambda^2 u_{yy} = 0,$$

where $\lambda = \pm 1$. This equation is implemented to describe slowly varying nonlinear waves in a dispersive medium (Peregrine 1996). The above written form shows that the KP equation is a generalization form of the KdV equation which is like the KdV equation; the KP equation is completely integrable. The KP equation was first discovered in 1970 by Kadomtsev and Petviashvili when they relaxed the restriction that the waves be strictly one-dimensional.

Example 10

Consider the solution $u(x, t)$ of the nonlinear partial differential equation

$$u_t + \varepsilon uu_x - \nu u_{xx} + \mu u_{xxx} = 0$$

where ε , ν and μ are positive parameters. This equation is called the Korteweg–de Vries Burgers' equation (KdVB) which is derived by Su and

Gardner (1969). KdVB is a model equation for a wide class of nonlinear systems in the weak nonlinearity and long wavelength approximations because it contains both damping and dispersion. KdVB has been constructed when including electron inertia effects in the description of weak nonlinear plasma waves. The KdVB equation possesses a steady-state solution which has been demonstrated to model weak plasma shocks propagating perpendicular to a magnetic field (Grad and Hu 1967). There are some other implementation places to use the KdVB equation. Examples are its use in the study of wave propagation through a liquid-filled elastic tube (Jonson 1970) and for a description of shallow water waves on a viscous fluid (Gardner et al. 1974; Jonson 1972).

Example 11

The nonlinear partial differential equation

$$(u_t + uu_x - qu_{xx} + pu_{xxx})_x + ru_{yy} = 0,$$

where p , q , r are real parameters. This equation is called the two-dimensional Korteweg–de Vries Burgers' (tdKdVB) equation. The tdKdVB equation is a model equation for a wide class of nonlinear wave models of fluids in an elastic tube, liquids with small bubbles and turbulence (Ma 1993; Parkas 1994).

Example 12

In this example we consider in the $(2 + 1)$ -dimension the potential Kadomtsev–Petviashvili equation,

$$u_{xt} + \frac{3}{2}u_x u_{xx} + \frac{1}{4}u_{xxx} + \frac{3}{4}u_{yy} = 0,$$

where the initial conditions $u(x, 0, t)$ and $u_y(x, 0, t)$ are given. Nonlinear phenomena play a crucial role in applied mathematics and physics. Obtaining the exact or approximate solutions of PDEs in physics and mathematics is important and it is still a hot spot to seek new methods to obtain new exact or approximate solutions (Debthnath 1983, 1997; Drazin and Johnson 1989). Different methods have been put forward to seek various exact solutions of multifarious physical models described by nonlinear PDEs.

Example 13

We consider the numerical solution to a problem involving a nonlinear partial differential equation of the form

$$u_t + uu_x + u_{xxx} - u_{xxxx} = 0,$$

this is called the Kawahara equation. The Kawahara equation occurs in the theory of magneto-acoustic waves in plasmas (Kawahara 1972) and in the theory of shallow water waves with surface tension (Hunter and Scheurle 1988).

Example 14

A Generalized Zakharov–Kuznetsov (gZK) equation (Debnath 1983, 1997; Drazin and Johnson 1989; Li et al. 2003; Whitham 1974) is proposed to understand the physical and scientific mechanisms in different physical and engineering problems. The gZK equation is as follows

$$u_t + b_1 u^\rho u_x + b_2 u^{2\rho} u_x + b_3 u_{xy} + b_4 u_{xxx} + b_5 u_{xyy} = 0,$$

where $b_1, b_2, b_3, b_4 \neq 0, b_5 \neq 0$ and $\rho \neq 0$ are constant parameters (Li et al. 2003). When $b_1 = 6, b_2 = b_3 = 0, b_4 = b_5 = 1$ and $\rho = 1$ the equation gZK is said to be Zakharov–Kuznetsov (ZK) which can be derived for Alfvén waves in a magnetized plasma at a special, critical angle to the magnetic field by means of an asymptotic multi-scale technique (Bridges and Reich 2001; Pelinovsky and Grimshaw 1996). If $b_1 = 0, b_2 = 1, b_3 = 0, b_4 = b_5 = 1$ and $\rho = 1$, then the equation gZK is said to be modified Zakharov–Kuznetsov (mZK) which represents an anisotropic two-dimensional generalization of the KdV equation and can be derived in a magnetized plasma for a small amplitude Alfvén wave at a critical angle to the undisturbed magnetic field. The radially symmetric positive solutions for the mZK equation have been computed (Parkes and Munro 1999; Munro and Parker 1997).

Example 15

Let's consider the Sharma–Tasso–Olver equation (STO) with its fission and fusion (Yan 2003)

$$u_t + 3\alpha u_x^2 + 3\alpha u^2 u_x + 3\alpha u u_{xx} + \alpha u_{xxx} = 0.$$

Attention has been focused on the STO equation in (Lian and Lou 2005; Wang et al. 2004) and the references therein due to its appearance in scientific applications (Wazwaz 2007d).

Example 16

Consider the Cahn–Hilliard equation

$$u_t + u_{xxx} = (u^3 - u)_{xx} + \beta u_x.$$

The Cahn–Hilliard equation is related to a number of interesting physical phenomena like spinodal decomposition, phase separation and phase-ordering dynamics. On the other hand this equation is very hard and difficult to solve. In this paper by considering a modified extended tanh method, we found some exact solutions of the Cahn–Hilliard equation. This equation is very crucial in materials science (Chan 1961; Chan and Hilliard 1958; Gurtin 1996). Many articles have focused on mathematical and numerical studies of this equation (Barrett and Blowey 1999; Garcke 2000; Li and Zhong 1998).

Future Directions

Nonlinear phenomena play a crucial role in applied mathematics and physics. Furthermore, when an original nonlinear equation is directly calculated, the solution will preserve the actual physical characters of solutions. Explicit solutions to the nonlinear equations are of fundamental importance. Various effective methods have been developed to understand the mechanisms of these physical models, to help physicians and engineers and to ensure knowledge for physical problems and its applications.

Many explicit exact methods have been introduced in the literature (Debnath 1983, 1997; Drazin and Johnson 1989; Inan and Kaya 2006; Khater et al. 1997; Wazwaz 2002, 2007a, b, c; Wazwaz and Helal 2004; Whitham 1974). These include the Bäcklund transformation, Hopf–Cole transformation, Generalized Miura

Transformation, the Inverse scattering method, Darboux transformation, Painleve method, homogeneous balance method, similarity reduction method, tanh method, Exp-function method, sine-cosine method and so on. There are also many numerical methods implemented for these equations (Geyikli and Kaya 2005a, b; Helal and Mehanna 2007; Inan and Kaya 2006, 2007; Kaya 2003, 2006; Kaya and Al-Khaled 2007; Kaya and El-Sayed 2003a, b, c; Polat et al. 2006; Shawagfeh and Kaya 2004; Ugurlu and Kaya 2022). These include the Finite elements method, finite difference methods and some approximate methods such as the Adomian decomposition method, Homotopy perturbation method, Variational perturbation method, Sinc-Galerkin method and so on.

There is still much work to be done by researchers in this field involving applications of nonlinear equations and exact and numerical implementations.

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Books and Reviews

- For another obtaining of the KdV equation for water waves, see Kevorkian and Cole (1981); one can see the work of the Johnson (1972) for a different water-wave application with variable depth, for waves on arbitrary shears in the work of Freeman and Johnson (1970) and Johnson (1980) for a review of one and two-dimensional KdV equations. In addition to these; one can see the book of Drazin and Johnson (1989) for some numerical solutions of nonlinear evolution equations. In the work of the Zabusky, Kruskal and Deam (F1965) and Eilbeck (F1981), one can see the motion pictures of soliton interactions. See a comparison of the KdV equation with water wave experiments in Hammack and Segur (1974)
- For further reading of the classical exact solutions of the nonlinear equations can be seen in the works: the Lax approach is described in Lax (1968); Calogero and Degasperis (1982, A.20), the Hirota's bilinear approach is developed in Matsuno (1984), the Bäcklund transformations are described in Rogers and Shadwick (1982); Lamb (1980, Chap. 8), the Painleve properties is discussed by Ablowitz and Segur (1981, Sect. 3.8), In the book of Dodd, Eilbeck, Gibbon and Morris (1982, Chap. 10) can found review of the many numerical methods to solve nonlinear evolution equations and shown many of their solutions.
- The following, referenced by the end of the paper, is intended to give some useful for further reading.

Shallow Water Waves and Solitary Waves

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Article Outline

Glossary

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Glossary

Deep water A surface wave is said to be in deep water if its wavelength is much shorter than the local water depth.

Internal wave An internal wave travels within the interior of a fluid. The maximum velocity and maximum amplitude occur within the fluid or at an internal boundary (interface). Internal waves depend on the density-stratification of the fluid.

Shallow water A surface wave is said to be in shallow water if its wavelength is much larger than the local water depth.

Shallow water waves Shallow water waves correspond to the flow at the free surface of a body of shallow water under the force of gravity, or to the flow below a horizontal pressure surface in a fluid.

Shallow water wave equations Shallow water wave equations are a set of partial differential equations that describe shallow water waves.

Solitary wave A solitary wave is a localized gravity wave that maintains its coherence and, hence, its visibility through properties of nonlinear hydrodynamics. Solitary waves have finite amplitude and propagate with constant speed and constant shape.

Soliton Solitons are solitary waves that have an elastic scattering property: they retain their shape and speed after colliding with each other.

Surface wave A surface wave travels at the free surface of a fluid. The maximum velocity of the wave and the maximum displacement of fluid particles occur at the free surface of the fluid.

Tsunami A tsunami is a very long ocean wave caused by an underwater earthquake, a submarine volcanic eruption, or by a landslide.

Wave dispersion Wave dispersion in water waves refers to the property that longer waves have lower frequencies and travel faster.

Definition of the Subject

The most familiar water waves are waves at the beach caused by wind or tides, waves created by throwing a stone in a pond, by the wake of a ship, or by raindrops in a river (see Fig. 1). Despite their familiarity, these are all different types of water waves. This entry only addresses shallow water waves, where the depth of the water is much smaller than the wavelength of the disturbance of the free surface. Furthermore, the discussion is focused on gravity waves in which buoyancy acts as the restoring force. Little attention will be paid to capillary effects, and capillary waves for which the primary restoring force is surface tension are not covered.

Although the history of shallow water waves (Bullough 1988; Craik 2004; Darrigol 2003) goes back to French and British mathematicians of the eighteenth and early nineteenth centuries, Stokes (1847) is considered one of the pioneers of hydrodynamics (see Craik 2005). He carefully derived the equations for the motion of incompressible, inviscid fluid, subject to a constant vertical gravitational force, where the fluid is bounded below by

Shallow Water Waves and Solitary Waves,

Fig. 1 Capillary surface waves from raindrops (Source: Socha (2007). Photograph courtesy of E. Scheller and K. Socha)



an impermeable bottom and above by a free surface. Starting from these fundamental equations and by making further simplifying assumptions, various shallow water wave models can be derived. These shallow water models are widely used in oceanography and atmospheric science.

This entry discusses shallow water wave equations commonly used in oceanography and atmospheric science. They fall into two major categories: Shallow water wave models with wave dispersion are discussed in section “[Completely Integrable Shallow Water Wave Equations](#).” Most of these are completely integrable equations that admit smooth solitary and cnoidal wave solutions for which computational procedures are outlined in section “[Computation of Solitary Wave Solutions](#).” Section “[Shallow Water Wave Equations of Geophysical Fluid Dynamics](#)” covers classical shallow water wave models without dispersion. Such hyperbolic systems can admit shocks. Section “[Water Wave Experiments and Observations](#)” addresses a few experiments and observations. The entry concludes with future directions in section “[Future Directions](#).”

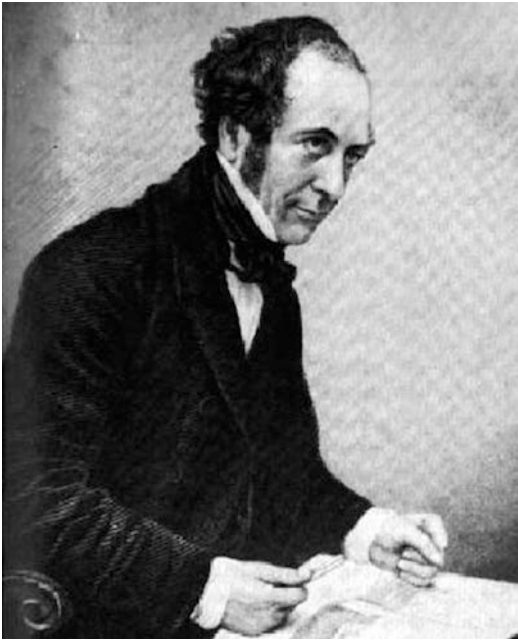
Introduction

The initial observation of a solitary wave in shallow water was made by John Scott Russell, shown in Fig. 2. Russell was a Scottish engineer and

naval architect who was conducting experiments for the Union Canal Company to design a more efficient canal boat.

In Russell’s (1844) own words: “I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.”

Russell built a water tank to replicate the phenomenon and research the properties of the solitary wave he had observed. Details can be found in a biography of John Scott Russell (1808–1882) by Emmerson (1977), and in review articles by Bullough (1988), Craik (2004), and Darrigol



Shallow Water Waves and Solitary Waves,
Fig. 2 John Scott Russell (Source: Emmerson (1977).
 Courtesy of John Murray Publishers)

(2003), who pay tribute to Russell's research of water waves.

In 1895, the Dutch professor Diederik Korteweg and his doctoral student Gustav de Vries (Korteweg and de Vries 1895) derived a partial differential equation (PDE) which models the solitary wave that Russell had observed. Parenthetically, the equation which now bears their names had already appeared in seminal work on water waves published by Boussinesq (1872, 1877) and Rayleigh (1876). The solitary wave was considered a relatively unimportant curiosity in the field of nonlinear waves. That all changed in 1965, when Zabusky and Kruskal realized that the KdV equation arises as the continuum limit of a one dimensional anharmonic lattice used by Fermi et al. (1955) to investigate "thermalization" – or how energy is distributed among the many possible oscillations in the lattice. Zabusky and Kruskal (1965) simulated the collision of solitary waves in a nonlinear crystal lattice and observed that they retain their shapes and speed after collision. Interacting solitary waves merely experience a phase shift, advancing the faster and

retarding the slower. In analogy with colliding particles, they coined the word "solitons" to describe these elastically colliding waves. A narrative of the discovery of solitons can be found in (Zabusky 2005).

Since the 1970s, the KdV equation and other equations that admit solitary wave and soliton solutions have been the subject of intense study (see, e.g., Dauxois and Peyrard 2004; Filippov 2000; Remoissenet 1999). Indeed, scientists remain intrigued by the physical properties and elegant mathematical theory of the shallow water wave models. In particular, the so-called completely integrable models which can be solved with the inverse scattering transform (IST). For details about the IST method, the reader is referred to Ablowitz et al. (1974), Ablowitz and Segur (1981), and Ablowitz and Clarkson (1991). The completely integrable models discussed in the next section are infinite-dimensional bi-Hamiltonian systems, with infinitely many conservation laws and higher-order symmetries, and admit soliton solutions of any order.

As an aside, in 1995, scientists gathered at Heriot-Watt University for a conference and successfully recreated a solitary wave but of smaller dimensions than the one observed by Russell 161 years earlier (see Fig. 3).

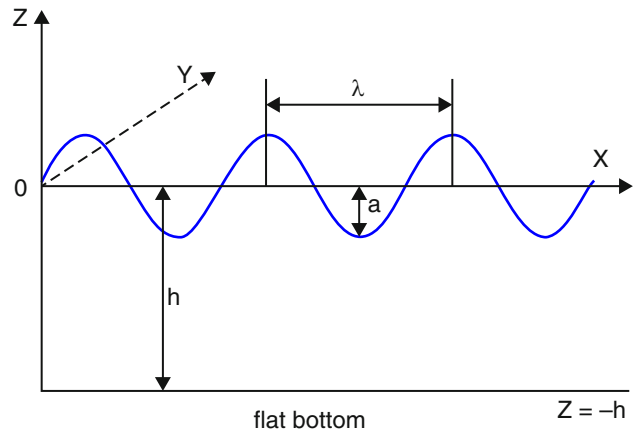
Completely Integrable Shallow Water Wave Equations

Starting from Stokes' (1847) governing equations for water waves, completely integrable PDEs arise at various levels of approximation in shallow water wave theory. Four length scales play a crucial role in their derivation. As shown in Fig. 4, the wavelength λ of the wave measures the distance between two successive peaks. The amplitude a measures the height of the wave, which is the varying distance between the undisturbed water to the peak of the wave. The depth of the water h is measured from the (flat) bottom of the water up to the quiescent free surface. The fourth length scale is along the Y -axis which is along the crest of the wave and perpendicular to the (X, Z) -plane.



Shallow Water Waves and Solitary Waves, Fig. 3 Recreation of a solitary wave on the Scott Russell Aqueduct on the Union Canal (Photograph courtesy of Heriot-Watt University)

Shallow Water Waves and Solitary Waves, Fig. 4 Coordinate frame and periodic wave on the surface of water



Assuming wave propagation in water of uniform (shallow) depth, i.e., h is constant, and ignoring dissipation, the model equations discussed in this section have a set of common features and limitations which make them mathematically tractable (Segur 2007b). They describe (i) long waves (or shallow water), i.e., $h \ll \lambda$, (ii) with relatively small amplitude, i.e., $a \ll h$, (iii) traveling in one direction (along the X -axis) or weakly two-dimensional (with a small component in the Y -direction). Furthermore, the small effects must

be comparable in size. For example, in the derivation of the KdV and Boussinesq equations one assumes that $\varepsilon = a/h = O(h^2/\lambda^2)$, where ε is a small parameter ($\varepsilon \ll 1$), and O indicates the order of magnitude.

The Korteweg-de Vries Equation

The KdV equation was originally derived to describe shallow water waves of long wavelength and small amplitude. In the derivation, Korteweg and de Vries assumed that all motion is uniform in

the Y-direction, along the crest of the wave. In that case, the surface elevation (above the equilibrium level h) of the wave, propagating in the X -direction, is a function only of the horizontal position X (along the canal) and of time T , i.e., $Z = \eta(X, T)$.

In terms of the physical parameters, the KdV equation reads

$$\begin{aligned} \frac{\partial \eta}{\partial T} + \sqrt{gh} \frac{\partial \eta}{\partial X} + \frac{3}{2} \frac{\sqrt{gh}}{h} \eta \frac{\partial \eta}{\partial X} \\ + \frac{1}{2} h^2 \sqrt{gh} \left(\frac{1}{3} - \frac{\mathcal{T}}{\rho g h^2} \right) \frac{\partial^3 \eta}{\partial X^3} = 0, \end{aligned} \quad (1)$$

where h is the uniform water depth, g is the gravitational acceleration (about 9.81 m/s^2 at sea level), ρ is the density, and $\mathcal{T}$ stands for the surface tension. The dimensionless parameter $\mathcal{T}/\rho g h^2$ is called the *Bond number* which measures the relative strength of surface tension and the gravitational force.

Keeping only the first two terms in Eq. 1, the speed of the associated linear (long) wave is $c = \sqrt{gh}$. This is indeed the maximum attainable speed of propagation of gravity-induced water waves of infinitesimal amplitude. The speed of propagation of the small-amplitude solitary waves described by Eq. 1 is slightly higher. According to Russell's empirical formula the speed equals $\sqrt{g(h+k)}$, where k is the height of the peak of the solitary wave above the surface of undisturbed water. As Bullough (Vreugdenhil 1988) has shown, Russell's approximate speed and the true speed of solitary waves only differ by a term of $O(k^2/h^2)$.

The KdV equation can be recast in dimensionless variables as

$$u_t + \alpha u u_x + u_{xxx} = 0, \quad (2)$$

where subscripts denote partial derivatives. The parameter α can be scaled to any real number. Commonly used values are $\alpha = \pm 1$ or $\alpha = \pm 4$.

The first linear term, u_t , describes the time evolution of the wave propagating in one direction. Therefore, Eq. 2 is called an *evolution* equation. The nonlinear term $\alpha u u_x$ accounts for

steepening of the wave, and the linear dispersive term u_{xxx} describes spreading of the wave. The linear first-order term $\sqrt{gh} \frac{\partial \eta}{\partial X}$ in Eq. 1 can be removed by an elementary transformation. Conversely, a linear term in u_x can be added to Eq. 2.

The nonlinear steepening of the water wave can be balanced by dispersion. If so, the result of these counteracting effects is a stable solitary wave with particle-like properties. A solitary wave has a finite amplitude and propagates at constant speed and without change in shape over a fairly long distance. This is in contrast to the concentric group of small-amplitude capillary waves, shown in Fig. 1, which disperse as they propagate.

The closed-form expression of a solitary wave solution is given by

$$\begin{aligned} u(x, t) = \frac{\omega - 4k^3}{\alpha k} \\ + \frac{12k^2}{\alpha} \text{sech}^2(kx - \omega t + \delta) \end{aligned} \quad (3)$$

$$= \frac{\omega + 8k^3}{\alpha k} - \frac{12k^2}{\alpha} \tanh^2(kx - \omega t + \delta), \quad (4)$$

where the wave number k , the angular frequency ω , and δ are arbitrary constants.

Requiring that $\lim_{x \rightarrow \pm \infty} u(x, t) = 0$ for all time leads to $\omega = 4k^3$. Then Eqs. 3 and 4 reduce to

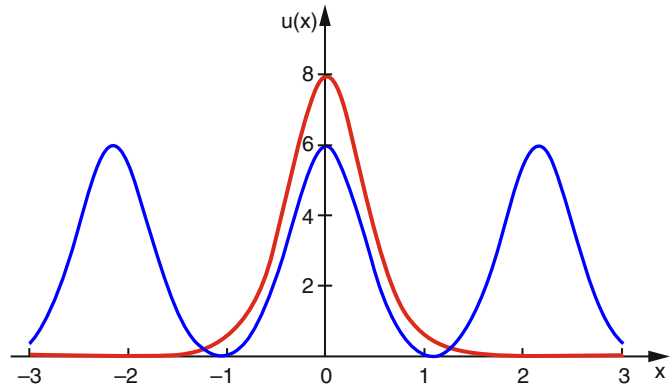
$$\begin{aligned} u(x, t) = \frac{12k^2}{\alpha} \text{sech}^2(kx - 4k^3 t + \delta) \\ = \frac{12k^2}{\alpha} [1 - \tanh^2(kx - 4k^3 t + \delta)]. \end{aligned} \quad (5)$$

The position of the hump-type wave at $t = 0$ is depicted in Fig. 5 for $\alpha = 6$, $k = 2$, and $\delta = 0$. As time changes, the solitary wave with amplitude $2k^2 = 8$ travels to the right at speed $v = \omega/k = 4k^2 = 16$. The speed is exactly twice the peak amplitude. So, the taller the wave the faster it travels, but it does so without change in shape. The reciprocal of the wavenumber k is a measure of the width of the sech-squared pulse.

As shown by Korteweg and de Vries (1895), Eq. 2 also has a simple periodic solution,

Shallow Water Waves and Solitary Waves,

Fig. 5 Solitary wave (red) and periodic cnoidal (blue) wave profiles



$$u(x, t) = \frac{\omega - 4k^3(2m - 1)}{\alpha k} + \frac{12k^2 m}{\alpha} \text{cn}^2(kx - \omega t + \delta; m), \quad (6)$$

which they called the *cnoidal wave* solution for it involves the Jacobi elliptic cosine function, cn , with modulus m ($0 < m < 1$). The wavenumber k gives the characteristic width of each oscillation in the “cnoid.”

Three cycles of the cnoidal wave are depicted in Fig. 5 at $t = 0$. The graph corresponds to $\alpha = 6$, $k = 2$, $m = 3/4$, $\omega = 16$, and $\delta = 0$. Using the property $\lim_{m \rightarrow 1} \text{cn}(\xi; m) = \text{sech}(\xi)$, one readily verifies that Eq. 6 reduces to Eq. 3 as m tends to 1. Pictorially, the individual oscillations then stretch infinitely far apart leaving a single-pulse solitary wave.

The celebrated KdV equation appears in all books and reviews about soliton theory. In addition, the equation has been featured in, e.g., Miles (1981) and Miura (1976).

Regularized Long-Wave Equations

A couple of alternatives to the KdV equation have been proposed. A first alternative,

$$u_t + u_x + \alpha u u_x - u_{xxt} = 0, \quad (7)$$

was proposed by (Benjamin et al. 1972). Hence, Eq. 7 is referred to as the BBM or regularized long-wave (RLW) equation, which has a solitary wave solution,

$$u(x, t) = \frac{\omega - k - 4k^2 \omega}{\alpha k} + \frac{12k\omega}{\alpha} \text{sech}^2(kx - \omega t + \delta), \quad (8)$$

was also derived by Peregrine (1966) to describe the behavior of an undular bore (in water), which comprises a smooth wavefront followed by a train of solitary waves. An undular bore can be interpreted as the dispersive analog of a shock wave in classical nondispersive, dissipative hydrodynamics (El 2007).

The linear dispersion relation for the KdV equation, $\omega = k(1 - k^2)$, can be obtained by substituting $u(x, t) = \exp[i(kx - \omega t + \delta)]$ into $u_t + u_x + u_{xxx} = 0$. The linear phase velocity, $v_p = \omega/k = 1 - k^2$, becomes negative for $|k| > 1$, thereby contradicting the assumption of uni-directional propagation. Furthermore, the group velocity $v_g = d\omega/dk = 1 - 3k^2$ has no lower bound which implies that there is no limit to the rate at which shorter ripples propagate in the negative x -direction.

The BBM equation, where $\omega = k/(1 + k^2)$, $v_p = 1/(1 + k^2)$, and $v_g = (1 - k^2)/(1 + k^2)^2$, was proposed to get around these shortcomings and to address issues related to proving the existence of solutions of the KdV equation. The dispersion relation of Eq. 7 has more desirable properties for high wave numbers, but the group velocity becomes negative for $|k| > 1$. In addition, the KdV and BBM equations are first order in time making it impossible to specify initial data for both u and u_t .

To circumvent these limitations, a second alternative,

$$u_t + u_x + \alpha uu_x + u_{xtt} = 0, \quad (9)$$

was proposed by Joseph and Egri (1977) and Jeffrey (1978). It is called the time regularized long-wave (TRLW) equation and its solitary wave solution is given by

$$u(x, t) = \frac{\omega - k - 4k\omega^2}{\alpha k} + \frac{12\omega^2}{\alpha} \operatorname{sech}^2(kx - \omega t + \delta). \quad (10)$$

The TRLW equation shares many of the properties of both the KdV and BBM equations, at the cost of a more complicated dispersion relation, $\omega = (-1 \pm \sqrt{1 + 4k^2})/2k$ with two branches. Only one of these branches is valid because the derivation of the TRLW equation shows that Eq. 9 is uni-directional, despite the presence of two time derivatives in u_{xtt} .

Bona and Chen (1999) have shown that the initial value problem for the TRLW equation is well-posed, and that for small-amplitude, long waves, solutions of Eq. 9 agree with solutions of either Eq. 2 or 7. As a matter of fact, all three equations agree to the neglected order of approximation over a long timescale, provided the initial data is properly imposed (see also Bona et al. 1981).

Fine-tuning the dispersion relation of the KdV equation comes at a cost. In contrast to Eq. 2, the RLW and TRLW equations are no longer completely integrable. Perhaps that is why these equations never became as popular as the KdV equation.

The Boussinesq Equation

The classical Boussinesq equation,

$$\eta_{TT} - c^2 \eta_{XX} - \frac{3c^2}{h} (\eta_X^2 + \eta \eta_{XX}) - \frac{c^2 h^2}{3} \eta_{XXXX} = 0, \quad (11)$$

was derived by Boussinesq (1871) to describe gravity-induced surface waves as they propagate at constant (linear) speed $c = \sqrt{gh}$ in a canal of uniform depth h .

In contrast to the KdV Eq. 11, has a second-order time-derivative term. Ignoring

all but the first two terms in Eq. 11, one obtains the linear wave equation, $\eta_{TT} - c^2 \eta_{XX} = 0$, which describes both left-running and right-running waves. However, Eq. 11 is *not* bidirectional because in the derivation Boussinesq used the constraint $\eta_T = -c\eta_X$, which limits Eq. 11 to waves traveling to the right. This crucial restriction is often overlooked in the literature.

In dimensionless form, the Boussinesq equation reads

$$u_{tt} - c^2 u_{xx} - \alpha u_x^2 - \alpha u u_{xx} - \beta u_{xxxx} = 0. \quad (12)$$

The values of the parameters c , $\alpha > 0$, and β do not matter, but the sign of β matters. Typically, one sets $c = 1$, $\alpha = 3$, and $\beta = \pm 1$.

A simple solitary wave solution of Eq. 12 is given by

$$u(x, t) = \frac{\omega^2 - c^2 k^2 - 4\beta k^4}{\alpha k^2} + \frac{12\beta k^2}{\alpha} \operatorname{sech}^2(kx - \omega t + \delta). \quad (13)$$

The equation with $\beta = 1$ is a scaled version of Eq. 11 and thus most relevant to shallow water wave theory. Mathematically, Eq. 12 with $\beta = 1$ is ill-posed, even without the nonlinear terms, which means that the initial value problem cannot be solved for arbitrary data. This shortcoming does not happen for Eq. 12 with $\beta = -1$, which is therefore nicknamed the “good” Boussinesq equation (McKean 1981). Nonetheless, the classical and good Boussinesq equations are completely integrable.

The “improved” or “regularized” Boussinesq equation (see, e.g., Bona et al. 2002) has $\beta = 1$ but u_{xtt} instead of u_{xxxx} , which improves the properties of the dispersion relation. Like Eq. 12, the regularized version describes uni-directional waves. The regularized Boussinesq equation and other alternative equations listed in the literature (see, e.g., Madsen and Schäffer 1999) are not completely integrable.

Bona et al. (2002, 2004) analyzed a family of Boussinesq systems of the form

$$\begin{aligned} w_t + u_x + (uw)_x + \alpha u_{xxx} - \beta w_{xxt} &= 0, \\ u_t + w_x + uu_x + \gamma w_{xxx} - \delta u_{xxt} &= 0, \end{aligned} \quad (14)$$

which follow from the Euler equations as first-order approximations in the parameters $\varepsilon_1 = a/h \ll 1$, $\varepsilon_2 = h^2/\lambda^2 \ll 1$, where the Stokes number, $S = \varepsilon_1/\varepsilon_2 = a\lambda^2/h^3 \approx 1$.

In Eq. 14 $w(x, t)$ is the nondimensional deviation of the water surface from its undisturbed position; $u(x, t)$ is the nondimensional horizontal velocity field at a height θh (with $0 \leq \theta \leq 1$) above the flat bottom of the water. The constant parameters α through δ in Eq. 14 satisfy the following consistency conditions: $\alpha + \beta = \frac{1}{2}(\theta^2 - \frac{1}{3})$ and $\gamma + \delta = \frac{1}{2}(1 - \theta^2) \geq 0$. Solitary wave solutions of various special cases of Eq. 14 have been computed by Chen (1998).

Boussinesq systems arise when modeling the propagation of long-crested waves on large bodies of water (such as large lakes or the ocean). The Boussinesq family Eq. 14 encompasses many systems that appeared in the literature. Special cases and properties of well-posedness of Eq. 14 are addressed by Bona et al. (2002, 2004).

1D Shallow Water Wave Equation

The so-called one-dimensional (1D) shallow water wave equation,

$$v_{xxt} + \alpha v v_t - v_t - v_x + \beta v_x \int_{-\infty}^x v_t(y, t) dy = 0, \quad (15)$$

can be derived from classical shallow water wave theory (see section “Shallow Water Wave Equations of Geophysical Fluid Dynamics”) in the Boussinesq approximation. In that approximation, one assumes that vertical variations of the static density, ρ_0 , are neglected, except the buoyancy term proportional to $d\rho_0/dz$, which is, in fact, responsible for the existence of solitary waves. The integral term in Eq. 15 can be removed by introducing the potential u . Indeed, setting $v = u_x$, Eq. 15 can be written as

$$u_{xxt} + \alpha u_x u_{xt} - u_{xt} - u_{xx} + \beta u_{xx} u_t = 0. \quad (16)$$

The equation is completely integrable and can be solved with the IST if and only if either $\alpha = \beta$

(Hirota and Satsuma 1976) or $\alpha = 2\beta$ (Ablowitz et al. 1974). When $\alpha = \beta$, Eq. 16 can be integrated with respect to x and thus replaced by

$$u_{xt} + \alpha u_x u_t - u_t - u_x = 0. \quad (17)$$

Closed-form solutions of Eq. 15, and in particular of Eq. 17, have been computed by Clarkson and Mansfield (1994).

The Camassa-Holm Equation

The CH equation, named after Camassa and Holm (1993, 1994),

$$\begin{aligned} u_t + 2\kappa u_x + 3uu_x - \alpha^2 u_{xxt} + \gamma u_{xxx} \\ - 2\alpha^2 u_x u_{xx} - \alpha^2 uu_{xxx} &= 0, \end{aligned} \quad (18)$$

also models waves in shallow water. In Eq. 18, u is the fluid velocity in the x -direction or, equivalent, the height of the water's free surface above a flat bottom, and κ , γ and α are constants. Retaining only the first four terms in Eq. 18 gives the BBM Eq. 7. Setting $\alpha = 0$ reduces Eq. 18 to the KdV equation.

The CH equation admits solitary wave solutions, but in contrast to the hump-type solutions of the KdV and Boussinesq equations, they are implicit in nature (see, e.g., Johnson 2003). In the limit $\kappa \rightarrow 0$, Eq. 18 with $\gamma = 0$, $\alpha = 1$ has a cusp-type solution of the form $u(x, t) = c \exp(-|x - ct - x_0|)$. The solution is called a peakon since it has a peak (or corner) where the first derivatives are discontinuous. The solution travels at speed $c > 0$ which equals the height of the peakon.

The Kadomtsev-Petviashvili Equation

In their 1970 study (Kadomtsev and Petviashvili 1970) of the stability of line solitons, Kadomtsev and Petviashvili (KP) derived a 2D generalization of the KdV equation which now bears their names. In dimensionless variables, the KP equation is

$$(u_t + \alpha uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0, \quad (19)$$

where y is the transverse direction. In the derivation of the KP equation, one assumes that the scale of variation in the y -direction which is along the

crest of the wave (as shown in Fig. 4) is much longer than the wavelength along the x -direction.

The solitary wave and periodic (cnoidal) solutions of Eq. 19 are, respectively, given by

$$u(x, t) = \frac{k\omega - 4k^4 + \sigma^2 l^2}{\alpha k^2} + \frac{12k^2}{\alpha} \operatorname{sech}^2(kx + ly - \omega t + \delta), \quad (20)$$

and

$$u(x, t) = \frac{k\omega - 4k^4(2m - 1) - \sigma^2 l^2}{\alpha k^2} + \frac{12k^2 m}{\alpha} \operatorname{cn}^2(kx + ly - \omega t + \delta; m). \quad (21)$$

As shown in Fig. 6, near a flat beach the periodic waves appear as long, quasilinear stripes with a cn-squared cross section. Such waves are typically generated by wind and tides.

The equation with $\sigma^2 = -1$ is referred to as KP1, whereas Eq. 19 with $\sigma^2 = 1$ is called KP2, which describes shallow water waves (Segur 2007a). Both KP1 and KP2 are completely integrable equations but their solution structures are fundamentally different (see, e.g., Scott 2005, pp. 489–490).

Shallow Water Wave Equations of Geophysical Fluid Dynamics

The shallow water equations used in geophysical fluid dynamics are based on the assumption $D/L \ll 1$, where D and L are characteristic values for the vertical and horizontal length scales of motion. For example, D could be the average depth of a layer of fluid (or the entire fluid) and L could be the wavelength of the wave.

The geophysical fluid dynamics community (see, e.g., Pedlosky 1987; Toro 2001; Vallis 2006) uses the following 2D shallow water equations,

$$u_t + uu_x + vu_y + gh_x - 2\Omega v = -gb_x, \quad (22)$$

$$v_t + uv_x + vv_y + gh_y + 2\Omega u = -gb_y, \quad (23)$$

$$h_t + (hu)_x + (hv)_y = 0, \quad (24)$$

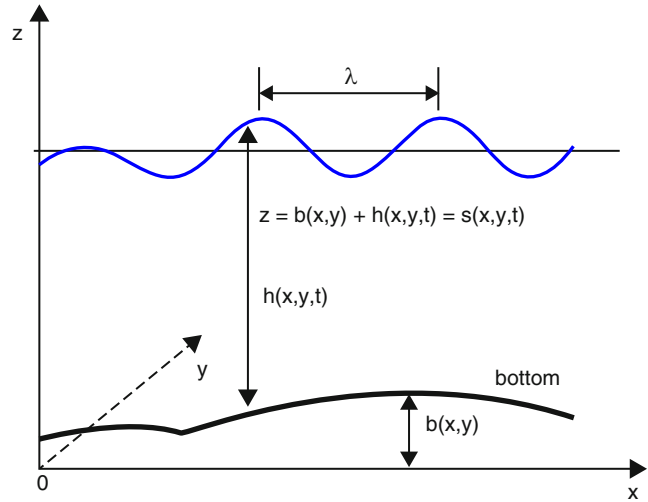
to describe water flows with a free surface under the influence of gravity (with gravitational acceleration g) and the Coriolis force due to the earth's rotation (with angular velocity Ω). As usual, $\mathbf{u} = (u, v)$ denotes the horizontal velocity of the fluid and $h(x, y, t)$ is its depth as shown in Fig. 7, $h(x, y, t)$ is the distance between the free surface $z = s(x, y, t)$ and the variable bottom $b(x, y)$. Hence,



Shallow Water Waves and Solitary Waves, Fig. 6 Periodic plane waves in shallow water, off the coast of Lima, Peru (Photograph courtesy of A. Segur)

Shallow Water Waves and Solitary Waves,

Fig. 7 Setup for the geophysical shallow water wave model



$s(x, y, t) = b(x, y) + h(x, y, t)$. Equations 22 and 23 express the horizontal momentum-balance; Eq. 24 expresses conservation of mass. Note that the vertical component of the fluid velocity has been eliminated from the dynamics and that the number of independent variables has been reduced by one. Indeed, z no longer explicitly appears in Eqs. 22, 23, and 24, where u , v , and h only depend on x , y , and t .

A shortcoming of the model is that it does not take into account the density stratification which is present in the atmosphere (as well as in the ocean). Nonetheless, Eqs. 22, 23, and 24 are commonly used by atmospheric scientists to model flow of air at low speed.

More sophisticated models treat the ocean or atmosphere as a stack of layers with variable thickness. Within each layer, the density is either assumed to be uniform or may vary horizontally due to temperature gradients. For example, Lavoie's rotating shallow water wave equations (see Dellar 2003),

consider only one active layer with layer depth $h(x, y, t)$, but take into account the forcing due to a horizontally varying potential temperature field $\theta(x, y, t)$. Vector $\mathbf{u} = u(x, y, t)\mathbf{i} + v(x, y, t)\mathbf{j}$ denotes the fluid velocity and $\mathbf{\Omega} = \Omega\mathbf{k}$ is the angular velocity vector of the Earth's rotation $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}$ is the gradient operator, and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the x , y , and z -axes.

Lavoie's equations are part of a family of multilayer models proposed by Ripa (1993) to study, for example, the effects of solar heating, freshwater fluxes, and wind stresses on the upper equatorial ocean. A study of the validity of various layered models has been presented by Ripa (1999). Obviously, the more sophisticated the models become the harder they become to treat with analytic methods so one has to apply numerical methods. Numerical aspects of various shallow water models in atmospheric research and beyond are discussed in, e.g., LeVeque (2002); Vreugdenhil (1988); Weiyan (1992).

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} \\ = -\nabla(h\theta) + \frac{1}{2}h\nabla\theta, \end{aligned} \quad (25)$$

$$h_t + \nabla \cdot (h\mathbf{u}) = 0, \quad (26)$$

$$\theta_t + \mathbf{u} \cdot (\nabla\theta) = 0, \quad (27)$$

Computation of Solitary Wave Solutions

As shown in section “Completely Integrable Shallow Water Wave Equations,” solitary wave solutions of the KdV and Boussinesq equations (and like PDEs) can be expressed as polynomials of the hyperbolic secant (sech) or tangent (tanh)

functions, whereas their simplest period solutions involve the Jacobi elliptic cosine (cn) function.

There are several methods to compute exact, analytic expressions for solitary and periodic wave solutions of nonlinear PDEs. Two straightforward methods, namely the direct integration method and the tanh-method, will be discussed in this section. Both methods seek traveling wave solutions. By working in a traveling frame of reference, the PDE is replaced by an ordinary differential equation (ODE) for which one seeks closed-form solutions in terms of special functions.

In the terminology of dynamical systems, the solitary wave solutions correspond to heteroclinic or homoclinic trajectories in the phase plane of a first-order dynamical system corresponding to the underlying ODE (see, e.g., Balmforth 1995). The periodic solutions (often expressible in terms of Jacobi elliptic functions) are bounded in the phase plane by these special trajectories, which correspond to the limit of infinite period and modulus one.

Other more powerful methods, such as the aforementioned IST and Hirota's method (see, e.g., Hirota 2004) deal with the PDE directly. These methods allow one to compute closed-form expressions of soliton solutions (in particular, solitary wave solutions) addressed elsewhere in the encyclopedia.

Direct Integration Method

Exact expressions for solitary wave solutions can be obtained by direct integration. The steps are illustrated for the KdV equation, Eq. 2. Assuming that the wave travels to the right at speed $v = \omega/k$, Eq. 2 can be put into a traveling frame of reference with independent variable $\xi = k(x - vt - x_0)$. This reduces Eq. 2 to an ODE, $-v\phi' + \alpha\phi\phi' + k^2\phi''' = 0$, for $\phi(\xi) = u(x, t)$. A first integration with respect to ξ yields

$$-v\phi + \frac{\alpha}{2}\phi^2 + k^2\phi'' = A, \quad (28)$$

where A is a constant of integration. Multiplication of Eq. 28 by ϕ' , followed by a second integration with respect to ξ , yields

$$-\frac{v}{2}\phi^2 + \frac{\alpha}{6}\phi^3 + \frac{k^2}{2}\phi'^2 = A\phi + B, \quad (29)$$

where B is an integration constant. Separation of variables and integration then leads to

$$\int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{a\phi^2 - b\phi^3 + \tilde{A}\phi + \tilde{B}}} = \pm \int_{\xi_0}^{\xi} d\xi, \quad (30)$$

where $a = v/k^2$, $b = \alpha/3k^2$, $\tilde{A} = 2A/k^2$, and $\tilde{B} = 2B/k^2$.

The evaluation of the elliptic integral in Eq. 30 depends on the relationship between the roots of the function $f(\phi) = a\phi^2 - b\phi^3 + \tilde{A}\phi + \tilde{B}$. In turn, the nature of the roots depends on the choice of $\tilde{A}$ and $\tilde{B}$. Two cases lead to physically relevant solutions.

Case 1

If the three roots are real and distinct, then the integral can be expressed in terms of the inverse of the cn function (see, e.g., Drazin 1989 for details). This leads to the cnoidal wave solution given in Eq. 6.

Case 2

If the three roots are real and (only) two of them coincide, then the tanh-squared solution follows. This happens when $\tilde{A} = \tilde{B} = 0$. Integrating both sides of Eq. 30 then gives

$$\int_{\phi_0}^{\phi} \frac{d\phi}{\phi\sqrt{a-b\phi}} = -\frac{2}{\sqrt{a}} \operatorname{Arctanh} \left[\frac{\sqrt{a-b\phi}}{\sqrt{a}} \right] + C = \pm(\xi - \xi_0). \quad (31)$$

where, without loss of generality, C and ξ_0 can be set to zero. Solving Eq. 31 for ϕ yields

$$\begin{aligned} \phi(\xi) &= \frac{a}{b} \left(1 - \tanh^2 \left(\frac{\sqrt{a}}{2} \xi \right) \right) \\ &= \frac{a}{b} \operatorname{sech}^2 \left(\frac{\sqrt{a}}{2} \xi \right). \end{aligned} \quad (32)$$

Returning to the original variables, one gets

$$\phi(\xi) = \frac{3v}{\alpha} \operatorname{sech}^2\left(\frac{\sqrt{v}}{2k}\xi\right), \quad (33)$$

or

$$u(x, t) = \frac{3v}{\alpha} \operatorname{sech}^2\left(\frac{\sqrt{v}}{2}(x - vt - x_0)\right), \quad (34)$$

where v is arbitrary. Setting $v = \omega/k = 4k^2$, where k is arbitrary, and $\delta = -kx_0$, one can verify that Eq. 34 matches Eq. 5.

The Tanh-Method

If one is only interested in tanh- or sech-type solutions, one can circumvent explicit integration (often involving elliptic integrals) and apply the so-called tanh-method. A detailed description of the method has been given by Baldwin et al. (2004). The method has been fully implemented in *Mathematica*, a popular symbolic manipulation program, and successfully applied to many nonlinear differential equations from soliton theory and beyond.

The tanh-method is based on the following observation: all derivatives of the tanh function can be expressed as polynomials in tanh. Indeed, using the identity $\cosh^2\xi - \sinh^2\xi = 1$, one computes $\tanh'\xi = \operatorname{sech}^2\xi = 1 - \tanh^2\xi$, $\tanh''\xi = -2\tanh\xi + 2\tanh^3\xi$, etc. Therefore, all derivatives of $T(\xi) = \tanh\xi$ are polynomials in T . For example, $T' = 1 - T^2$, $T'' = -2T + 2T^3$, and $T''' = -2T + 8T^2 - 6T^4$.

By applying the chain rule, the PDE in $u(x, t)$ is then transformed into an ODE for $U(T)$ where $T = \tanh\xi = \tanh(kx - \omega t + \delta)$ is the new independent variable. Since all derivatives of T are polynomials of T , the resulting ODE has polynomial coefficients in T . It is therefore natural to seek a polynomial solution of the ODE. The problem thus becomes algebraic. Indeed, after computing the degree of the polynomial solution, one finds its unknown coefficients by solving a nonlinear algebraic system.

The method is illustrated using Eq. 2. Applying the chain rule (repeatedly), the terms of Eq. 2 become $u_t = -\omega(1 - T^2)U'$, $u_x = k(1 - T^2)U'$, and

$$\begin{aligned} u_{xxx} &= k^3(1 - T^2) \\ &\quad \left[-2(1 - 3T^2)U' - 6T(1 - T^2)U'' + (1 - T^2)^2U'''\right], \end{aligned} \quad (35)$$

where $U(T) = U(\tanh(kx - \omega t + \delta)) = u(x, t)$, $U' = dU/dT$, etc. Substitution into Eq. 2 and cancellation of a common $1 - T^2$ factor yields

$$\begin{aligned} -\omega U' + \alpha k U U' - 2k^3(1 - 3T^2)U' \\ - 6k^3T(1 - T^2)U'' + k^3(1 - T^2)^2U''' = 0 \end{aligned} \quad (36)$$

This ODE for $U(T)$ has polynomial coefficients in T . One therefore seeks a polynomial solution

$$U(T) = \sum_{n=0}^N a_n T^n, \quad (37)$$

where the integer exponent N and the coefficients a_n must be computed.

Substituting T^N into Eq. 36 and balancing the highest powers in T gives $N = 2$. Then, substituting

$$U(T) = a_0 + a_1 T + a_2 T^2 \quad (38)$$

into Eq. 36, and equating to zero the coefficients of the various power terms in T , yields

$$\begin{aligned} a_1(\alpha a_2 + 2k^2) &= 0, \\ \alpha a_2 + 12k^2 &= 0, \\ a_1(\alpha k a_0 - 2k^3 - \omega) &= 0, \\ \alpha k a_1^2 + 2\alpha k a_0 a_2 - 16k^3 a_2 - 2\omega a_2 &= 0. \end{aligned} \quad (39)$$

The unique solution of this nonlinear system for the unknowns a_0 , a_1 , and a_2 is

$$a_0 = \frac{8k^3 + \omega}{\alpha k}, \quad a_1 = 0, \quad a_2 = -\frac{12k^2}{\alpha}. \quad (40)$$

Finally, substituting Eq. 40 into Eq. 38 and using $T = \tanh(kx - \omega t + \delta)$ yields Eq. 4.

The solitary wave solutions and cnoidal wave solutions presented in section “[Completely Integrable Shallow Water Wave Equations](#)” have been

automatically computed with a Mathematica package (Baldwin et al. 2004) that implements the tanh-method and variants.

A review of numerical methods to compute solitary waves of arbitrary amplitude can be found in Vanden-Broeck (2007).

Water Wave Experiments and Observations

Through a series of experiments in a hydrodynamic tank, Hammack investigated the validity of the BBM equation (Hammack 1973) and KdV equation as models for long waves in shallow water (Hammack and Segur 1974, 1978a, 1978b) and long internal waves (Segur and Hammack 1982). Their research addressed the question: Would an initial displacement of water, as it propagates forward, eventually evolve in a train of localized solitary waves (solitons) and an oscillatory tail as predicted by the KdV equation? Based on the experimental data, they concluded that (i) the KdV dynamics only occurs if the waves travel over a long distance, (ii) a substantial amount of water must be initially displaced (by a piston) to produce a soliton train, (iii) the water volume of the initial wave determines the shape of the leading wave in the wave train, and (iv) the initial direction of displacement (upward or downward piston motion) determines what happens later. Quickly raising the piston causes a train of solitons to emerge; quickly lowering the piston causes all wave energy to distribute into the oscillatory tail, as predicted by the theory.

Several other researchers have tested the validity of the KdV equation and variants in laboratory experiments (see, e.g., Helfrich and Melville 2006; Remoissenet 1999). Bona et al. (1981) give an in-depth evaluation of the BBM Eq. 7 with and without dissipative term u_{xx} . Their study includes (i) a numerical scheme with error estimates, (ii) a convergence test of the computer code, (iii) a comparison between the predictions of the theoretical model and the results of laboratory experiments. The authors note that it is important to include dissipative effects to obtain reasonable agreement between the forecast of the model and the empirical results.

Water tank experiments in conjunction with the analysis of actual data allow researchers to judge

whether or not the KdV equation can be used to model the dynamics of tsunamis (see Segur 2007a). Tsunami research intensified after the December 2004 tsunami devastated large coastal regions of India, Indonesia, Sri Lanka, and Thailand, and killed nearly 300,000 people.

Apart from shallow water waves near beaches, the KdV equation and its solitary wave solution also apply to internal waves in the ocean. Internal solitary waves in the open ocean are slow waves of large amplitude that travel at the interface of stratified layers of different density. Stratification based on density differences is primarily due to variations in temperature or concentration (e.g., due to salinity gradients). For example, absorption of solar radiation creates a near surface thin layer of warmer water (of lower density) above a thicker layer or colder, denser water. The smaller the density change, the lower the wave frequency, and the slower the propagation speed. If the upper layer is thinner than the lower one, then the internal wave is a wave of depression causing a downward displacement of the fluid interface.

Internal solitary waves are ubiquitous in stratified waters, in particular, whenever strong tidal currents occur near irregular topography. Such waves have been studied since the 1960s. An early, well-documented case deals with internal waves in the Andaman Sea, where Perry and Schimke (1965) found groups of internal waves up to 80 m high and 2000 m long on the main thermocline at 500 m in 1500 m deep water. Their measurements were confirmed by Osborne and Burch (1980) who showed that internal waves in the Andaman Sea are generated by tidal flows and can travel over hundreds of kilometers.

Strong internal waves can affect biological life and interfere with underwater navigation. Understanding the behavior of internal waves can aid in the design of offshore production facilities for oil and natural gas.

The near-surface current associated with the internal wave locally modulates the height of the water surface. Hence, the internal wave leaves a “signature” or “footprint” at the sea surface in the form of a packet of solitary waves (sometimes called current rips or tide rips). These visual manifestations appear as long, quasilinear stripes in

satellite imagery or photographs taken during space flights. Over 50 case studies and hundreds of images of oceanic internal waves can be found in “An Atlas of Internal Solitary-like Waves and Their Properties” (Jackson 2004).

Figure 8 shows a photograph of three solitary wave packets which are the surface signature of internal waves in the Strait of Gibraltar. The photograph was taken from the Space Shuttle on October 11, 1984. Spain is to the North, Morocco to the South. Alternate solitary wave packets move toward the northeast or the southeast. The amplitude of these waves is of the order of 50 m; their wavelength is in the range of 500–2000 m. The separation between the packets is approximately 30 km. Waves of longer wavelengths and higher amplitudes have traveled the furthest. The number of oscillations within each packet increases as time goes on. Solitary wave packets can reach 200 km into the Western Mediterranean sea and travel for more than 2 days before dissipating. An in-depth study of solitary waves in the Strait of Gibraltar can be found in Farmer and Armi (1988).

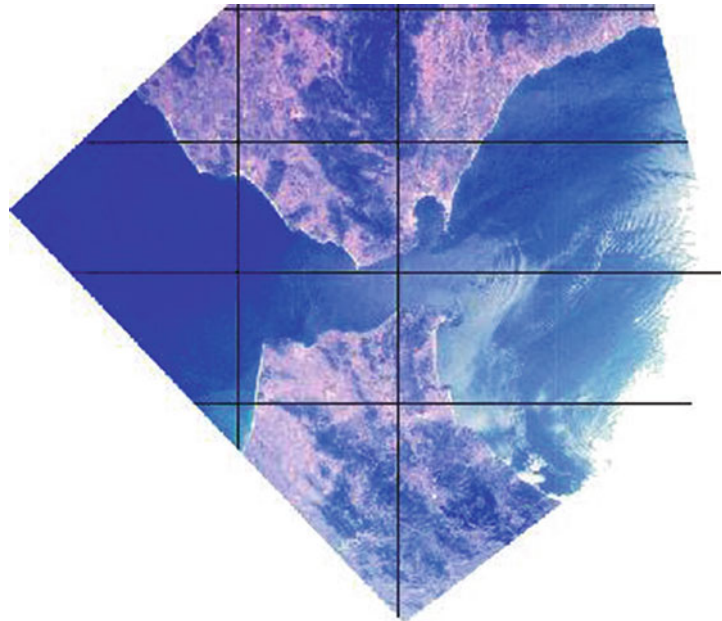
The KdV model is applicable to stratified fluids with two layers and internal solitary waves if (i) the ratio of the amplitude a to the upper layer

depth h is small, and (ii) the wavelength λ is long compared with the upper layer depth. More precisely, $a/h = O(h^2/\lambda^2) \ll 1$. A detailed discussion of internal solitary waves and additional references can be found in Garrett and Munk (1979), Grimshaw (1997, 2001, 2005), Helfrich and Melville (2006), Apel et al. (2007), and Pelinovsky et al. (2007). The last three papers discuss a variety of other theoretical models including the extended KdV equation (also known as Gardner’s equation or combined KdV-modified KdV equation) which contains both quadratic and cubic nonlinearities. Solitary wave solutions of the extended KdV equation can be found in Scott (2005, p. 856), Helfrich and Melville (2006), and Apel et al. (2007). A review of laboratory experiments with internal solitary waves was published by Ostrovsky and Stepanyants (2005).

As discussed in the review paper by Staquet and Sommeria (2002), internal gravity waves also occur in the atmosphere, where they are often caused by wind blowing over topography and cumulus convective clouds. Internal gravity waves reveal themselves as unusual cloud patterns, which are the counterpart of the solitary wave packets on the ocean’s surface.

Shallow Water Waves and Solitary Waves,

Fig. 8 Three solitary wave packets generated by internal waves from sills in the Strait of Gibraltar. Original image STS41G-34-81 courtesy of the Earth Sciences and Image Analysis Laboratory, NASA Johnson Space Center (<http://eol.jsc.nasa.gov>). Ortho-rectified, color adjusted photograph courtesy of Global Ocean Associates



Future Directions

For many shallow water wave applications, the full Euler equations are too complex to work with. Instead, various approximate models have been proposed. Arguably, the most famous shallow water wave equations are the KdV and Boussinesq equations.

The KdV equation was originally derived to describe shallow water waves in a rectangular channel. Surprisingly, the equation also models ion-acoustic waves and magneto-hydrodynamic waves in plasmas, waves in elastic rods, midlatitude and equatorial planetary waves, acoustic waves on a crystal lattice, and more (see, e.g., Scott 2003, 2005; Scott et al. 1973). The KdV equation has played a pivotal role in the development of the Inverse Scattering Transform and soliton theory, both of which had a lasting impact on twentieth-century mathematical physics.

Historically, the classical Boussinesq equation was derived to describe the propagation of shallow water waves in a canal. Boussinesq systems arise when modeling the propagation of long-crested waves on large bodies of water (e.g., large lakes or the ocean). As Bona et al. (2002) point out, a plethora of formally equivalent Boussinesq systems can be derived. Yet, such systems may have vastly different mathematical properties. The study of the well-posedness of the nonlinear models is of paramount importance and is the subject of ongoing research.

Shallow water wave theory allows one to adequately model waves in canals, surface waves near beaches, and internal waves in the ocean (see Apel et al. 2007). Due to their widespread occurrence in the ocean (see Jackson 2004), solitary waves and “solitary wave packets” (solitons) are of interest to oceanographers and geophysicists. The (periodic) cnoidal wave solutions are used by coastal engineers in studies of sediment movement, erosion of sandy beaches, interaction of waves with piers and other coastal structures.

Apart from their physical relevance, the knowledge of solitary and cnoidal wave solutions of nonlinear PDEs facilitates the testing of numerical solvers and also helps with stability analysis.

Shallow water wave models are widely used in atmospheric science as a paradigm for

geophysical fluid motions. They model, for example, inertia-gravity waves with fast timescale dynamics, and wave-vortex interactions and Rossby waves associated with slow advective-timescale dynamics.

This entry has reviewed commonly used shallow water wave models, with the hope of bridging two research communities: one that focuses on nonlinear equations with dispersive effects; the other on nonlinear hyperbolic equations without dispersive terms. Of common concern are the testing of the theoretical models on measured data and further validation of the equations with numerical simulations and laboratory experiments. A fusion of the expertise of both communities might advance research on water waves and help to answer open questions about wave breaking, instability, vorticity, and turbulence. Of paramount importance is the prevention of natural disasters, ecological ravage, and damage to man-made structures due to a better understanding of the dynamics of tsunamis, steep waves, strong internal waves, rips, tidal currents, and storm surges.

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Soliton Perturbation

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Article Outline

Glossary

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Introduction

Methods for Soliton Solutions

Soliton Perturbation

Variational Approach

Variational Iteration Method

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Glossary

Soliton A soliton is a nonlinear pulse-like wave that can exist in some nonlinear systems. The isolated wave can propagate without dispersing its energy over a large region of space; collision of two solitons leads to unchanged forms, solitons also exhibit particle like properties.

Soliton perturbation theory The theory is used to study the solitons that are governed by the various nonlinear equations in presence of the perturbation terms.

Homotopy perturbation method The homotopy perturbation method is a useful tool to the search for solitons without the requirement of presence of small perturbations. In this method, a homotopy is constructed with a homotopy parameter, p . When $p = 0$, it becomes a nonlinear wave equation such as a KdV equation with a known soliton solution; when $p = 1$, it turns out to be the original nonlinear equation. To change p from zero to

unity, one must only change from a trial soliton to the solved soliton.

Variational iteration method The variational iteration method is a new method for obtaining soliton-type solutions of various nonlinear wave equations. The method begins with a soliton-type solution with some unknown parameters which can be determined after few iterations. The iteration formulation is constructed by a general Lagrange multiplier which can be identified optimally via variational theory.

Exp-function method The exp-function method is a new method for searching for both soliton-type solutions and periodic solutions of nonlinear systems. The method assumes that the solutions can be expressed in arbitrary forms of the exp-function.

Definition of the Subject

The soliton is a kind of nonlinear wave. There are many equations of mathematical physics which have solutions of the soliton type. The first observation of this kind of wave was made in 1834 by John Scott Russell (1844). In 1895, the famous KdV equation, which possesses soliton solutions, was obtained by D. J. Korteweg and H. de Vries (1895), who established a mathematical basis for the study of various solitary phenomena.

From a modern perspective, the soliton is used as a constructive element to formulate the complex dynamical behavior of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasmas to shock waves, from tornados to the Great Red Spot of Jupiter, from traffic flow to the Internet, from Tsunamis to turbulence (Eilbeck 2007). More recently, solitary waves are of key importance in the quantum fields: on extremely small scales and at very high observational resolution equivalent to a very high energy, space-time resembles a stormy ocean and particles and their interactions have soliton-type solutions (El Naschie 2004a).

Introduction

The soliton was first discovered in 1834 by John Scott Russell, who observed that a canal boat stopping suddenly gave rise to a solitary wave which traveled down the canal for several miles, without breaking up or losing strength. Russell named this phenomenon the ‘soliton’.

In a highly informative as well as entertaining article (Russell 1844) J.S. Russell gave an engaging historical account of the important scientific observation:

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

His ideas did not earn attention until 1965 when N.J. Zabusky and M.D. Kruskal began to use a finite difference approach to the study of KdV equation (1965), and various analytical methods also led to a complete understanding of Solitons, especially the inverse scattering transform proposed by Gardner et al. (1967) in 1967. The significance of Russell’s discovery was then fully appreciated. It was discovered that many phenomena in physics, electronics and biology can be described by the mathematical and physical theory of the ‘Soliton’.

The particle-like properties of solitons (Bode et al. 2002) also caught much attention, and were proposed as models for elementary particles (Braun et al. 2005). More recently it has been realized that some of the quantum fields which are used to describe particles and their interactions

also have solutions of the soliton type (Ahufinger et al. 2007).

Methods for Soliton Solutions

The investigation of soliton solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. There are many analytical approaches to the search for soliton solutions, such as soliton perturbation, tanh-function method, projective approach, F-expansion method, and others (Ye et al. 2006; Bogning et al. 2005; El-Sabbagh and Ali 2005; Shen and Xu 2004; Sheng 2007).

Soliton Perturbation

We consider the following perturbed nonlinear evolution equation (Yu and Yan 2006; Herman 2005)

$$u_T + N(u) = \varepsilon R(u), \quad 0 < \varepsilon \ll 1. \quad (1)$$

When $\varepsilon = 0$, we have the unperturbed equation

$$u_T + N(u) = 0, \quad (2)$$

which is assumed to have a soliton solution.

When $\varepsilon \neq 0$, but $0 < \varepsilon \ll 1$, we can use perturbation theory (Yu and Yan 2006; Herman 2005), and look for approximate solutions of Eq. (1), which are close to the soliton solutions of Eq. (2). Using multiple time scales (a slow time τ and a fast time t , such that $\partial_T = \partial_\tau + \varepsilon \partial_t$), we assume that the soliton solution can be expressed in the form

$$u(x, T) = u_0(\xi, \tau) + \varepsilon u_1(\xi, \tau, t) + \varepsilon^2 u_2(\xi, \tau, t) + \dots \quad (3)$$

where $\xi = x - ct$, and τ is a slow time and t is a fast time.

Substituting Eq. (3) into Eq. (1) and then equating like-powers of ε , we can obtain a series of linear equations for u_i ($i = 0, 1, 2, 3, \dots$).

In most cases the nonlinear term $R(u)$ in Eq. (1) plays an important role in understanding various solitary phenomena, and the coefficient ε is not limited to a “small parameter”.

Variational Approach

Recently, variational theory and homotopy technology have been successfully applied to the search for soliton solutions (He and Wu 2006a; He 2005) without requiring the small parameter assumption. Both variational and homotopy technologies can lead to an extremely simple and elementary, but rigorous, derivation of soliton solutions.

Considering the KdV equation

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad (4)$$

we seek its traveling wave solutions in the following frame

$$\begin{aligned} u(x, t) &= U(\xi) \quad v(x, t) = V(\xi), \\ \xi &= x - ct, \end{aligned} \quad (5)$$

where c is angular frequency. Substituting Eq. (5) into Eq. (4) yields

$$-cu' - 6uu' + u''' = 0, \quad (6)$$

where a prime denotes the differential with respect to ξ .

Integrating Eq. (6) yields the result

$$-cu - 3u^2 + u'' = 0. \quad (7)$$

By the semi-inverse method (He 2004), the following variational formulation is established

$$J = \int_0^\infty \left(\frac{1}{2} cu^2 + u^3 + \frac{1}{2} \left(\frac{du}{d\xi} \right)^2 \right) d\xi. \quad (8)$$

The semi-inverse method is a powerful mathematical tool to the search for variational formulae for real-life physical problems.

By the Ritz method, we search for a solitary wave solution in the form

$$u = p \operatorname{sech}^2(q\xi), \quad (9)$$

where p and q are constants to be further determined.

Substituting Eq. (9) into Eq. (8) results in

$$\begin{aligned} J &= \int_0^\infty \left[\frac{1}{2} c^2 \operatorname{sech}^4(q\xi) + p^3 \operatorname{sech}^6(q\xi) \right. \\ &\quad \left. + \frac{1}{2} (4p^2 q^2 \operatorname{sech}^4(q\xi) \tanh^2(q\xi)) d\xi \right] \\ &= \frac{cp^2}{2q} \int_0^\infty \operatorname{sech}^4(z) dz + \frac{p^3}{q} \int_0^\infty \operatorname{sech}^6(z) dz \\ &\quad + 2p^2 q \int_0^\infty \{ \operatorname{sech}^4(z) \tanh^4(z) dz \} \\ &= \frac{cp^2}{3q} + \frac{8p^3}{15q} + \frac{4p^2 q}{15} \end{aligned} \quad (10)$$

Making J stationary with respect to p and q results in

$$\frac{\partial J}{\partial p} = \frac{2cp}{3q} + \frac{24p^2}{15q} + \frac{8pq}{15} = 0, \quad (11)$$

$$\frac{\partial J}{\partial q} = -\frac{cp^2}{3q^2} - \frac{8p^3}{15q^2} + \frac{4p^2}{15} = 0, \quad (12)$$

or simplifying

$$5c + 12p + 4q^2 = 0, \quad (13)$$

$$-5c - 8p + 4q^2 = 0. \quad (14)$$

From Eqs. (13) and (14), we can easily obtain the following relations:

$$p = -\frac{1}{2}c, \quad q = \sqrt{\frac{c}{4}}. \quad (15)$$

So the solitary wave solution can be approximated as

$$u = -\frac{c}{2} \operatorname{sech}^2 \sqrt{\frac{c}{4}} (x - ct - \xi_0), \quad (16)$$

which is the exact solitary wave solution of KdV equation (4).

The preceding analysis has the virtue of utter simplicity. The suggested variational approach can be readily applied to the search for solitary wave solutions of other nonlinear problems, and

the present example can be used as paradigms for many other applications in searching for solitary wave solutions of real-life physics problems.

Variational Iteration Method

The variational iteration method (He 1999) is an alternative approach to soliton solutions without the requirement of establishing a variational formulation for the discussed problems (He and Wu 2006a; Abulwafa et al. 2007; Inc 2007; Soliman 2006; Abdou and Soliman 2005). As an illustrating example, we consider the $K(3,1)$ equation in the form (He and Wu 2006a):

$$u_t + u^2 u_x + u_{xxx} = 0. \quad (17)$$

According to the variational iteration method, its iteration formulation can be constructed as follows

$$\begin{aligned} u_{n+1}(x, t) \\ = u_n(x, t) - \int_0^t \{ (u_n)_t + u_n^2 (u_n)_x + (u_n)_{xxx} \} dt. \end{aligned} \quad (18)$$

To search for its compacton-like solution, we assume the solution has the form

$$u_0(x, t) = \frac{a \sin^2(kx + wt)}{b + c \sin^2(kx + wt)}, \quad (19)$$

where a , b , k , and w are unknown constants further to be determined after few iterations (He and Wu 2006a).

Homotopy Perturbation Method

The homotopy perturbation method (He 2000) provides a simple mathematical tool for searching for soliton solutions without any small perturbation (He 2005; Ganji and Rafei 2006). Considering the following nonlinear equation

$$\begin{aligned} \frac{\partial u}{\partial t} + au \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} + N(u) = 0, \\ a > 0, b > 0, \end{aligned} \quad (20)$$

we can construct a homotopy in the form

$$\begin{aligned} (1-p) \left\{ \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right\} \\ + p \left\{ \frac{\partial u}{\partial t} + au \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} + N(u) \right\} = 0. \end{aligned} \quad (21)$$

When $p = 0$, we have

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad (22)$$

a well-known KdV equation whose soliton solution is known. When $p = 1$, Eq. (21) turns out to be the original equation. According to the homotopy perturbation method, we assume

$$u = u_0 + pu_1 + p^2u_2 + \dots \quad (23)$$

Substituting Eq. (23) into Eq. (21), and proceeding with the same process as the traditional perturbation method does, we can easily solve u_0 , u_1 and other components. The solution can be expressed finally in the form

$$u = u_0 + u_1 + u_2 + \dots \quad (24)$$

The homotopy perturbation method always stops before the second iteration, so the solution can be expressed as

$$u = u_0 + u_1 \quad (25)$$

for most cases.

Parameter-Expansion Method

The parameter-expansion method (Shou and He 2007; He 2001, 2002; Xu 2007) does not require one to construct a homotopy. To illustrate its solution procedure, we write Eq. (20) in the form

$$\frac{\partial u}{\partial t} + au \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} + 1 \cdot N(u) = 0. \quad (26)$$

Supposing that the parameters a , b , and 1 can be expressed in the forms

$$a = a_0 + pa_1 + p^2a_2 + \dots \quad (27)$$

$$b = b_0 + pb_1 + p^2b_2 + \dots \quad (28)$$

$$1 = pc_1 + p^2c_2 + \dots \quad (29)$$

where p is a book keeping parameter, $p = 1$. Substituting Eqs. (23), (27), (28) and (29) into Eq. (26) and proceeding the same way as the perturbation method, we can easily obtain the needed solution.

The exp-function method (He and Wu 2006b; He and Abdou 2007; Wu and He 2008) provides us with a straightforward and concise approach to obtaining generalized solitary solutions and periodic solutions and the solution procedure, with the help of Matlab or Mathematica, is utterly simple. Consider a general nonlinear partial differential equation of the form

$$F(u, u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, u_{zz}, u_{tt}, u_{xy}, u_{xt}, u_{yt}, \dots) = 0. \quad (30)$$

Using a transformation

$$\eta = ax + by + cz + dt, \quad (31)$$

we can re-write Eq. (30) in the form of the following nonlinear ordinary differential equation:

$$G(u, u', u'', u''', \dots) = 0, \quad (32)$$

where a prime denotes a derivation with respect to η .

According to the exp-function method, the traveling wave solutions can be expressed in the form

$$u(\eta) = \frac{\sum_{n=-k}^l a_n \exp(n\eta)}{\sum_{m=-i}^j b_m \exp(m\eta)}, \quad (33)$$

where i, j, k , and l are positive integer which could be freely chosen, a_n and b_m are unknown constants to be determined. The solution procedure is illustrated in (He and Abdou 2007).

Future Directions

It is interesting to point out the connection of catastrophe theory to loop soliton chaos, and finally to chaotic Cantorian spacetime (El Naschie 2004b, c).

El Naschie (2004b, c) studied the Eguchi–Hanson gravitational instant on solution and its interpretation by 't Hooft in the context of a quantum gravitational Hilbert space, as an event and a possible solitonic “extended” particle. Transferring a certain solitonic solution of Einstein’s field equations in Euclidean “real” space–time to the mathematical infinitely-dimensional Hilbert space, it is possible to observe a new non-standard process by which a definite mass can be assigned to massless particles. Thus by invoking Einstein’s gravity in “solitonic” gauge theory and viceversa, an alternative explanation for how massless particles acquire mass is found, which is also in harmony with the basic structure of our standard model as it stands at present (El Naschie 2004b, c).

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Solitons and Compactons

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Glossary

Compacton A compacton is a special solitary traveling wave that, unlike a soliton, does not have exponential tails.

Compacton-like solution A compacton-like solution is a special wave solution which can be expressed by the squares of sinusoidal or cosinoidal functions.

Generalized soliton A generalized soliton is a soliton with some free parameters. Generally a generalized soliton can be expressed by exponential functions.

Soliton A soliton is a stable pulse-like wave that can exist in some nonlinear systems. The soliton, after a collision with another soliton, eventually emerges unscathed.

Definition of the Subject

Soliton and compacton are two kinds of nonlinear waves. They play an indispensable and vital role in all ramifications of science and technology, and are used as constructive elements to formulate the complex dynamical behavior of wave systems throughout science: from hydrodynamics to nonlinearoptics, from plasmas to shock waves, from tornados to the Great Red Spot of Jupiter, from traffic flow to Internet, from Tsunamis to turbulence. Morerecently, soliton and compacton are of key importance in the quantum fields and nanotechnology especially in nanohydrodynamics.

Introduction

Solitary waves were first observed by John Scott Russell in 1895, and were studied by D. J. Korteweg and H. de Vries in 1895. arespecial solitons with finite wavelength. It was Philip Rosenau and his colleagues who first found compactons in 1993. Please refer to ► [“Soliton Perturbation”](#) for detailed information.

Solitons

A soliton is a special solitary traveling wave that after a collision with another soliton eventually emerges unscathed. Solitons aresolutions of partial differential equations that model phenomena like water waves or waves along a weakly anharmonic mass-spring chain.

The Korteweg–de Vries (KdV) equation is the generic model for the study of nonlinear waves in fluid dynamics, plasma and elastic media. KdV equation is one of the most fundamental equations in nature and plays a pivotal role in nonlinear phenomena. We consider the KdV equation in the form

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \quad (1)$$

Its solitary traveling wave solution can be solved as

$$u(x, t) = \frac{1}{2} c \operatorname{sech}^2 \left\{ \frac{1}{2} c^{1/2} (x - ct) \right\}. \quad (2)$$

The bell-like solution as illustrated in Fig. 1 is called a soliton.

We re-write Eq. (2) in an equivalently form:

$$u(\xi) = p \operatorname{sech}^2(q\xi) = \frac{4p}{e^{2q\xi} + e^{-2q\xi} + 2}. \quad (3)$$

where $u(x, t) = u(\xi)$, $\xi = x - ct$, c is the wave velocity.

It is obvious that

$$\lim_{\xi \rightarrow \infty} u(\xi) = 0 \quad \text{and} \quad \lim_{\xi \rightarrow -\infty} u(\xi) = 0. \quad (4)$$

The soliton has exponential tails, which are the basic character of solitary waves. The soliton obeys a superposition-like principle: solitons passing through one another emerge unmodified, see Fig. 2.

Compactons

Now consider a modified version of KdV equation in the form

$$u_t + (u^2)_x + (u^2)_{xxx} = 0. \quad (5)$$

Introducing a complex variable ξ defined as $\xi = x - ct$, where c is the velocity of traveling wave, integrating once, we have

$$-cu + u^2 + (u^2)_{\xi\xi} = D, \quad (6)$$

where D is an integral constant.

To avoid singular solutions, we set $D = 0$. We re-write Eq. (6) in the form

$$v_{\xi\xi} + v - cv^{1/2} = 0, \quad (7)$$

where $u^2 = v$.

In case $c = 0$, we have periodic solution: $v(\xi) = A \cos \xi + B \sin \xi$. Periodic solution of nonlinear oscillators can be approximated by sinusoidal function. It helps understanding if an equation can be classified as oscillatory by direct inspection of its terms.

We consider two common order differential equations whose exact solutions are important for physical understanding:

$$u'' - k^2 u = 0, \quad (8)$$

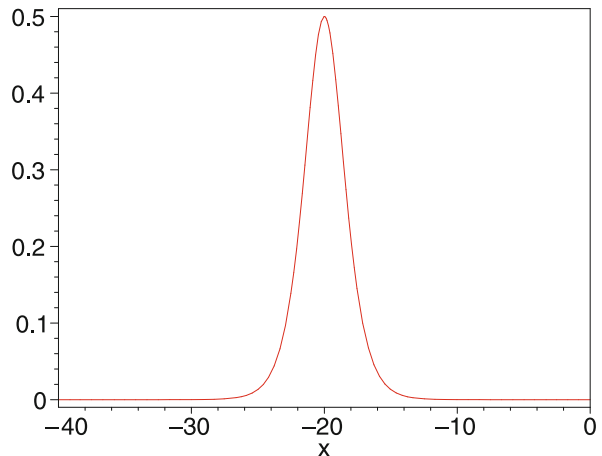
and

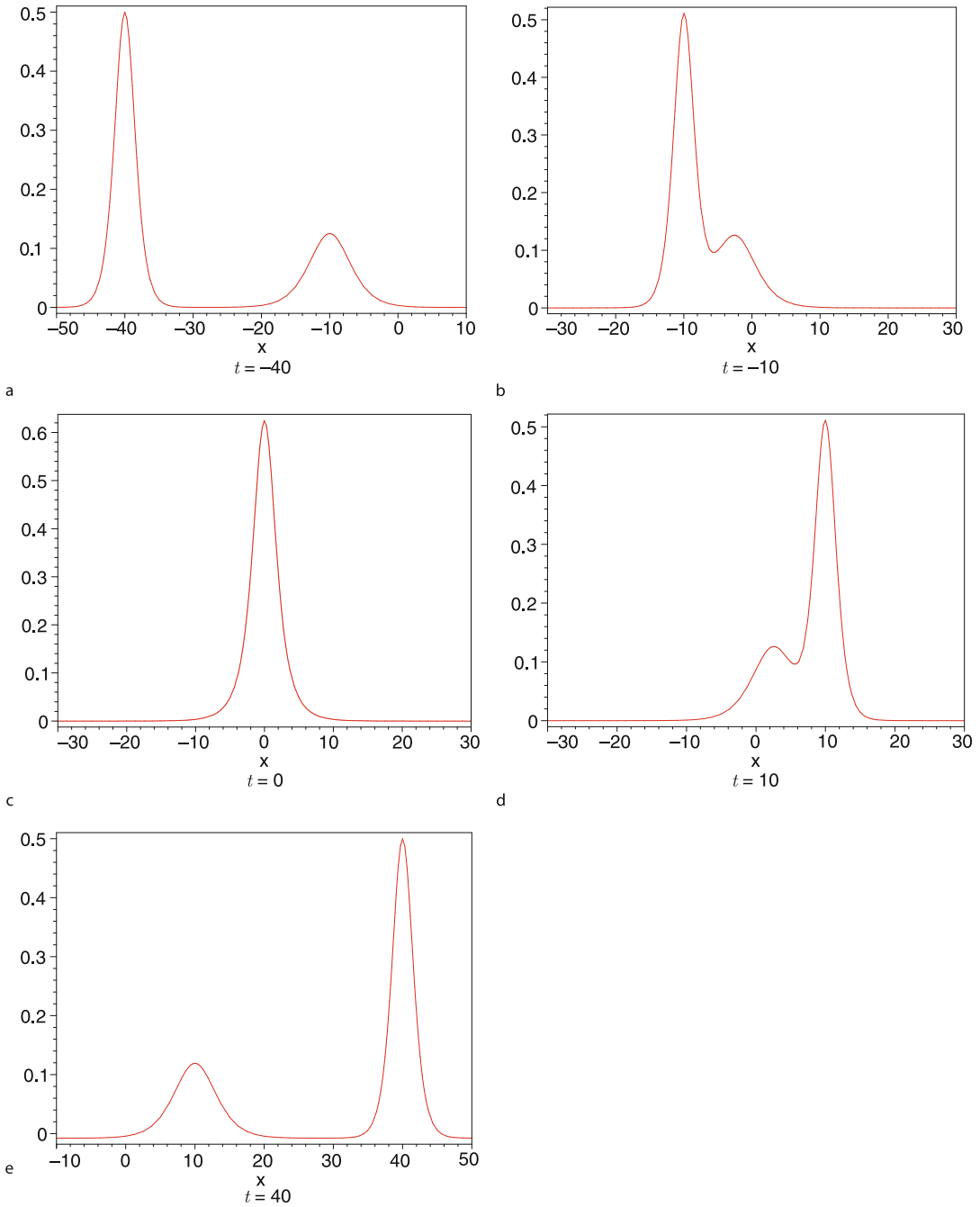
$$u'' + \omega^2 u = 0. \quad (9)$$

Both equations have linear terms with constant coefficients.

The crucial difference between these two very simple equations is the sign of the coefficient of

Solitons and Compactons, Fig. 1 Bell-like solitary wave





Solitons and Compactons, Fig. 2 Collision of two solitary waves

u in thesecond term. This determines whether the solutions are exponential or oscillatory. The general solution of Eq. (8) is

$$u = Ae^{kt} + Be^{-kt}. \quad (10)$$

The second Eq. (9) has a positive coefficient of u , and in this case the general solution reads

$$u = A \cos \omega t + B \sin \omega t. \quad (11)$$

This solution describes an oscillation at the angular velocity ω .

Equation (7) behaves sometimes like an oscillator when $1 - cv^{-1/2} > 0$, i. e., $u = v^{1/2}$ has a periodic solution, we assume v can be expressed in the form

$$v = u^2 = A^2 \cos^4 \omega \xi. \quad (12)$$

Substituting Eq. (12) into Eq. (7) results in

$$\begin{aligned} 12A^2\omega^2 \cos^2 \omega \xi - 16A^2\omega^2 \cos^4 \omega \xi \\ + A^2 \cos^4 \omega \xi - cA \cos^2 \omega \xi \\ = 0. \end{aligned} \quad (13)$$

We, therefore, have

$$\begin{aligned} 12A^2\omega^2 - cA &= 0 \\ -16A^2\omega^2 + A^2 &= 0. \end{aligned} \quad (14)$$

Solving the above system, Eq. (14), yields

$$\omega = \frac{1}{4}, \quad A = \frac{4}{3}c. \quad (15)$$

We obtain the solution in the form

$$u = v^{1/2} = \frac{4c}{3} \cos^2 \left[\frac{1}{4}(x - ct) \right]. \quad (16)$$

By a careful inspection, v can tend to a very small value or even zero, as a result, $1 - cv^{-1/2}$

tends to negative infinite, and Eq. (7) behaves like Eq. (8) with $k \rightarrow \infty$, the exponential tails vanish completely at the edge of the bell-shape (see Fig. 3):

$$u = \begin{cases} \frac{4c}{3} \cos^2 \left[\frac{1}{4}(x - ct) \right], & |x - ct| \leq 2\pi \\ 0, & \text{otherwise.} \end{cases} \quad (17)$$

This is a compact wave. Unlike solitons (Fig. 4), compacton does not have exponential tails (Fig. 3).

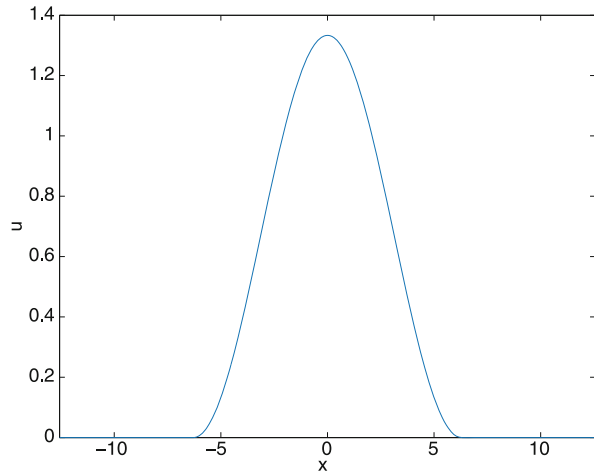
Generalized Solitons and Compacton-Like Solutions

Solitary solutions have tails, which can be best expressed by exponential functions. We can assume that a solitary solution can be expressed in the following general form

$$u(\eta) = \frac{\sum_{n=-c}^d a_n \exp(m\eta)}{\sum_{m=-p}^q b_m \exp(m\eta)}, \quad (18)$$

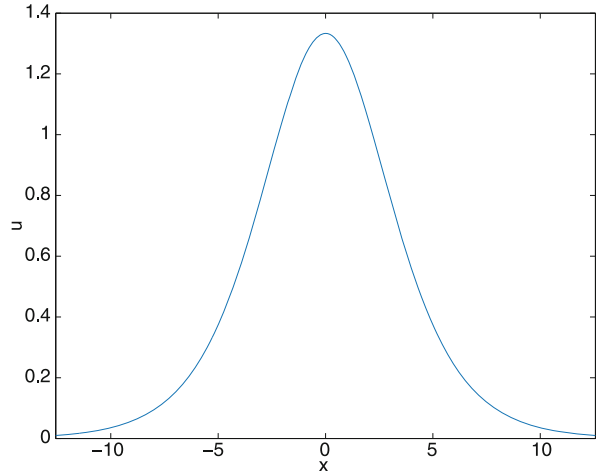
where c, d, p , and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants. The unknown constants can

Solitons and Compactons,
Fig. 3 Compacton wave without tails



Solitons and Compactons,

Fig. 4 Solitary wave with two tails



be easily determined using Matlab, the method is called the Exp-function method.

We consider the modified KdV equation in the form:

$$u_t + u^2 u_x + u_{xxx} = 0. \quad (19)$$

Using a transformation: $u(x, t) = u(\xi)$, $\xi = kx + \omega t$, we have

$$\omega u' + ku^2 u' + k^3 u''' = 0, \quad (20)$$

where prime denotes the differential with respect to ξ .

We suppose that the solution of Eq. (20) can be expressed as

$$u(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \cdots + b_{-q} \exp(-q\xi)}. \quad (21)$$

To determine values of c , d , p and q , we balance the linear term of highest order in Eq. (20) with the highest order nonlinear term. According to the homogeneous balance principle, we obtain the result $c = p$ and $d = q$. For simplicity, we set $c = p = 1$ and $d = q = 1$, so Eq. (21) reduces to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{\exp(p\xi) + b_0 + b_{-1} \exp(-\xi)}. \quad (22)$$

Substituting Eq. (22) into Eq. (20), and by the help of Matlab, clearing the denominator and setting the coefficients of power terms like $\exp(j\xi)$, $j = 1, 2, \dots$ to zero yield a system of algebraic equations, solving the obtained system, we obtain the following exact solutions:

$$\begin{cases} a_0 = a_1 b_0 + \frac{3k^2 b_0}{a_1}, & a_{-1} = \frac{b_0^2 (3k^2 + 2a_1^2)}{8a_1}, \\ b_{-1} = \frac{b_0^2 (3k^2 + 2a_1^2)}{8a_1^2}, & \omega = -ka_1^2 - k^3, \end{cases} \quad (23)$$

where a_1 and b_0 are free parameters, which depends upon the initial conditions and/or boundary conditions. The property that stability may depend on initial/boundary conditions is characteristic only for nonlinear systems.

The relationship between wave speed and frequency is

$$\omega = -ka_1^2 - k^3. \quad (24)$$

Note that the value of a_1 is determined from the initial/boundary conditions, so frequency or wave speed may not independent of initial/boundary conditions.

Then, the closed form solution of Eq. (19) reads

$$\begin{aligned}
 u(x, t) &= \frac{\left(a_1 \exp [kx - (ka_1^2 + k^3)t] + a_1 b_0 + \frac{3k^2 b_0}{a_1} + \frac{b_0^2(3k^2 + 2a_1^2)}{8a_1} \exp [-kx + (ka_1^2 + k^3)t] \right)}{\left(\exp [kx - (ka_1^2 + k^3)t] + b_0 + \frac{b_0^2(3k^2 + 2a_1^2)}{8a_1^2} \exp [-kx + (ka_1^2 + k^3)t] \right)} \\
 &= a_1 + \frac{\frac{3k^2 b_0}{8}}{\left(\exp [kx - (ka_1^2 + k^3)t] + b_0 + \frac{b_0^2(3k^2 + 2a_1^2)}{8a_1^2} \exp [-kx + (ka_1^2 + k^3)t] \right)}
 \end{aligned} \tag{25}$$

Generally a_1, b_0 and k are real numbers, and the obtained solution, Eq. (25), is a generalized soliton solution.

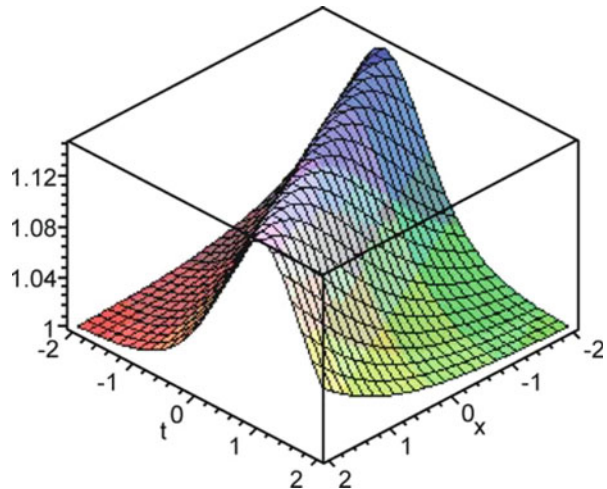
If we choose $k = 1, a_1 = 1, b_0 = \sqrt{8/5}$, Eq. (25) becomes

$$u(x, t) = 1 + \frac{3\sqrt{1/40}}{\exp [x - 2t] + \sqrt{8/5} + \exp [-x + 2t]}. \tag{26}$$

The bell-like solution is illustrated in Fig. 5.

Solitons and Compactons,

Fig. 5 Propagation of a solution with respect to time



In case k is an imaginary number, the obtained solitary solution can be converted into periodic

solution or compact-like solution. We write $k = iK$, Eq. (25) becomes

$$u(x, t) = a_1 + \frac{-\frac{3K^2 b_0}{8}}{\left((1+p) \exp [Kx - (Ka_1^2 - K^3)t] + b_0 + i(1-p) \frac{b_0^2(3k^2 + 2a_1^2)}{8a_1^2} \exp [-Kx + (Ka_1^2 - K^3)t] \right)}, \quad (27)$$

where $p = \frac{b_0^2(-3K^2 + 2a_1^2)}{8a_1^2}$.

If we search for a periodic solution or compact-like solution, the imaginary part in the denominator of Eq. (27) must be zero, that requires that

$$1 - p = 1 - \frac{b_0^2(-3K^2 + 2a_1^2)}{8a_1^2} = 0. \quad (28)$$

Solving b_0 from Eq. (28), we obtain

$$b_0 = \pm \sqrt{\frac{8}{-3K^2 + 2a_1^2}}. \quad (29)$$

Substituting Eq. (29) into Eq. (27) results in a periodic solution, which reads

$$u(x, t) = a_1 + \frac{\pm 3K^2 \sqrt{\frac{2}{-3K^2 + 2a_1^2}}}{\cos [Kx - (Ka_1^2 - K^3)t] \pm \sqrt{\frac{2}{-3K^2 + 2a_1^2}}} \quad (30)$$

or a generalized compact-like solution:

$$u(x, t) = \begin{cases} a_1 + \frac{\pm 3K^2 \sqrt{\frac{2}{-3K^2 + 2a_1^2}}}{\left(\cos [Kx - (Ka_1^2 - K^3)t] \pm \sqrt{\frac{2}{-3K^2 + 2a_1^2}} \right)}, \\ a_1 + 3K^2, \text{ otherwise} \end{cases} \quad (31)$$

$$|Kx - (Ka_1^2 - K^3)t| \leq \frac{\pi}{2}$$

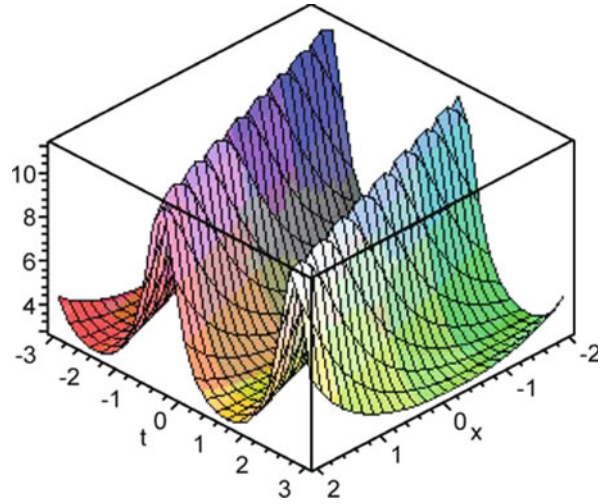
where a_1 and K are free parameters, and it requires that $2a_1^2 > 3K^2$. If we choose $k = 1$, $a_1 = 1$, $b_0 = \sqrt{8/5}$, Eq. (30) becomes

$$u(x, t) = 1 + \frac{3\sqrt{2}}{\cos [x - 3t] + \sqrt{2}}. \quad (32)$$

The periodic solution is illustrated in Fig. 6.

Solitons and Compactons,

Fig. 6 Periodic solution



Now we give an heuristical explanation of why Eq. (19) behaves sometimes periodically and sometimes compaton-like.

We re-written Eq. (20) in form

$$u'' + \frac{\omega}{k^3}u + \frac{1}{3k^2}u^3 = 0. \quad (33)$$

It is a well-known Duffing equation with a periodic solution for all $\omega > 0$ and $k > 0$.

Actually in our study ω can be negative, we re-write Eq. (33) in the form

$$u'' - \frac{\omega}{k^3}u + \frac{1}{3k^2}u^3 = 0, \quad \omega > 0. \quad (34)$$

This equation, however, has not always a periodic solution. We use the parameter-expansion method to find its period and the condition to be an oscillator. In order to carry out a straightforward expansion like that in the classical perturbation method, we need to introduce a parameter, λ , because none appear explicitly in this equation. To this end, we seek an expansion in the form

$$u = u_0 + \lambda u_1 + \lambda^2 u_2 + \lambda^3 u_3 + \dots \quad (35)$$

The parameter λ is used as a bookkeeping device and is set equal to unity.

The coefficients of the linear term and non-linear term can be, respectively, expanded in a similar way:

$$-\frac{\omega}{k^3} = \Omega^2 + m_1\lambda + m_2\lambda^2 + \dots \quad (36)$$

$$\frac{1}{3k^2} = n_1\lambda + n_2\lambda^2 + \dots, \quad (37)$$

where m_i and n_i are unknown constants to be further determined.

Interpretation of why such expansions work well is given by (He 2006a).

Substituting Eqs.(35)–(37) to (34), we have

$$\begin{aligned} & (u_0 + \lambda u_1 + \lambda^2 u_2 + \dots)'' \\ & + (\Omega^2 + m_1\lambda + m_2\lambda^2 + \dots)(u_0 + \lambda u_1 + \lambda^2 u_2 + \dots) \\ & + (n_1\lambda + n_2\lambda^2 + \dots) \cdot (u_0 + \lambda u_1 + \lambda^2 u_2 + \dots)^3 \\ & = 0 \end{aligned} \quad (38)$$

and equating coefficients of like powers of λ , we obtain **Coefficient of λ^0**

$$u_0'' + \Omega^2 u_0 = 0. \quad (39)$$

Coefficient of λ^1

$$u_1'' + \Omega^2 u_1 + m_1 u_0 + n_0 u_0^3 = 0. \quad (40)$$

The solution of Eq. (39) is

$$u_0 = A \cos \Omega t. \quad (41)$$

Substituting u_0 into (40) gives

$$u_1'' + \Omega^2 u_1 + A \left(m_1 + \frac{3}{4} n_0 A^2 \right) \cos \Omega t + \frac{1}{4} n_0 A^3 \cos 3\Omega t = 0. \quad (42)$$

No secular term in u_1 requires that

$$m_1 + \frac{3}{4} n_0 A^2 = 0 \text{ or } A = 0. \quad (43)$$

If the first-order approximate solution is searched for, then we have

$$-\frac{\omega}{k^3} = \Omega^2 + m_1 \quad (44)$$

$$\frac{1}{3k^2} = n_1. \quad (45)$$

We finally obtain the following relationship

$$\Omega^2 = -\frac{\omega}{k^3} + \frac{1}{4k^2} A^2. \quad (46)$$

To behave like an oscillator requires that

$$-\frac{\omega}{k^3} + \frac{1}{4k^2} A^2 > 0 \quad (47)$$

or

$$\frac{\omega}{k} < \frac{1}{4} A^2. \quad (48)$$

The amplitude A may strongly depend upon initial/boundary conditions which may determine the wave type of a nonlinear equation.

Now we approximate Eq. (34) in the form

$$u'' + \frac{1}{k^2} \left(-\frac{\omega}{k} + \frac{A^2 \cos^2 \Omega t}{3} \right) u = 0. \quad (49)$$

In case $|\Omega t| \rightarrow \pi/2$, the above equation behaves exponentially, resulting in a compact-like wave as discussed above.

Future Directions

It is interesting to identify the conditions for a nonlinear equation to have solitary, or periodic, or compacton-like solutions. In most open literature, many papers on soliton and compacton are focused themselves on a special solution with either a soliton or a compacton without considering the initial/boundary conditions, which might be vital important for its actual wave type.

Solitons and compactons for difference-differential equations (e. g. Lotka–Volterra-like problems) have been caught much attention due to the fact that discrete spacetime may be the most radical and logical viewpoint of reality (refer to E-infinity theory detailed concept). For small scales, e.g., nano scales, the continuum assumption becomes invalid, and difference equations have to be used for space variables.

Fractional differential model is another compromise between the discrete and the continuum, and can best describe solitons and compactons.

Many interesting phenomena arise in nanohydrodynamics recently, such as remarkably excellent thermal and electric conductivity, and extremely extraordinary fast flow in nanotubes. Consider a single compacton wave along a nanotube, and its wavelength is as same as the diameter of the nanotubes, under such a case, almost no energy is lost during the transportation, resulting in extremely extraordinary fast flow in the nanotubes.

The physical understanding of the transformation $k = iK$ is also worth further studying.

Cross-References

► [Soliton Perturbation](#)

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Solitons: Historical and Physical Introduction

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Article Outline

Glossary
Definition of the Subject
Introduction
Historical Discovery of Solitons
Physical Properties of Solitons and Associated Applications
Mathematical Methods Suitable for the Study of Solitons
Future Directions
Bibliography

Glossary

Breaking waves As waves increase in height through the shoaling process, the crest of the wave tends to speed up relative to the rest of the wave. Waves break when the speed of the crest exceeds the speed of the advance of the wave as a whole.

Crystal lattice A geometric arrangement of the points in space at which the atoms, molecules, or ions of a crystal occur.

Deep water Water sufficiently deep that surface waves are little affected by the ocean bottom. Water deeper than one-half the surface wavelength is considered deep water.

Fluxon Quantum of magnetic flux.

Freak wave Single waves which result from a local focusing of wave energy. They are of

considerable danger to mariners because of their unexpected nature.

Geostrophic adjustment The process by which an unbalanced atmospheric flow field is modified to geostrophic equilibrium, generally by a mutual adjustment of the atmospheric wind and pressure fields depending on the initial horizontal scale of the disturbance.

Geostrophic equilibrium A state of motion of an inviscid fluid in which the horizontal Coriolis force exactly balances the horizontal pressure force at all points of the field.

Hydraulic jump A sudden turbulent rise in water level, such as often occurs at the foot of a spillway when the velocity of rapidly flowing water is instantaneously slowed.

Katabatic wind Most widely used in mountain meteorology to denote a downslope flow driven by cooling at the slope surface during periods of light larger-scale winds.

Lightning Lightning is a transient, high-current electric discharge.

Plasma Hot, ionized gas.

Shallow water Water depths less than or equal to one-half of the wavelength of a wave.

Solitary wave Localized wave that propagates along one space direction only, with undeformed shape.

Soliton Large-amplitude pulse of permanent form whose shape and speed are not altered by collision with other solitary waves, the exact solution of a nonlinear equation.

Spillway A feature in a dam allowing excess water to pass without overtopping the dam.

Thermocline A layer in which the temperature decreases significantly (relative to the layers above and below) with depth.

Synoptic scale Used with respect to weather systems ranging in size from several hundred kilometers to several thousand kilometers.

Thunder The sound emitted by rapidly expanding gases along the channel of a lightning discharge.

Thunderstorm In general, a local storm invariably produced by a cumulonimbus cloud and always accompanied by lightning and thunder,

usually with strong gusts of wind, heavy rain, and sometimes with hail.

Tidal bore Tidal wave that propagates up a relatively shallow and sloping estuary or river, in a solitary wave form. The leading edge presents an abrupt rise in level, sometimes with continuous breaking and often immediately followed by several large undulations. The tidal bore is usually associated with high tidal range and a sharp narrowing and shoaling at the entrance. It is also called *pororoca* (Brazilian) and *mascaret* (French).

Troposphere The portion of the atmosphere from the earth's surface to the tropopause, that is, the lowest 10–20 km of the atmosphere.

Tsunami Long-period ocean wave generated by an earthquake or a volcanic explosion.

Definition of the Subject

The interest in nonlinear physics has grown significantly over the last 50 years. Although numerous nonlinear processes had been previously identified, the mathematic tools of nonlinear physics had not yet been developed. The available tools were linear, and nonlinearities were avoided or treated as perturbations of linear theories. The solitary water wave, experimentally discovered in 1834 by John Scott Russell, led to numerous discussions. This hump-shape localized wave that propagates along one space direction with undeformed shape has spectacular stability properties. John Scott Russell carried out many experiments to obtain the properties of this wave. The theories which were based on linear approaches concluded that this kind of wave could not exist. The controversy was resolved by J. Boussinesq (1871) and by Lord Rayleigh (1876) who showed that if dissipation is neglected, the increase in local wave velocity associated with finite amplitude is balanced by the decrease associated with dispersion, leading to a wave of permanent form. A model equation describing the unidirectional propagation of long waves in water of relatively shallow depth with a localized solution representing a single hump as discovered by

Russell was obtained by Korteweg and de Vries (1895). This equation has become very famous and is now known as the Korteweg-de Vries equation or KdV equation. These results were not considered very important when they were obtained. A remarkable numerical discovery was made by E. Fermi et al. (1955) as they studied the flow of incoherent energy in a solid modeled by a one-dimensional lattice of equal masses connected by nonlinear springs. The initial injected energy was not shared among all the degrees of freedom of the lattice, but returned almost entirely to the original excited mode. The explanation was given only 10 years later by Zabusky and Kruskal (1965) from numerical solutions of the KdV equation used to model in the continuum approximation a nonlinear atomic lattice with periodic boundary conditions. This led to the concept of solitons and ultimately to the development of integrable systems. The term soliton was chosen as it is a localized wave which propagates preserving its shape and velocity as a particle would propagate. A soliton is then a large-amplitude pulse of permanent form whose shape and speed are not altered by collision with other solitary waves, and it is the exact solution of a nonlinear equation. The mathematical features of solitons are often well developed in soliton literature since they are at the origin of very nice theoretical developments such as the powerful inverse scattering transform which solves a complex nonlinear equation through a series of linear steps. Nevertheless, the physics of solitons is very rich, and it is a very actual research topic in numerous fields, for example, in hydrodynamics, in optics, in electricity, in solid physics, in chemistry, and in biology. Nonlinearity and dispersion are very common in macroscopic physics as well as in microscopic physics. Nonlinearity tends to localize the signals, while dispersion spreads them. These two opposite effects sometimes compensate and the regime of solitons takes place. Real physical systems are only approximately described by the equations of the theory of solitons, but a remarkable feature of solitons is their very high stability relative to perturbations. The soliton concept is

now firmly established. They are widely accepted as a structural basis for viewing and understanding the dynamic behavior of complex nonlinear systems.

Introduction

The first experimental observation of a solitary wave was made in August 1834 by a Scottish engineer named John Scott Russell (1808–1882). He was mounted on a horse along the Union Canal linking Edinburgh with Glasgow when he saw a rounded smooth well-defined heap detach itself from the prow of a boat brought to rest and continue its course without change of shape or diminution of speed. Scott Russell reported his observation in the British Association Report (Russell 1844) as follows:

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

Russell wrote that this August day in 1834 was the happiest in his life. In July 1995, an international gathering of scientists witnessed a recreation of the solitary wave observed by Russell on an aqueduct of the Union Canal. This aqueduct was named after this occasion *Scott Russell aqueduct*; the scientists were attending a conference on nonlinear waves in physics and biology at Heriot-Watt University (Edinburgh), near the canal.

Throughout his life, Russell remained convinced that “his” wave that he initially called the “wave of translation” and later the “great solitary wave” was

of fundamental importance. He therefore carried out extensive experiments in wave tanks and established the velocity formula for solitary waves:

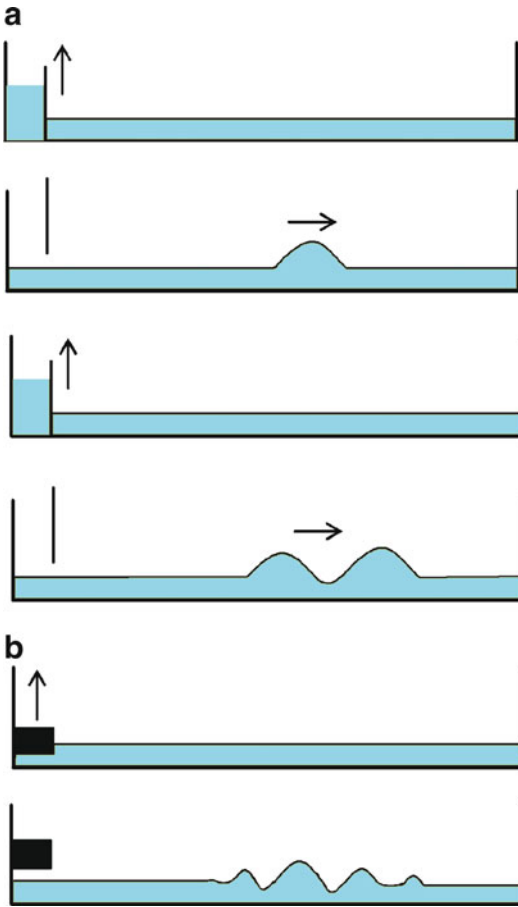
$$V_s = \sqrt{g(d + A_s)} \quad (1)$$

where g is the acceleration due to gravity, d the undisturbed water depth, and A_s the amplitude of the solitary wave. The “quasi-cycloidal” form of the wave was found to be independent of the way it was generated. Russell (1837) described the induced movement of fluid particles: “By the transit of the wave the particles of the fluid are raised from their places, transferred forwards in the direction of the motion of the wave, and permanently deposited at rest in a new place at a considerable distance from their original position.” He also deduced from his numerous experiments that two solitary waves can cross each other “without change of any kind.” Figure 1 shows a schematic view of his experiments in tanks, adapted from Remoissenet (1999). An initial elevation of water may induce one or two solitary waves depending on the relation between its height and length (Fig. 1a); the case where an initial depression does not evolve into solitary waves but leads to an oscillatory wave train of gradually increasing length and decreasing amplitude is shown in Fig. 1b.

In section “[Historical Discovery of Solitons](#),” the historical discovery of solitons is presented. The physical concept of solitons and the associated applications are described in section “[Physical Properties of Solitons and Associated Applications](#).” In section “[Mathematical Methods Suitable for the Study of Solitons](#),” the mathematical methods suitable for the study of solitons are considered. Finally, section “[Future Directions](#)” is devoted to future directions.

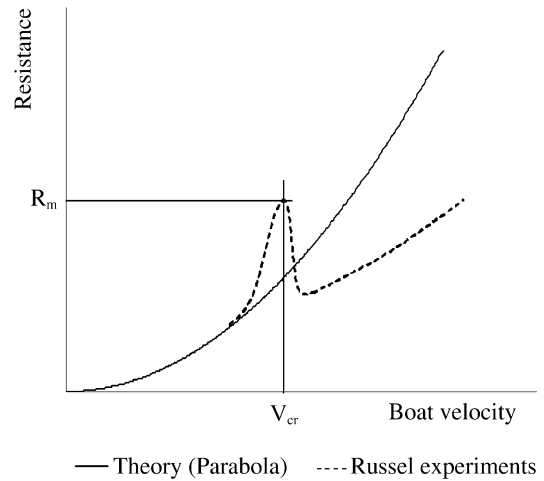
Historical Discovery of Solitons

After Russell had carried out his observation of the solitary wave in August 1834, he was put in charge by the Union Canal Company of determining the efficiency of canals for the transport of steam-driven barges. He worked on the resistance of boats towed through water of finite depth.



Solitons: Historical and Physical Introduction, Fig. 1 Schematic view of Scott Russell's experiments

A resistance proportional to the square of the velocity V_b of the boat was predicted by the theory at that time. Convinced of the imperfection of this theory, Russell performed numerous experiments in the summers of 1834 and 1835 (Darrigol 2003). Several boats were towed in canals of various depths at velocities ranging from 3 to 15 miles per hour. The resistance was measured by a dynamometer. Resistance curves such as the one depicted in Fig. 2 were obtained, with a resistance local maximum R_m for a boat velocity V_{cr} and a growing deviation from the theoretical curve for increasing values of boat velocity. The critical velocity V_{cr} was found to depend on the water depth. Russell noticed that this critical velocity was the same as the velocity of the solitary wave for the given water depth d and gave a qualitative



Solitons: Historical and Physical Introduction, Fig. 2 Resistance as a function of towing velocity according to Russell

explanation for the diminution of flow resistance for increasing values of the boat velocity just above the critical velocity V_{cr} . For this description, he considered the variation of the shape of the water surface around the moving boat with the velocity. For velocities smaller than the critical value, the water level is raised around the prow where a wave of displacement is generated, leading to an inclination of the boat. The resulting effect is an augmentation of the section of immersion and of the corresponding flow resistance. When the velocity of the boat reaches the critical value, the wave of displacement has the velocity of a solitary wave, and the vessel travels on this solitary wave. For velocities greater than the critical value, the boat stays on the wave summit leading to a smaller flow resistance. Russell (1839) tried a quantitative explanation of this process, but this entry contained several errors. However, Russell was a very fine observer; he was far ahead of his time in considering the fundamental importance of solitary waves. He mentioned (Russell 1839) that “a large or high wave had a greater velocity than a small one. When a small wave preceded a large one, the latter invariably overtook the other, and when the large wave was before the less, their mutual distance invariably became greater.” He showed that the shape of solitary waves depends on their height, the width

of the wave decreasing for increasing values of height. It is also fascinating that Russell had observed the collision properties of solitary waves (Bullough 1988). Nevertheless, the observations of Russell induced numerous critical comments. Airy (1845) wrote in his treatise “Tides and Waves”: “We are not disposed to recognize this wave (discovered by Scott Russell) as deserving the epithets ‘great’ or ‘primary,’ and we conceive that ever since it was known that the theory of shallow waves of great length was contained in the equation $\partial^2 X / \partial t^2 = gd \partial^2 X / \partial x^2$ the theory of the solitary wave has been perfectly well known.” In the equation quoted by Airy, t is the time and x is the horizontal coordinate of a fluid particle. The wave velocity in shallow water at the first order of approximation, when the ratio H/d between the wave height H and the water depth may be neglected, is $V_0 = \sqrt{gd}$, as found by Lagrange in 1786; this velocity differs from the formulation proposed by Russell for solitary waves (Eq. 1). Airy mentioned further: “Some experiments were made by Mr. Russell on what he calls a negative wave. But (we know not why) he appears not to have been satisfied with these experiments and had omitted them in his abstract. All of the theorems of our IVth section, without exception, apply to these as well as to positive waves, the sign of the coefficient only being changed.” Also, according to Airy (1845) and Lamb (1879), long waves could not propagate in a canal of rectangular section without changing their shape. The controversy arose because the dispersion is neglected by the nonlinear shallow water theory. Stokes (1847) considered Russell results and tried to obtain them analytically; however, his linear or weakly nonlinear approach did not permit him to retrieve the results of Russell, which made him doubtful about them. In order to distinguish his “great wave” from other waves, Russell (1844) introduced four orders of waves:

1. Waves of translation. These waves involve mass transfer. The solitary waves observed by Russell are included in this order.
2. Oscillatory waves. These are the waves which can be the most often observed; they do not involve mass transfer.

3. Capillary waves. The surface tension effects are important for those waves.
4. Corpuscular waves. Those waves are rapid successions of solitary waves.

Russell was most concerned with the first order, but he also carried out experiments with waves of the second and third order. The fourth order (corpuscular waves) suggested by Russell had been seriously criticized, and the manuscripts he submitted on this order were never published due to ignorance of mechanics principles.

The great difficulties encountered by Russell to prove the importance of his findings strongly disappointed him. Tired of discussions, he stopped his work on solitary waves and started to build large steam ships with great success. A detailed biography of Russell may be found in Emerson (1977).

A French mathematician, Joseph Boussinesq, knew of the existence of Russell’s observations and also of detailed experiments performed by Bazin (1865) in the long branch of the Canal de Bourgogne close to Dijon (France), which confirmed Russell’s observations. Boussinesq tried to obtain a solution of Euler’s equations compatible with Russell’s and Bazin’s results. He developed the horizontal and vertical velocity components of fluid velocity in a rectangular channel in power of the distance from the bed. Including nonlinear terms which had been neglected by Lagrange, Boussinesq (1871) obtained a solution with the properties of the solitary wave observed by Russell. In particular, the shape of the wave was found to be given by

$$\eta = A_S \operatorname{sech}^2 \left[\sqrt{\frac{3A_S}{4d^3}} (x - V_S t) \right] \quad (2)$$

where η is the free surface displacement and $\operatorname{sech} x = 1 / \cosh x$; the velocity of the wave V_S is given by Eq. 1. The article of Boussinesq (1871) had been almost ignored in England, and 5 years after its publication, Lord Rayleigh (1876) independently obtained the solitary wave profile. Following Rayleigh’s work and including the effects of surface tension, the Amsterdam professor of

mathematics Diederik Johannes Korteweg and his doctoral student Gustav de Vries derived a model equation (Korteweg and de Vries 1895) which describes the unidirectional propagation of long waves in water of relatively shallow depth. This famous equation is now known as the Korteweg-de Vries equation or KdV equation, and it has the following form:

$$\frac{\partial \eta}{\partial t} + V_0 \frac{\partial \eta}{\partial x} + \frac{3}{2} \frac{V_0}{d} \eta \frac{\partial \eta}{\partial x} + \frac{1}{6} V_0 d^2 \frac{\partial^3 \eta}{\partial x^3} = 0. \quad (3)$$

Korteweg and de Vries showed that this nonlinear wave equation has a localized solution which represents a single hump of positive elevation corresponding to the solitary wave observed by Scott Russell. The KdV equation had been in fact formulated in 1877 by Boussinesq in an impressive paper (Boussinesq 1877), but this author did not use directly this formulation. Russell's experiments had found their theory, and one could think that many scientists would rapidly extend the results of this theory. However, the KdV equation had a quiet life for many decades.

In the early 1950s, the physicists Enrico Fermi, John Pasta, and Stan Ulam could use one of the first computers, the "Maniac," to work on systems without closed analytic solutions. In particular, they studied the way a crystal evolves toward the thermal equilibrium through the numerical simulation of the dynamics of a one-dimensional lattice consisting of N equal masses (atoms) connected by nonlinear springs. The problem is described by the Hamiltonian:

$$H = \sum_{i=0}^{N-1} \frac{1}{2} p_i^2 + \sum_{i=0}^{N-1} \frac{1}{2} K (x_{i+1} - x_i)^2 + \frac{K\alpha}{3} \times \sum_{i=0}^{N-1} (x_{i+1} - x_i)^3 \quad (4)$$

where x_i is the atom i displacement, p_i the corresponding momentum, K the constant of the quadratic potential, and $\alpha (\ll 1)$ a coefficient of nonlinear interaction. The boundary condition $x_0 = x_N = 0$ is assumed. Considering a normal mode decomposition

$$A_k = \sqrt{\frac{2}{N}} \sum_{i=0}^{N-1} x_i \sin(ik\pi/N), \quad (5)$$

the Hamiltonian (Eq. 4) may be written in the following way:

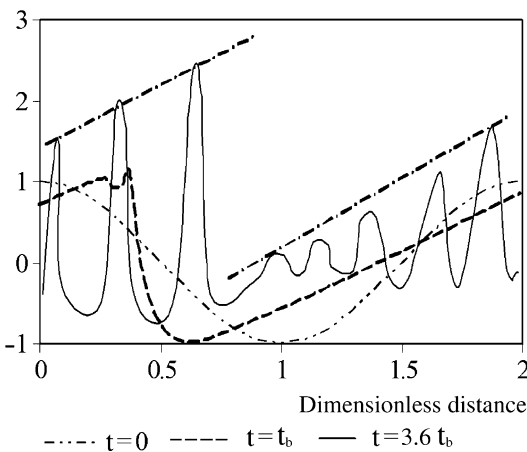
$$H = \frac{1}{2} \sum_k \left(\dot{A}_k^2 + \omega_k^2 A_k^2 \right) + \alpha \sum_{k,l,m} c_{klm} A_k A_l A_m \quad (6)$$

where the pulsations ω_k are given by $\omega_k^2 = 4K \sin^2(k\pi/2N)$ and c_{klm} are constants. When the masses have very small displacements from the equilibrium positions, the dynamics of the lattice may be described by a superposition of normal modes. It was supposed that if only one mode (the fundamental mode $k = 1$ in the circumstances) is excited in a crystal lattice, nonlinear interactions would lead to energy equipartition or thermal equilibrium. This was suggested by the results at the beginning of the calculations since the modes 2, 3, ... were successively excited. However, after 157 fundamental periods, almost all the energy returned to the fundamental mode. Furthermore, the energy returned almost periodically to the original excited mode leading to a near recurrence process, now known as the FPU paradox. Fermi et al. (1955) did not get the expected result, and the modeled crystal lattice did not approach energy equipartition. This remarkable discovery is at the origin of the comprehension of the concept of the soliton and of the developments of "numerical experiments." Fermi died just after this work; this had a rather inhibitory effect on the significance of this result, and no paper was published after the preprint "Studies of nonlinear problems" (Fermi et al. 1955). The results were communicated to the scientific community by Ulam through several conferences. Complementary studies (Ford 1961; Jackson 1963) confirmed that the introduction of nonlinearity in a system does not guarantee the equipartition of energy. In 1960, the KdV equation was rederived by Gardner and Morikawa who worked on collisionless hydromagnetic waves. These authors attracted the attention of Zabusky and Kruskal on the KdV equation, and the FPU

paradox was solved a few years later in terms of solitons (Zabusky and Kruskal 1965). Zabusky and Kruskal considered the following motion equations corresponding to the Hamiltonian (Eq. 4):

$$\ddot{x}_i = K(x_{i+1} + x_{i-1} - 2x_i) + K\alpha \left[(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2 \right]. \quad (7)$$

Taking into account that the recurrence occurs before the excitation of high-order modes, these authors carried out an asymptotic analysis in the continuum approximation and obtained an equation including dispersive and nonlinear terms which correspond to the KdV equation. This equation, obtained in a totally different physical context as the one to explain the observations of Scott Russell, is at the origin of the explanation of the FPU paradox. As depicted in Fig. 3, the numerical experiments of Zabusky and Kruskal showed that the initial sinusoidal condition evolves into step fronts and then into a finite number of short pulses which are solitons. These solitons travel around the lattice with velocities depending on their amplitude; they collide preserving their individual shapes and velocities with only a small change in their phases. As time increases, there is an instant at which the solitons collide at the same point, and the initial state comes close to recurrence. The



Solitons: Historical and Physical Introduction, Fig. 3 Evolution of an initially periodic profile from Zabusky and Kruskal numerical analysis of the KdV equation. The breaking time for the wave profile is t_b

recurrence period calculated by Zabusky and Kruskal closely approximates the actual FPU recurrence period, showing that the KdV equation is a suitable approximation of the FPU system. The system had to be considered as totally nonlinear to be solved, and the linear normal modes of the system could not be considered. The pulse-like waves were called solitons by Zabusky and Kruskal to indicate the remarkable quasi-particle properties of these very stable solitary waves. They were at first called solitons to introduce them into the family of particle names such as electron and neutron, but the name soliton was a trademark and was therefore not suitable. The concept of solitons has spread over numerous branches of physics, such as optical physics, biological physics, astronomy, particle physics, condensed matter, fluid dynamics, and ferromagnetism. The numerical experiments of Zabusky and Kruskal led to the development of integrable systems. Taniuti and Wei (1968) and Su and Gardner (1969) have shown that a large class of nonlinear evolution equations may be reduced to consideration of the KdV equation which may be regarded as a very important canonical form. A famous method, the so-called inverse scattering transform (IST), was proposed by Gardner et al. (1967) to integrate the KdV equation. This procedure may be considered as the nonlinear analogue of the Fourier transform method suitable to linear dispersive systems.

Physical Properties of Solitons and Associated Applications

Properties of Solitons

Let us consider the KdV equation (Eq. 3). Introducing the variable $X = x - V_0 t$, this equation may be written in the following way:

$$\frac{1}{V_0} \frac{\partial \eta}{\partial t} + \frac{3}{2d} \eta \frac{\partial \eta}{\partial X} + \frac{d^2}{6} \frac{\partial^3 \eta}{\partial X^3} = 0. \quad (8)$$

Using the dimensionless variables $\phi = \eta/d$, $\xi = X/X_0$, and $\tau = t/t_0$ where X_0 and t_0 are, respectively, a typical length and time, the KdV equation may be formulated in its standard form:

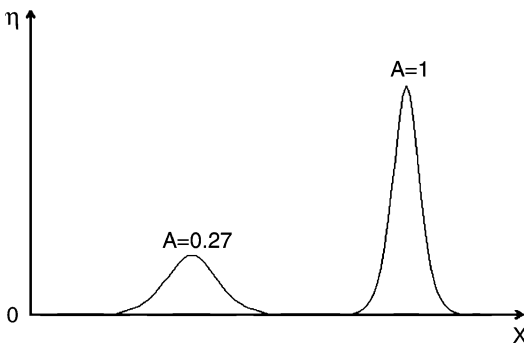
$$\frac{\partial \phi}{\partial \tau} + 6\phi \frac{\partial \phi}{\partial \xi} + \frac{\partial^3 \phi}{\partial \xi^3} = 0. \quad (9)$$

This equation has many solutions, as all non-linear equations. Among these solutions, there are the spatially localized solutions:

$$\phi = A \operatorname{sech}^2 \left[\sqrt{\frac{A}{2}} (\xi - 2A\tau) \right], \quad (10)$$

where A is a positive constant. These solutions quantitatively correspond to the observations of Scott Russell. In particular, the width of the soliton decreases for increasing values of its amplitude, as depicted in Fig. 4. In dimensional variables and in a fixed referential, the corresponding solution is given by Eq. 2.

The KdV equation appears widely in physics, each time that waves propagate in a weakly non-linear and weakly dispersive medium. The non-linear term $\phi(\partial\phi/\partial\xi)$ in this equation causes the steepness of the wave form. In a dispersive medium, the Fourier components of a pulse propagate at different velocities, inducing a spreading of the pulse. The dispersive term $\partial^3\phi/\partial\xi^3$ in the KdV equation makes the wave form spread. The soliton solution of the KdV equation results from the balance of nonlinearity and dispersion. This balance is generally very stable, which explains the numerous applications of the theory of solitons, even if the real physical situations (where friction and defects take place) are only approximately described by the



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Fig. 4 Comparison between two solutions of the KdV equation with different amplitude A

soliton equations and in particular by the KdV equation. It would be strictly speaking more appropriate to use the terminology quasi-soliton in physics; however, bearing in mind that the word soliton has a quite ideal connotation in the context of systems with exact solutions, we will use the word soliton. The conservation of the shape and of the velocity of the soliton after a collision is a manifestation of stability. The high stability of solitons relative to perturbations, combined with the fact that they can spontaneously emerge in a physical system in which energy is supplied, leads to numerous applications of solitons in macroscopic and microscopic physics. Nevertheless, a soliton may sometimes vanish. For example, this is the case when such a wave propagates in a media where the water depth decreases during the wave propagation. The dispersive effect progressively decreases when the nonlinear effect increases, leading to the wave breaking as waves on a beach. The vanishing of a soliton may also result from energy dissipation. Marin et al. (2005) have shown that sand ripples may form when solitons propagate in shallow water over a sandy bed and that a strong interaction between the free surface and the bed occurs. This interaction leads to a decrease of the soliton amplitude, which may result under certain circumstances in the disappearance of the soliton. Let us now consider some of the most typical soliton applications.

Solitons in Fluid Mechanics

In the hydrodynamic field, a very nice case where solitons take place is the tidal bore, also called hydraulic jump in translation and known as the mascaret in France. A tidal bore is a positive wave of translation which can occur in a river with a gently sloping bottom and a broad funnel-shaped estuary, where a difference of more than 6 m between high and low water takes place (Lynch 1982). The tidal bore appears as a single wave or a series of waves which propagate upstream. At a given instant, it is localized at the upstream extremity of the flood tide propagation in the intertidal zone. The flow direction is toward the sea before the tidal bore arrival and may be in the opposite direction after its occurrence. Such bores are distributed widely throughout the world. The

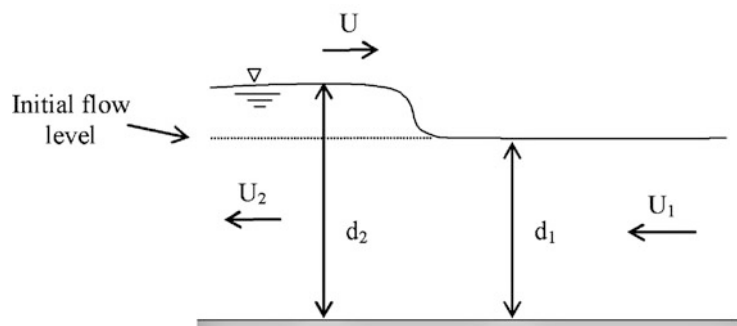
most powerful can be seen in the Amazon River in South America, where it is called *pororoca*, which means gigantic noise. An approximately 5-m-high wave propagates with a velocity of about 30 m/s, and surfers take advantage of this exceptional wave to organize competitions. In England, bores may be contemplated on several rivers, the most famous occurring in the Severn River, near Bristol. In France, bores take place mainly in the Mont Saint-Michel Bay, in the Gironde, Dordogne, and Garonne rivers. The mascaret of the Seine River almost disappeared between 1960 and 1970 because of the construction of dikes and the dredging works which had been carried out in order for large boats to reach the city of Rouen. Increasing the water depth induces a decrease of the nonlinear effects which become insufficient to balance the dispersion. In Canada, several bores are observed in Fundy Bay. In Asia, an important bore occurs in the Qiantang River, while the one in the Pungue River in Africa can propagate upstream over more than 80 km. Two types of bores may be observed: the breaking bore and the undulated bore, depending on the value of the Froude number F_r , defined by $F_r = (U_1 + U) / \sqrt{gd_1}$, where U_1 is the flow velocity at the depth d_1 before the passage of the bore and U the bore velocity (Fig. 5). In Fig. 5, d_2 depicts the water depth after the passage of the bore and U_2 the flow velocity at the depth d_2 . For a value of F_r lower than about 1.5, the bore is undulated, and for a value of F_r greater than 1.5, the breaking bore occurs. Most of the bores are undulated bores. The tidal bore is a very important turbulent phenomenon for the intertidal zone of a river, inducing a significant mixing of waters and complex

motions of sediment on the riverbed (Donnelly and Chanson 2002). River bores have been known for many centuries, but only qualitative observations were carried out. Quantitative measurements came much later, mainly because solitons are inherently nonlinear when only linear processes were considered.

Long waves such as tsunamis often behave like solitary waves (Liu et al. 1991). In particular, the run-up and shoreward inundation are often simulated using solitary waves (Lo and Shao 2002). The tsunamis may be described by the KdV equation. The theory of very long shallow water waves is valid since the tsunamis have a wavelength which is generally greater than 100 km and they propagate in oceans whose mean depth is about 4000 m. Their propagation velocity is greater than $V_0 = \sqrt{gd}$, that is, greater than approximately 700 km/h. When they approach a coast, their velocity decreases and their height increases to satisfy the equation of energy conservation, and they finally break inducing often very significant damages on the shore. Most tsunamis have a seismic origin, such as the dramatic one which occurred in the Indian Ocean on 26 December 2004 and which led to the death of more than 220,000 people. These solitons may also result from a quick arrival in the sea of volcanic products, through a similar process to that in Scott Russell's experiments depicted in Fig. 1. This was the case for the tsunami generated in 1883 by the eruption of Krakatau in Indonesia, when approximately 36,000 people died. The inundation phase for a tsunami highly depends on the bottom bathymetry. The breaking can be progressive and structural damages are mainly caused by

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Fig. 5 Sketch of the propagation of a positive wave of translation



inundation (Dias and Dutykh 2007). The breaking can also be explosive and induce the formation of a plunging jet. When the breaking takes place in very shallow water, the tsunami amplitude becomes so high that an undulation appears on the long wave, which develops into a progressive bore (Chanson 2005). This turbulent front is of the same type as the wave which occurs when a dam breaks, and the water can rise very rapidly, typically from 0 to 3 m in 1.5 min. The event of December 2004 has shown that the available models are not able to predict accurately the wave run-up heights under severe conditions.

Helal and El-Eissa (1996) presented the connection between shallow water waves and the KdV equation. More recently, Mei and Li (2004) considered the propagation of long waves over a randomly rough seabed. These authors have shown analytically and numerically that, assuming a randomness height-to-mean depth ratio comparable to the one of mean depth to the characteristic wavelength, disorder causes diffusion, leading to spatial attenuation of the wave amplitude. Mei and Li (2004) proposed a modified KdV equation which includes terms representing the effects of disorder on amplitude attenuation. After a propagation over a region of finite length, the transmitted wave is a pulse which disintegrates into several small solitons of vanishing energy.

Hydrodynamic solitons may also be observed in deep water. Stokes showed in 1847 through the use of a small amplitude expansion of a sinusoidal wave that periodic waves of finite amplitude are possible in deep water. The Stokes waves are unstable to infinitesimal modulation perturbations (Benjamin and Feir 1967). Zakharov and Shabat (1972) were the first to show that an initial wave packet may evolve into a number of envelope solitons and a dispersive tail, the envelope solitons consisting of a carrier wave modulated by an envelope signal. This was experimentally verified by Yuen and Lake (1975), by carrying out detailed experiments in deep water in a wave tank. The stability properties of envelope solitons were also considered. The freak waves, also called rogue waves, extreme waves, or giant waves, may also have soliton-like shape. These waves appear

surprisingly in the sea (“wave from nowhere”) in deep or shallow water and can cause severe damages to ships and fixed ocean structures. They are characterized by their brief occurrence in space and time, resulting from a local focusing of wave energy (Fochesato et al. 2007). The main features of their physical processes may be obtained by performing numerical simulations in the framework of classical nonlinear evolution equations, such as the KdV equation (Kharif and Pelinovsky 2003).

Oceanographers have also observed internal solitary waves in many regions around the world’s oceans (Ostrovsky and Stepanyants 1989). Most of the observed solitons propagate at the interface between the thermocline and the deep sea (Michallet and Barthélemy 1998). They are mainly excited by tidal flows over bottom topography (Gerkema and Zimmerman 1994), and they contribute to significant vertical mixing. Brandt et al. (1997) developed a rotationless Boussinesq-like model for the generation and propagation of internal waves; the results of the simulations are in good agreement with large solitary waves recorded with airborne radar images. Michallet and Barthélemy (1998) carried out experiments in a 3-m-long flume to study interfacial long waves in a two-immiscible fluid system involving water and petrol. The comparison between the experiments and nonlinear theories shows that the KdV solitary waves match the experiments for small amplitude waves for all layer thickness ratios. When the crest is close to the critical level, which is approximately located at mid-depth, the large-amplitude waves are well predicted by a modified KdV equation including both quadratic and cubic nonlinear terms (KdV-mKdV equation). However, only very few laboratory or field measurements of internal solitary waves are available, due to the prohibitive cost of obtaining precise measurements in the field and to the difficulty of generating and measuring these waves in the laboratory.

Blood pressure pulses may be considered as solitons (Peyrard and Dauxois 2004). There is a balance between the nonlinearity due to the blood flow and the dispersion connected to the elasticity of the artery. Previous studies (Hashizume 1985,

1988; Yomosa 1987) have shown that the dynamics of weakly nonlinear pressure waves in a thin nonlinear elastic tube filled with an incompressible fluid may be governed by the KdV equation. A Boussinesq-like equation was obtained by Paquerot and Remoissenet (1994) to describe the blood pressure propagation in large arteries, in the limit of an ideal fluid and for slowly varying arterial parameters.

Several methods have been tested to generate hydrodynamic solitons in a flume; the method used by Scott Russell has been described in section “Introduction” (Fig. 1). Hammack and Segur (1974) showed experimentally and theoretically that from any net positive volume of water above the still water level, at least one solitary wave will emerge followed by a train of dispersive waves. Maxworthy (1976) and Weidman and Maxworthy (1978) pushed the liquid in the horizontal direction by a single displacement of a piston. The generation of solitary waves using a piston-type wave maker has been studied in detail by Goring (1978). Guizien and Barthélemy (2002) have proposed an experimental procedure to generate solitary waves in a flume using a piston-type wave maker from Rayleigh’s (1876) solitary wave solution. The advantage of this method is that solitary waves are rapidly established with very little loss of amplitude in the initial stage of the propagation of the solitary waves. The generation of solitons using a wave flume in resonant mode, without an absorbing beach, has been considered by Marin et al. (2005). Surface waves are produced in shallow water by an oscillating paddle at one end of the flume; a near-perfect reflection takes place at the other end. The frequency of the oscillating paddle was chosen close to the resonant frequency of the mode whose wavelength is equal to the flume length. For small values of the amplitude of displacement of the oscillating paddle, only standing harmonic waves are generated in the flume. For values of this amplitude greater than a critical value, pulses propagating from one end of the channel to the other end are excited on the background of the standing harmonic wave. From one to four pulses, propagating in each direction of the flume could be generated over the time period of the flow, depending on the frequency

and the amplitude of horizontal displacement of the oscillating paddle. These pulses were identified as solitons (one pulse) and bound states of solitons (several pulses). The generation of solitons in the flume results from the excitation of high harmonics. The spatiotemporal properties of solitons generated in this way were studied in detail by Ezersky et al. (2006). Space-time diagrams have been constructed to highlight the spatiotemporal dynamics of nonlinear fields for two solitons colliding in the resonator. Period doublings and the multistability of the nonlinear waves, i.e., the occurrence of different regimes for the same values of the control parameters but under different initial conditions, have been shown. It is important to keep in mind that the solitary waves correspond to the limit of cnoidal waves of period tending to infinity. The velocity of cnoidal waves V_{cn} is known to depend on the so-called elliptic parameter m and is given by the formula (11):

$$V_{cn} = \left[1 - \frac{A_s}{2d} + \frac{A_s}{md} \left(1 - \frac{3E(m)}{2K(m)} \right) \right] \sqrt{gd} \quad (11)$$

where $K(m)$ and $E(m)$ are complete elliptic functions of the first and second kind. The parameter m , which depends on the wave amplitude-to-width ratio, is responsible for the shape of the wave with $m = 0$ corresponding to the harmonic wave and $m = 1$ to the soliton. Strictly speaking, the waves generated in a flume used in resonant mode are not exactly solitons, but cnoidal waves with an elliptic parameter very close to 1; the value of this parameter was found to be 0.9996 for the tests carried out by Ezersky et al. (2006). The shapes of pulses are almost indistinguishable for $m = 1$ and $m = 0.9996$, but the pulse repetition rates differ appreciably, the period tending to infinity as $m \rightarrow 1$.

Atmospheric solitons also exist, such as the Morning Glory Cloud of the Gulf of Carpentaria in northern Australia, which can be seen as a spectacular propagating roll cloud. Atmospheric solitons are horizontally propagating nonlinear internal gravity waves that can travel over large distances with minimal change in form (Rottman

and Grimshaw 2001). The solitary waves that have been observed in the atmosphere can be divided into two classes: those that propagate in a fairly shallow stratified layer near the ground and those that occupy the entire troposphere. The generation mechanisms appear to be quite different for these two classes of waves. The waves that occupy the lower part of the troposphere mainly involve a gravity current such as a thunderstorm outflow, a katabatic wind, a sea breeze front, or a downslope windstorm, interacting with a low-level stable layer. The motion of the gravity current produces perturbations that are trapped in the low-level stable layer and eventually evolve over time into a series of solitary waves. For the tropospheric scale waves, they are very probably generated by synoptic scale features such as large-scale convective systems and geostrophic adjustments. For the two types of atmospheric solitons, there must exist some feature in the atmosphere that serves to prevent the wave energy from propagating away in the vertical direction. These trapping mechanisms are either deep layers of the atmosphere with very low values of the buoyancy frequency or critical layers. The KdV equation, eventually enhanced with higher-order nonlinearity, gives a reasonable agreement with the observations (Rottman and Grimshaw 2001).

Solitons in Nonlinear Transmission Lines

Let us consider the elementary LC network depicted in Fig. 6, with linear inductors L and nonlinear capacitors C . The differential capacitance $C(V_n)$ is supposed to depend nonlinearly on the voltage across the n th capacitor V_n :

$$C(V_n) = \frac{dQ_n(V_n)}{dV_n} \quad (12)$$

where $Q_n(V_n)$ is the charge stored in the n th capacitor. Following Kirchhoff's law, we have

$$V_{n-1} - V_n = \frac{d\phi_n}{dt}, \quad I_n - I_{n+1} = \frac{dQ_n}{dt} \quad (13)$$

where the magnetic flux ϕ_n is related to the current I_n by the relation: $\phi_n = LI_n$. Using Eqs. 12 and 13, it is easy to obtain

$$\frac{d^2 Q_n}{dt^2} = \frac{1}{L} (V_{n+1} + V_{n-1} - 2V_n), \quad n = 1, 2, \dots \quad (14)$$

Assuming the following capacitance-voltage relation,

$$C(V_n) = C_0(1 - 2aV_n) \quad (15)$$

where a is a small nonlinear coefficient, we find

$$LC_0 \frac{d^2 V_n}{dt^2} - LC_0 a \frac{d^2 V_n^2}{dt^2} = (V_{n+1} + V_{n-1} - 2V_n), \quad n = 1, 2, \dots \quad (16)$$

The system (Eq. 16) of nonlinear equations cannot be solved analytically. The continuum limit is used to get approximate solutions; setting the position $x = n\delta$ where δ is a hypothetical cellule length, we get

$$\frac{\partial^2 V}{\partial t^2} - \frac{\delta^2}{LC_0} \frac{\partial^2 V}{\partial x^2} = \frac{\delta^4}{12LC_0} \frac{\partial V}{\partial x^4} + a \frac{\partial^2 V^2}{\partial t^2}. \quad (17)$$

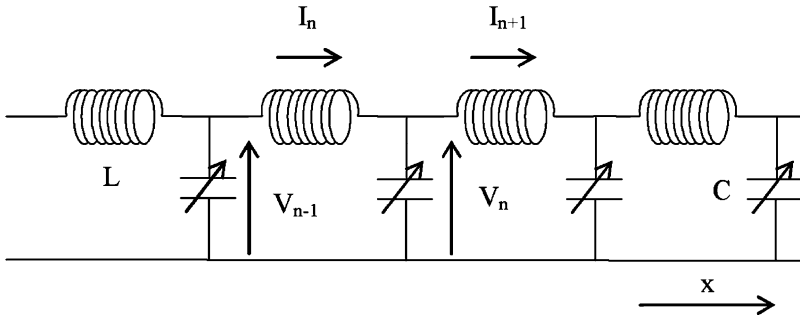
This weakly dispersive and nonlinear wave equation describes waves that can travel both to the left and to the right; the dispersive nature is due to the discreteness of the electrical network. A localized wave solution of Eq. 17 that does not change its shape as it propagates with constant velocity v may be found (Remoissenet 1999):

$$V = \frac{3(v^2 - v_0^2)}{2av^2} \operatorname{sech}^2 \left(\frac{\sqrt{3(v^2 - v_0^2)}}{v_0} \left(n - \frac{v}{\delta} t \right) \right) \quad (18)$$

where $v_0 = \delta/\sqrt{LC}$. This solitary wave solution represents a pulse with amplitude:

$$V_m = \frac{3}{2a} \frac{v^2 - v_0^2}{v^2} \quad (19)$$

which depends on the velocity v . The width L_w at half height of this pulse is given by



Solitons: Historical and Physical Introduction, Fig. 6 Elementary circuit of an electrical network with linear inductance's L and nonlinear capacitance's C . In another area of physics, transmission lines with nonlinear elements are found to propagate in a soliton mode. Non-linear electrical transmission lines are simple experimental

devices to observe and consider quantitatively the propagation and properties of solitons. The propagation of these waves is used for picosecond impulse generation and broadband millimeter-wave frequency multiplication (Rodwell et al. 1994)

$$L_w = 1.76 \frac{v_0}{\sqrt{3(v^2 - v_0^2)}}. \quad (20)$$

The wave forms of this solitary wave and of the KdV soliton are similar, and the width L_w depends on the amplitude V_m :

$$V_m L_w^2 = C^t \quad (21)$$

where C^t is a constant.

Solitons in Plasmas

Solitons may propagate in plasmas; this has received much attention because of a possible relevance to final state configurations in fusion devices. The combination of dispersion and non-linearity in plasmas may be described by the KdV equation. A direct analogy between the equations for shallow water and those for plasmas has been mentioned by Gardner and Morikawa (1960). However, the KdV equation is not always adapted to the excitations which can be generated in plasmas by electromagnetic waves, since wave packets are often produced instead of pulses characterizing the solutions of the KdV equation. Another type of soliton equation is then more suitable, the nonlinear Schrödinger equation (NLS); this second class of soliton equation will be considered in subsection “Solitons in Optical Fibers.”

Solitons in a Chain of Pendulums

Hydrodynamic solitons, solitons in nonlinear transmission lines, are nontopological solitons, because the system returns to its initial state after the passage of the wave. There is also another type of soliton, the kink solitons which are topological solitons, since the structure of the system is sometimes modified after the passage of the wave. A mechanical system consisting of a chain of pendulums, each pendulum being elastically connected to its neighbors by springs, is a typical example for which topological solitons can merge. It is easy to show that this mechanical system may be described by the following equation:

$$\frac{\partial^2 \theta}{\partial t^2} - c_0^2 \frac{\partial^2 \theta}{\partial x^2} + \omega_0^2 \sin \theta = 0 \quad (22)$$

where θ is the angle of rotation of the pendulums, x is the axis along which the pendulums are distributed, $c_0^2 = l_1^2 \beta / I_i$, l_1 being the distance between two pendulums, β the torque constant of a section of spring between two pendulums, I_i the moment of inertia of a single pendulum of mass m and length l_2 , and $\omega_0^2 = mg l_2 / I_i$. The first term in Eq. 22 represents the inertial effects of the pendulums, the second term corresponds to the restoring torque due to the coupling between pendulums, and the third term represents the gravitational torque. Equation 22 is called the Sine-

Gordon (SG) equation (Lamb 1971); it can be totally integrated and admits exact soliton solutions. It has become a very famous equation containing dispersion and nonlinearity which is used to model various phenomena in physics. It constitutes the third class of nonlinear equation leading to solitons.

Fluxons in a Josephson Tunnel Junction

Solitons arise also in a Josephson tunnel junction, which is a junction between two superconductors. These junctions are very attractive for information processing and storage (Ustinov 1998). The physical quantity of interest in the long Josephson junction is a quantum of magnetic flux, or a fluxon, which has a soliton behavior. The relevant nonlinear equation which describes the physical processes is the Sine-Gordon equation. This equation is particularly suitable in solid physics.

Solitons in Optical Fibers

One of the fundamental applications of solitons concerns optical fibers. In optical fiber communication using linear pulses, dispersion and losses in the fiber limit the information capacity which can be transported and the distance of transmission. Hasegawa and Tappert (1973) first proposed to balance the dispersive effect by nonlinearity using soliton-based communications. The Kerr effect, in other words the nonlinear change of the refractive index of the fiber material, is used, and an initial optical pulse may form a nonlinear stable pulse, an optical soliton, which is in fact an envelope soliton. Hasegawa and Tappert proposed the previously mentioned nonlinear Schrödinger equation for the description of solitons propagating in optical fibers. The first observations of these solitons were made in 1980 by Mollenauer, Stolen, and Gordon (Mollenauer et al. 1980). In real media, the light intensity of the soliton decreases due to losses in the fiber, and suitable reshaping of the pulse is required. Numerous theoretical and experimental studies on nonlinear guided waves have been carried out, because of their wide range of applications (Hao et al. 2004; Kruglov et al. 2003). Inserting an optical fiber in a laser cavity, a “soliton laser” may be obtained, a femtosecond laser

which is now in use. A real revolution in international communications is taking place at present, based on optical soliton technology. It is anecdotal that a fiber-optic cable linking Edinburgh and Glasgow now runs beneath the path from which John Scott Russell made his initial observations and along the aqueduct which now bears his name.

The NLS equation may be obtained from the Sine-Gordon equation. Previously considered systems as water waves on the free surface or chains of pendulums have also weak amplitude solutions, plane waves which have the form:

$$\theta = \psi e^{i(qx - \omega t)} + \text{c.c.} \quad (23)$$

where ψ is the wave amplitude, q is the wave number, ω is the pulsation, and c.c. denotes the complex conjugate of a complex number. When the wave amplitude is sufficiently increased for the nonlinearity to take place, modulation can spontaneously arise because harmonics are generated by the nonlinearity. The initial wave may split in wave packets whose properties correspond to the ones of solitons. These solitons consisting of a carrier wave which is modulated by an envelope signal are called envelope solitons. It can be shown (Peyrard and Dauxois 2004) that the evolution of the envelope ψ is described by the NLS equation:

$$i \frac{\partial \psi}{\partial t} + P \frac{\partial^2 \psi}{\partial x^2} + Q |\psi|^2 \psi = 0 \quad (24)$$

where P and Q are coefficients which depend on the physical problem, as the significance of the variables t and x . The NLS equation is formally similar to the Schrödinger equation of quantum mechanics:

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} - U\psi = 0. \quad (25)$$

The potential U in Eq. 25 is proportional to the absolute square of the wave envelope ψ , m is the mass of the quasi-particle, and $\hbar = h/2\pi$ where h is the Planck constant. The potential represents the self-trapping of the wave energy.

Solitons in Solid Physics

Solitons are involved in the atomic structure of matter. At the beginning of the twentieth century, a central question in solid-state physics was how a plastic deformation of a metal takes place in the corresponding crystal lattice. This problem induced numerous discussions, and it was suggested that if one atom in the lattice is pulled away, a neighbor atom follows the one which had been moved, jumping into the new hole, that is the new free lattice space. Then, a dislocation runs through the crystal. This proposition was improved in 1939 by Frenkel and Kontorova (1939) who developed a model where not only each atom moves alone over one lattice distance, but also many atoms form a dislocation line generating a wave of translation. The process of traveling of the dislocation line through the crystal was assumed to occur without losses of energy. This is equivalent to Russell's solitary wave of translation which travels over long distance without change of form and velocity. This led Frenkel and Kontorova to make an analogy between the dislocation line and the soliton, and they obtained the one-soliton solution of the Sine-Gordon equation.

The solitons permit us to interpret properties of dielectric materials. In other respects, magnetic materials are interesting examples to experimentally verify in a very accurate way the theory of solitons at the atomic scale (Mireska and Steiner 1991). The concept of soliton in polymer physics constitutes a very nice case of an interdisciplinary approach. Their occurrence was suggested in 1988 by theoretician physicists (Heeger et al. 1988). Many studies have since then been carried out in chemistry and experimental physics. Alan J. Heeger, Alan G. MacDiarmid, and Hideki Shirakawa received the Nobel Prize in chemistry in October 2000 for their works on the electric conduction of conductive plastics which is performed by solitons.

Solitons in Biology

Interest in the physical modeling of biological processes has grown significantly since the beginning of the 1990s. Nonlinear localization phenomena have been recently shown in systems very close to biological systems (Edler and

Hamm 2002). The notion of the soliton is now used to explain the dynamics of biological macromolecules such as proteins and the molecule ADN; an approach to the dynamics of the molecule ADN can be obtained using the envelope soliton solution of the NLS equation (Peyrard and Dauxois 2004). Intense researches are presently being carried out on this subject. The great size of biological molecules allows collective behaviors of atom groups, which are with non-linearity important components for the existence of solitons.

Mathematical Methods Suitable for the Study of Solitons

Solitons may appear in many fields, such as fluid mechanics, solid-state physics, plasma physics, optical fibers, and biology, as shown in section "Physical Properties of Solitons and Associated Applications." The description of these physical systems often leads to dispersive and nonlinear equations which are not presently solvable. It is then important to refer to the great classes of soliton equations, which are idealized models for numerous systems. These great classes of soliton equations correspond to the Korteweg-de Vries equation, known as the KdV equation (Eq. 9), the Sine-Gordon equation (SG; Eq. 22), and the nonlinear Schrödinger equation (NLS, Eq. 24). These three famous equations can be totally integrated. They are not completely dissociated. For instance, it is possible to obtain the NLS equation from the SG equation and from the KdV equation. The soliton solutions represent only the first approximation of the physical properties of real systems. Dissipation and a weak spatial or temporal variation of the physical parameters in real physical systems are examples of phenomena which are very common and which are not taken into account in the soliton equations. The modeling has to be adapted to the physical system and to the excitation conditions of this system. In the case of plasmas, the KdV equation is more suitable for a description of the system when intense pulses excite it, and the NLS equation is more adapted for sinusoidal perturbations.

There are two broad theoretical approaches available to obtain the solutions of the great classes of soliton equations, the analytic and the algebraic methods. Analytic approaches include the powerful inverse scattering transform (IST) (Gardner et al. 1967) and the remarkable Hirota method (Hirota 1971). One-soliton solutions or multi-soliton solutions can be obtained. Moreover, the IST method can predict the number of solitons that can emerge from an initial disturbance applied to a physical system (Newell 1985). This method shows how the solitons have a role in nonlinear normal modes as the Fourier modes for a linear equation (Lakshmanan 1997). There are also the Bäcklund transformation method (Rogers and Shadwick 1982), the direct linearizing transform method (Ablowitz and Clarkson 1991), the Painlevé analysis (Lakshmanan and Sahadevan 1993), and the method of position-like solutions (Jaworski and Zagrodzinski 1995). The algebraic methods involve the Lie group theoretical method (Olver 1986), the direct algebraic method (Martnez Alonso and Olmedilla Morino 1995), and the tangent hyperbolic method (Malfliet 1992).

The theoretical methods cannot always be used since they need some conditions to make them applicable. Numerical methods can be applied (Helal 2002). Zabusky and Kruskal (1965) were the pioneers in studying in 1965 the KdV equation numerically, by using the leapfrog method as an explicit finite difference scheme. Many methods have since then been used: a split-step Fourier method (Tappert 1974), the hopscotch method (Greig and Morris 1976), a pseudo-spectral method (Fornberg and Whitham 1978), spectral methods (the Galerkin and Chebyshev methods) (Helal 2001; Sanz-Serna and Christie 1981), the finite element method (Taha and Ablowitz 1984), and Fourier collocation calculations (Canuto et al. 1988).

Future Directions

Solitons may occur in the macroscopic and microscopic scales of physics. This results from their very general existence conditions, the coexistence of dispersion and nonlinearity. The concept of

solitons is not restricted to a specific field of physics, and the investigation of solitons remains a very active research area. We have previously mentioned (section “[Physical Properties of Solitons and Associated Applications](#)”) that one important application of solitons concerns optic fibers. The propagation of optical solitons through nonlinear fibers has been extensively studied; however, in real media, the dynamics of these solitons and the conditions for their generation are significantly affected by various inhomogeneities in the media. The problem of nonlinear wave propagation in the form of solitons in inhomogeneous optical fiber media is actually not well understood, despite the wide range of applications (Hao et al. 2004). The investigation of nonlinear processes in coupled optical waveguides is another research direction for the future design of optical computers and sensor elements. Recent advances have shown that solitons have a great potential for the improvement of optical systems which demand fast and reliable data transfer. In the hydrodynamic field, the tsunamis often behave like solitary waves. The dramatic tsunami which occurred in the Indian Ocean on 26 December 2004 has clearly shown that the actual numerical models do not accurately predict the water propagation on the coastal plains. An important direction for future research is the three-dimensional (3D) simulation of tsunami breaking along a coast. The 3D simulation of the interaction of a tsunami with different beach profiles, with and without obstacles, has also to be considered. This will permit the design of protecting devices and the setting up of security zones. In biology, nonlinear localization phenomena have been proved in model systems close to biological systems, but the challenge is open for biological molecules. Other important application fields for solitons concern magneto-electronics and secure communications. The formations of spatiotemporal patterns on perturbations of soliton systems are important problems to be studied in the years to come. The existence of instabilities in the solitary wave solutions of the great classes of soliton equations has to be tackled. Nonlinear excitations in systems whose spatial dimension is greater than one raise many questions. Among these

questions, there are the problems of the possible stable structures, their collision properties, and the effect of external forces. New applications will certainly emerge from these studies.

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Solitons Interactions

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Glossary

Solitary wave A localized structure which travels with constant speed and shape in otherwise homogeneous medium.

Soliton In the classical nomenclature, a soliton is a nonlinear spatially or temporally localized entity (solitary wave, impulse, wave packet, kink, etc.) of permanent form that retains its identity and shape in interactions with other similar entities. Also frequently used term for solutions of the relevant nonlinear partial differential equations approximately describing such physical entities.

Soliton (colloquially as well as in certain domains) Used for denoting any long-lived localized structures (e.g. solitary waves, self-trapped optical beams, localized vortices) which exhibit low energy radiation and approximately keep their shape.

Soliton solution A solution, usually in closed form, of the nonlinear partial differential equations that exhibit properties of a single soliton. A multi-soliton solution exhibits properties and collective behavior of groups of solitons.

Elastic interaction A generalization of the concept of elastic collisions of ideal mechanical bodies: interaction of localized entities in which the dynamically significant properties of the counterparts such as the total energy, linear and angular momentum, and topological charge or spin are conserved. For low-dimensional and scalar solitons denotes the case in which all the properties of interacting solitons are completely restored after interaction.

Inelastic interaction Interaction of soliton-like structures that modifies one or more dynamically significant properties of the counterparts.

Phase shift A change of the position of a soliton owing to interactions with other solitons compared to the location in which it would be in the absence of other solitons.

Resonant interaction A specific form of soliton interactions leading to the fusion of two or more solitons into a new soliton. Generally associated with an infinite phase shift of the counterparts.

Definition of the Subject

The concept of solitons reflects one of the most important developments in science of the twentieth century: the nonlinear description of the world. It is customary to start an introduction to solitons by recalling that it is not easy to give a comprehensive and precise definition of a soliton. Frequently, a soliton is explained as a spatially localized (solitary) wave with spectacular stability properties. Although the combination of simultaneously being a localized structure propagating while mostly keeping its shape as waves do, and surviving for a long time in realistic (that is, nonlinear) conditions, already is a fascinating property in itself, yet another key quality defines

whether a particular entity is a soliton. The distinction is made based on the way in which two or more of these objects interact with each other.

The classical nomenclature associates the term soliton with (i) nonlinear and (ii) localized entities, which (iii) have a permanent form and (iv) retain their identity in interactions with other entities from the same class (e.g. Drazin and Johnson (1989); also the entry ► [“Solitons and Compactons”](#)). The fundamental property of a soliton is thus to retain its identity in nonlinear interactions. In low-dimensional systems (understood here as environments admitting solitons in one spatial dimension and line solitons in two dimensions), a soliton’s amplitude, shape, and velocity are restored after each interaction, whereas in more complex systems this request embraces all dynamically significant qualities. In other words, a structure is a soliton if and only if its *interactions* are *fully elastic*. Thus, the interaction of solitons is always an intrinsic topic whenever solitons are considered.

In the nonlinear world, linear superposition does not hold and nonlinear interaction takes place. Mathematically, classes of entities are called linear, whenever any linear combination of entities belongs to the same class. Among structures corresponding to the solutions of certain equations, exclusively those that satisfy linear equations are denoted as linear ones. Any linear combination of solutions of a linear equation also solves this equation. The principle of linear superposition is that the resultant structure is simply the sum of the parties and it implies that the individual linear wave shapes are perfectly restored after meeting with each other.

The property of elasticity of soliton interactions has a fundamentally different nature, because, as a rule, a linear superposition of soliton solutions is not a solution of the governing nonlinear equation. This property distinguishes nonlinear *solitary* traveling waves (that is, spatially or temporally localized pulses, whereas the relevant single-wave solution of the underlying equation may be completely stable) from *solitons*, which may be observed and analyzed as separate entities but which always express property (iv) should similar entities appear in the system.

The propagation and interaction of both solitons and solitary waves in realistic conditions (that is, in the presence of dissipation, external fields, and in homogeneities of the medium) frequently results in a certain loss of energy (usually in the form of radiation of relatively shortwaves); in certain cases collapse or fission of the structure, or a net exchange of energy between the waves occurs. Even if a solitary wave travels without radiation losses in an ideal environment, it may only be called a soliton if neither radiation loss nor net energy exchange occurs when it meets a similar structure; otherwise a perfect re-emerging of each soliton after the interaction would be impossible. More generally, no net changes of any dynamically or topologically significant quantities are permitted in elastic collisions. Only changes of certain dynamically less significant properties such as the exact location (phase) of the solitons are common in elastic interactions. Solitary waves or structures possessing a selection of properties of solitons are sometimes called quasi-solitons.

Although the first evidence of solitons was extracted from observations in nature, the theory of solitons and their interactions has been developed very much in terms of the mathematical theory of the relevant nonlinear partial differential equations (PDEs) of the motion. The definitions of solitons and their interactions are commonly formulated in terms of solutions to such equations. It is even customary to speak of solitary waves and solitons, and of their interactions, having in mind the relevant exact solutions to these equations. Therefore, in mathematical terms, a soliton – either a solitary wave or a more complex entity – is a solution of a nonlinear PDE that elastically interacts with other similar solutions. In the contemporary nomenclature it is also customary to associate solitons and their interactions with numerical (multi-soliton) solutions to these equations.

Interactions between solitons are the most fascinating features of soliton phenomena. The most instructive quality of soliton interactions is the universality of many of their features that exists in spite of the extremely diverse physical origins of the counterparts. The presentation of the basic

conceptual ideas involving soliton interactions, a discussion of several particular cases of such interactions and an overview of observations of the relevant phenomena in natural conditions are mostly limited to effects occurring in the elastic interactions of one-dimensional and line solitons, in which the solitons behave very much like ideal mechanical bodies and which serve as the kernel of the classical concept of solitons and their interactions. Some spectacular features of nearly elastic interactions in higher dimensions are interpreted in terms of generalizations of the classical elastic interactions. Finally, a selection of practical applications of specific features of soliton interactions is again discussed from the viewpoint of line solitons.

Introduction: Key Equations, Milestones, and Methods

Integrable Equations

The quality of some nonlinear PDEs to admit soliton solutions is associated with the property of integrability. Integrable equations usually admit an infinite number of integrals (constants) of motion. Many nonintegrable equations possess localized shape-preserving traveling wave solutions that resemble solitons. However, only integrable equations have the universal property of possessing exact multi-soliton solutions that reflect perfectly elastic interactions between individual solitons. Thus, the integrability of the underlying equation is a primary constituent of the classical soliton interactions. Arnold (1998) defines, for example, a soliton as a solitary wave (solution) of an integrable PDE having additional stability and robustness features which are inherited directly from the integrability of this PDE. Nonlinear interactions even between the greatly different (but physically relevant) solutions of integrable equations are also elastic.

Among the variety of integrable PDEs that admit multi-soliton solutions, examples related to the Korteweg–de Vries (KdV) equation, the (focusing) nonlinear Schrödinger (NLS) equation, the integrable Boussinesq equation, the sine-Gordon (SG), and the Kadomtsev–Petviashvili (KP)

equation will be used below. Below we shall refer to the listed equations as the classical soliton equations. Since the KdV and the KP equations represent a large number of different physical systems and their solutions can be easily visualized in terms of the easily accessible environment of shallow-water waves, solutions to and results from studies of these equations will be largely used for illustrations of the explanations below.

The rapid development of the theory of solitons has led to the discovery of many integrable equations which show multi-soliton solutions and describe soliton interactions (see, e.g. Arnold (1998) and the entry ► [“Partial Differential Equations that Lead to Solitons”](#)). A number of analytical methods have been developed to obtain the (multi-) soliton solutions of integrable equations, starting from the late 1960s. A major tool is the Inverse Scattering Transform (IST), first developed by Gardner, Green, Kruskal and Miura (1967, 1974) for the KdV equation. The method consists of associating to the evolution equation a Sturm–Liouville problem. In the case of the KdV equation the Sturm–Liouville equation is just the time independent Schrödinger equation of quantum mechanics, where the potential is the wave function of the KdV equation. The single- and multi-soliton solutions are associated with the discrete spectrum of eigenvalues. A generalization of the Inverse Scattering Transform, known as the AKNS method (named after Ablowitz, Kaup, Newell and Segur (1974) who developed the method), applies to a large class of integral equations. See the entry ► [“Inverse Scattering Transform and the Theory of Solitons”](#) for detailed information.

At the same time, Hirota (1971) found that soliton solutions can be obtained by reducing, through a suitable transformation, the evolution equation to a bilinear form. Using some properties of the bilinear operator, it is possible to find the multi-soliton solution of an integrable equation. The proof of integrability of a particular PDE is usually nontrivial. The experience with the Hirota method, applied to a variety of nonlinear wave equations, is that exact multi-soliton solutions are found in many cases when it is not known if the equation is integrable. It is natural to expect that the existence of such solutions reflects the

integrability in some sense; however, no proof of this conjecture is known at this time.

Mathematically speaking, it is always possible to build an integrable equation representing certain features of an observed phenomenon. Nevertheless, only a few of these equations can be properly derived, normally using asymptotic expansions, from the corresponding primitive equations describing physical phenomena. Even the classical soliton equations only approximately describe the natural phenomena. A more exact description of the natural processes requires the introduction of additional contributions to these equations. Such contributions usually destroy the property of integrability and lead to nonelasticity of interactions of soliton-like entities. Consequently, most of the times solitons are just a good approximation of solitary waves in nature. This limitation is to some extent balanced by the fact that in many cases integrable soliton-admitting equations adequately represent the basic features of physical phenomena even far beyond their formal scope of validity. In spite of such intrinsic limitations, the tools developed in soliton theory have allowed researchers to reach a very deep understanding of some physical phenomena which would have hardly been explained by other means.

Milestones

Several milestones of the solitons' history (► [“Solitons: Historical and Physical Introduction”](#)) are particularly significant to soliton interactions. Already the famous first observations in 1834 of shallow-water solitons (Russell 1844) implicitly involved certain effects created through soliton interactions. The observed wave ahead of a ship probably had a straight crest, as the one reproduced in 1995 in the Union Canal near Edinburgh (see, e.g. p. 11 in Dauxois and Peyrard (2006)). It is a remarkable feature of ship-induced solitons in relatively narrow and shallow channels that a straight-crested upstream soliton exists ahead of a two-dimensional (2D) disturbance, whereas the solitons generated ahead of a ship sailing in wider shallow water areas have curved crests. The straight crest of the Russell's soliton and of her sister structures are thus not

intrinsically one-dimensional (1D) structured solitons. They are formed owing to a specific mechanism of soliton interactions as described below. The KP equation, allowing a simple but adequate description of the crest-straightening phenomenon, was derived even later than the word soliton was coined.

A substantial step towards the concept of solitons was made in the mid-1950s by Fermi, Pasta and Ulam (1955) who numerically analyzed the evolution of phonons in an anharmonic lattice. From the mathematical point of view this problem can be described by a discretization of the KdV equation. Surprisingly at the time, the process did not lead to an energy equipartition among the modes, although nonlinearity generally tends to create such an equipartition. Instead, a sort of recurrence, an early evidence of the elastic soliton interactions, was observed.

The basic features of soliton interactions were first demonstrated for media described by the KdV equation (Ikezi et al. 1970; Zabusky and Kruskal 1965). The evolution of a sinusoidal initial wave with periodic boundary conditions gives rise to an ensemble of solitary waves that move with different speeds and gradually overtake each other. Originally eight of them were mentioned; later revisitations increased the count to nine (Osborne and Bergamasco 1986). The difference in the number of solitons is not principal since small-amplitude solitons only become evident through additional phase shifts in extremely long-term simulations (Engelbrecht and Salupere 2005; Salupere et al. 2002). One surprise of the system was that after a very long time the whole profile reappears. The other, conceptually much more striking effect, was the persistence of the waves: after a certain phase of interaction with each other, they continued thereafter as if there had been no interaction at all. This persistence, which reflects the essence of the soliton interactions, led Zabusky and Kruskal to invent the name “soliton”.

Solitons and new features of their interactions were later discovered in a large number of different physical systems. The universal principle generalizing the physical forces affecting the birth of all solitons is that the propagating disturbance (a pulse in a fiber, a wave-packet on a water surface, a self-

focusing optical beam, a localized vortex, or a vortex dipole, etc.) is captured in a potential well jointly induced by its motion and by virtue of the nonlinearity of the underlying system. The solitons can thus be interpreted as the bound states of the relevant potential wells, and soliton interactions – as interactions between the bound states of a jointly induced potential well, or between bound states of different wells located at close proximity. This concept of potential well becomes vividly evident in the inverse scattering theory due to the conservation of the eigenvalues of the underlying spectral problem. This universality explains why, in spite of the immense diversity of the solitons and the underlying physical systems, the interactions (collisions) between solitons in all of these systems follow the same principles.

Classical Soliton-Admitting Equations and Appearance of Solitons

The particular appearance of soliton interactions may vary a lot, depending on the physical system and the nature of the solitons. The Russell soliton is an example of nontopological solitons that often become evident in the form of traveling waves and that cannot exist in rest. On the contrary, single topological solitons (that interpolate between two different states of the system which have the same energy) may exist in rest.

A localized entity consists of an infinite number of Fourier harmonics and generally experiences (linear) spreading or wave radiation. A localized entity of permanent form therefore may only appear if the spreading effects are balanced by a certain restoring force that evidently can only be of a nonlinear nature. In other words, all solitons correspond to a certain balance between spreading and nonlinear effects, the latter becoming evident, for example, as nonlinear steepening of the wave shape in the KdV system, or focusing of diffractive structure in nonlinear optics.

The physical meaning of spreading depends on the particular system. Dispersion is responsible for the spreading of pulses in all media in which the group velocity depends on the wave properties. The classical examples of solitons are surface (Russell 1844), internal (Davis and Acrivos 1967) and Rossby solitons (Redekopp and Weidman

1978) in geophysical systems. Similar structures occur in a large variety of environments such as drift solitons or ion-acoustic solitons in plasma (Washimi and Taniuti 1966), or solitons in optical fibers (Hasegawa and Tappert 1973).

Many examples of dispersive solitons and their interactions occur in the framework of the KdV equation and its generalizations. It is a characteristic equation governing weakly nonlinear, spatially one-dimensional (1D) waves whose phase speed attains a simple maximum for infinitely long waves

$$\tilde{\eta}_t + \frac{3c_0}{2h}\tilde{\eta}\tilde{\eta}_x + \frac{c_0h^2}{6}\tilde{\eta}_{xxx} = 0, \quad (1)$$

in nondimensional variables $\eta_t + 6\eta\eta_x + \eta_{xxx} = 0$,

where η is the relative water surface elevation in the shallow water environment (Fig. 1), h is the still water depth, and c_0 is the maximum phase speed (celerity) of linear waves at this depth. Its generalization to the case of solitons propagating in slightly different directions

$$(\eta_t + 6\eta\eta_x + \eta_{xxx})_x + 3\eta_{yy} = 0, \quad (2)$$

was derived by Kadomtsev and Petviashvili (1970). Named the KP equation after them, it describes features of a so-called weakly 2D environment (also called 1.5-dimensional systems). The non-dimensional (x, y, t, η) and physical variables $(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\eta})$ in Eq. (2) are related as follows: $x = \sqrt{\varepsilon}(\tilde{x} - \tilde{t}\sqrt{gh})/h$, $y = \varepsilon\tilde{y}/h$, $t = \sqrt{\varepsilon^3gh}\tilde{t}/h$, $\eta = 3\tilde{\eta}/(2\varepsilon h) + O(\varepsilon)$ whereas $\varepsilon = |\tilde{\eta}_{\max}|/h \ll 1$.

Since the KdV equation only admits nontopological elevation solitons, the KdV/KP framework misses several basic features of soliton interactions, such as collisions of solitons with different topological charge and interactions of solitons of different type. These classes of dispersive solitons are represented by the sine-Gordon (SG) equation

$$\theta_{tt} - c_0^2\theta_{xx} + \omega_0^2\sin\theta = 0, \quad (3)$$

where θ in many applications has the meaning of a certain 2π -periodic angle and ω_0 is the minimum angular frequency of linear waves. This equation,

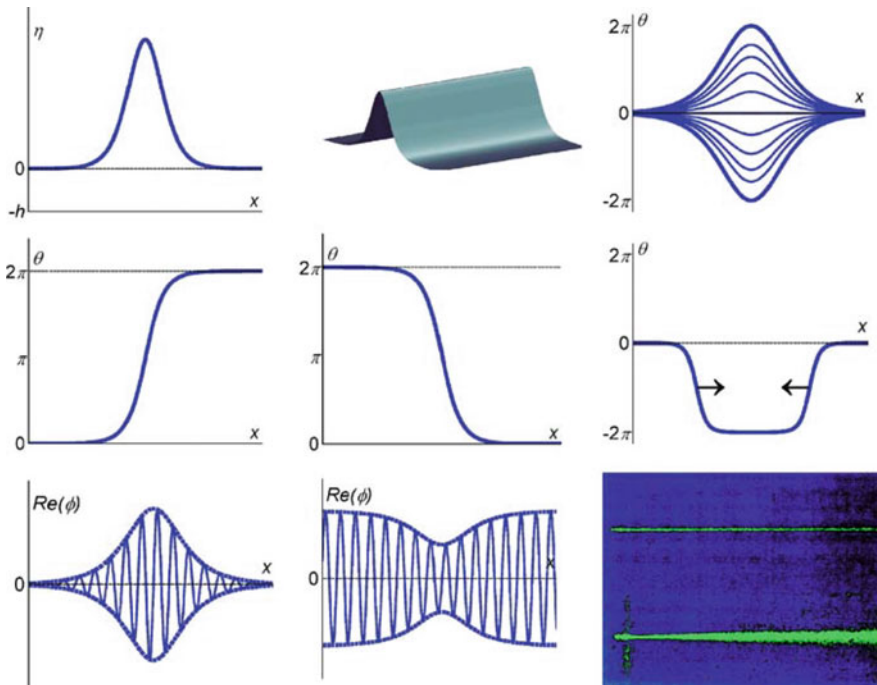
with its name stemming from a play of words regarding its form which is similar to the Klein–Gordon equation, arose already in the 19th century in studies of certain surfaces. It grew greatly in importance when it was realized that it led to kink and antikink solutions with the collisional properties of solitons (Perring and Skyrme 1962). Its major field of application is solid state physics, yet it also appears in a number of other physical environments such as the propagation of fluxons in Josephson junctions (a junction between two superconductors), the motion of rigid pendulums attached to a stretched wire, and dislocations in crystals. Its kink solutions (Fig. 1)

$$\theta_s = 4 \arctan \exp \left(\pm \frac{\omega_0}{c_0} \frac{x - vt - x_0}{\sqrt{1 - v^2/c_0^2}} \right), \quad (4)$$

where x_0 is the initial position of the single soliton at the x -axis and $v < c_0$ is the speed of translation of

the soliton, represent so-called topological solitons that interpolate between two different states of the system, which have the same energy and may exist in rest (e.g. a frozen (anti)kink, $v = 0$). The SG equation is a simple model admitting breather solitons and soliton pairs with different topological charges. The bulk topological charge is an invariant of the system. In a more complex manner the topological charge becomes evident in the theory of optical spatial solitons, where it can be interpreted in terms of spin of elementary particles. The kink and antikink solutions also represent the possibility of the existence of solitons and anti-solitons. Both attractive and repulsive interactions can be vividly demonstrated in this framework and, after all, interactions of physically meaningful solitons of different types are possible.

Another key equation for dispersive waves is the nonlinear Schrödinger (NLS) equation that has a nondimensional form



Solitons Interactions, Fig. 1 A selection of solitons. From left to right: top – KdV traveling wave soliton, KP plane (line) soliton, SG breather; middle – kink, antikink, and a kink-antikink pair; bottom – bright and gray envelope soliton, and a top view photograph of a 10- μ m-wide optical spatial soliton propagating in a strontium barium niobate

photorefractive crystal and the same beam diffracting naturally. The bottom right image is from (Stegeman and Segev 1999). Optical spatial solitons and their interactions: Universality and diversity. (Reprinted with permission from AAAS)

$$i\psi_t + p\psi_{xx} + q|\psi|\psi^2 = 0 \quad (5)$$

and only has soliton solutions in the focusing case $pq > 0$. This is a universal equation describing the evolution of complex wave envelopes ψ in a dispersive weakly nonlinear medium. It applies, among other phenomena, to deep water waves. The studies of its soliton solutions and their interactions also date back to the 1960s (Zakharov 1968; Zakharov and Shabat 1972). Its major application in the context of soliton interactions is nonlinear optics, where its different versions describe the propagation and interactions of both dispersive solitons (e.g. envelope solitons in optical fibers) and diffractive optical spatial solitons.

The basic reason for the existence of optical solitons is that the optical properties (refractive index or absorption) of some materials are modified by the presence of light. This feature introduces nonlinearity into the system and alters the propagation of optical pulses either in space or in time. A dispersive optical soliton is formed when the linear dispersion effects are balanced by a sort of lensing (of short light pulses of permanent form that are called temporal solitons in the optical nomenclature) in an appropriate material (for example, an optical fiber). They were predicted by Hasegawa and Tappert (1973) and first observed by Mollenauer et al. (1980). The basic properties of their interactions (Hasegawa 1989) are analogous to those occurring for KdV or SG solitons. There is continuous interest for their studies because of their applications, e.g. in long-distance optical communication systems.

The generic example of diffractive solitons, evolution and interactions of a part of which are described by the NLS equation in two space dimensions, are optical spatial solitons. The natural tendency to broaden in space owing to the diffraction of optical beams (Fig. 1) can be balanced, for example, due to the optical Kerr effect that consists of a local refractive index change induced by light. As a first approximation, this change is assumed to take place instantaneously and to be proportional to the local intensity of light. In the focusing case (when an increase of the intensity of light causes an increase in the refractive index), the light-induced lensing can

make an optical beam stable with respect to perturbations in both its width and intensity. The resulting beam in a 2D or 3D medium is an example of a diffractive soliton. Interaction of such beams occurs owing to overlapping of the modified regions of the medium. Usually such an interaction is local and has a long range only under specific conditions (Rotschild et al. 2006).

The first studies suggesting the existence of such solitons in nonlinear optical Kerr media date back to the 1960s (Askar'yan 1962; Chiao et al. 1964). In some cases (such as the 1D Kerr solitons in a planar medium) they are classical solitons and described by a fully integrable equation (the NLS equation in two space dimensions (Zakharov and Shabat 1972)). Self-trapped light beams and more complex structures in a 2D and 3D medium attracted the attention of researchers starting from the 1980s, when the appropriate materials were found (Barthelemy et al. 1985).

In multi-dimensional media, both diffraction and dispersion affect the propagation of solitary waves. A classical example of a diffractive-dispersive system is the motion of short-crested waves on the water surface. Most of the analysis of soliton interactions in this system has been made for effectively 1D solitons with infinitely long crests (so-called plane or line solitons); the effects of diffraction being implicitly neglected in their propagation and interactions. A practically realized example of diffractive-dispersive solitons are short pulses of incoherent (or white) light that is self-trapped both in the direction of its propagation and in the transversal direction(s), and that propagate changing neither their shape nor their length (Mitchell and Segev 1997).

Extended Definitions

The classical nomenclature of low-dimensional solitons and their interactions was formulated several decades ago, and it is not surprising that many later discoveries have led to attempts to extend its content. A part of the extensions add several consistent features (such as the conservation of linear and angular momentum, and topological charge or spin during the soliton interactions); however,

in several schools substantial generalizations of the term soliton and of the interpretation of the soliton interactions have been introduced. The only component of the classical nomenclature kept in all the interpretations is the essential role of nonlinearity. Since it is virtually impossible to reflect all the extensions, mostly the material and results following the classical definition are presented below.

The solitons are commonly interpreted as a subset of traveling solitary waves. The classical nomenclature favors structures that are stationary in a proper coordinate system; yet it does not exclude the resting topological solitons (e.g. single-kink solutions to the sine-Gordon equation).

A principal extension of the classical definition of solitons that caused quite a serious discussion in the 1970s (Newell and Redekopp 1977) consists of accounting for the *resonant interactions*. They may result in the fusion of several solitons into a new soliton or, equivalently, in the decay of solitons into counterparts (Zakharov et al. 1980). Although the number of solitons is not conserved, all the parties of the resonant system are solitons at the limit of exact resonance and no energy radiation occurs. It is commonly accepted now as a variation (or a limiting case) of elastic interactions.

The classical soliton interactions preserve the shape of the solitons. The shape is understood in a wide sense; e.g. the shape of the envelope of the NLS solitons is preserved as well as the time-dependent shape of breathers that have an internal oscillation. This quality, which is universal for all scalar solitons, has a more general interpretation in the case of vector (composite) *solitons*. For example, the otherwise elastic interactions of Manakov vector solitons exhibit a shape-changing nature (Radhakrishnan et al. 1997) in an integrable system consisting of two coupled NLS equations (Lakshmanan and Rajasekhar 2003). This property has a paramount practical importance and bright perspectives e.g. in optical soliton-based computations (Steiglitz 2001, see also entry “Optical Computing”), being one of the key options of practical applications of specific features of soliton interactions.

An obvious reason for a wider view on soliton interactions is that the relatively simple integrable

equations admitting soliton solutions stem from certain asymptotic expansions and only approximately describe the processes in nature. A more exact representation of the physical processes through inclusion of higher order terms of those expansions generally destroys the integrability of these equations. In many cases, though, the perturbing terms are small. The influence of a small dissipative perturbation is usually obvious: it brakes the moving solitons and/or damps the oscillating solitons. Such structures are sometimes called dissipative solitons although such a name is somewhat self-contradicting. Effects due to conservative (Hamiltonian) perturbations are much richer in content. Usually they do not destroy or brake solitons, but they may, e.g., render otherwise elastic soliton interactions to inelastic ones owing to extra wave radiation. A well-known example of the effect of perturbations is dissipation-induced annihilation of a kink-antikink pair in a perturbed sine-Gordon equation that otherwise would lead to a breather (Pedersen et al. 1984). A monumental survey of the relevant problems is presented in (Kivshar and Malomed 1989); see also entry ► “Soliton Perturbation”.

Since none of the physical and numerical environments perfectly matches the governing equations, solitary waves both in realistic conditions and in numerical simulations always are approximations to solitons. In numerical experiments practically ideal solitons can be represented and errors basically occur due to purely computational inaccuracies. Laboratory experiments with solitons encounter problems with perfect excitation of even single solitons and moreover with controlling their interaction properties; however the relevant results are of fundamental importance in establishing the adequacy, the limits and the practical applicability of the properties of solitons and their interactions.

In the theory of elementary particles frequently a soliton is defined merely as a localized solution (resp. entity) of permanent form and the constraint of its survival in collisions is released (e.g. Rebhi (1979)). A popular and deep example is the quantum field theory, where e.g. the Yang–Mills field equations admit solutions localized in space (which represent very heavy elementary particles), and also solutions, localized in time as well as in

space (instantons). This position has certain support by the conjecture that the classical soliton equations can be interpreted as reductions of self-dual Yang–Mills equations and thus the classical solitons form a subset of the above solutions.

A similar position is also widely adopted in studies of optical beams and vortices. A large part of such beams are described by nonintegrable equations (Segev 1998) and thus lie beyond the classical soliton nomenclature. Many features of their behavior and interactions, however, demonstrate a striking similarity with that of classical solitons, and exhibit universal properties (Stegeman and Segev 1999), resembling the similar universality of the classical solitons interactions. Soliton-like beams organized by different nonlinear effects were mostly called solitary waves until themid-1990s. The modern nonlinear optics nomenclature treats all self-trapped optical beams as solitons (Segev and Stegeman 1998) although their interactions are not always elastic. A selection of results of studies of their interactions that give a flavor of the richness of soliton interaction phenomena in 3D media is presented below.

The constraint of energy conservation both in the motion and the interactions of the solitons is frequently disregarded in studies of long-lived structures which exhibit reasonable radiation (Flierl et al. 1980) but that still survive in certain collisions. On the other hand, other highly

interesting solitary structures such as (Rossby) dipole modons (Larichev and Reznik 1976, 1983) do not radiate energy but only survive under certain conditions. A certain amount of results and concepts in this domain is presented to highlight the overlapping of the properties of soliton interactions and interactions of other long-living structures.

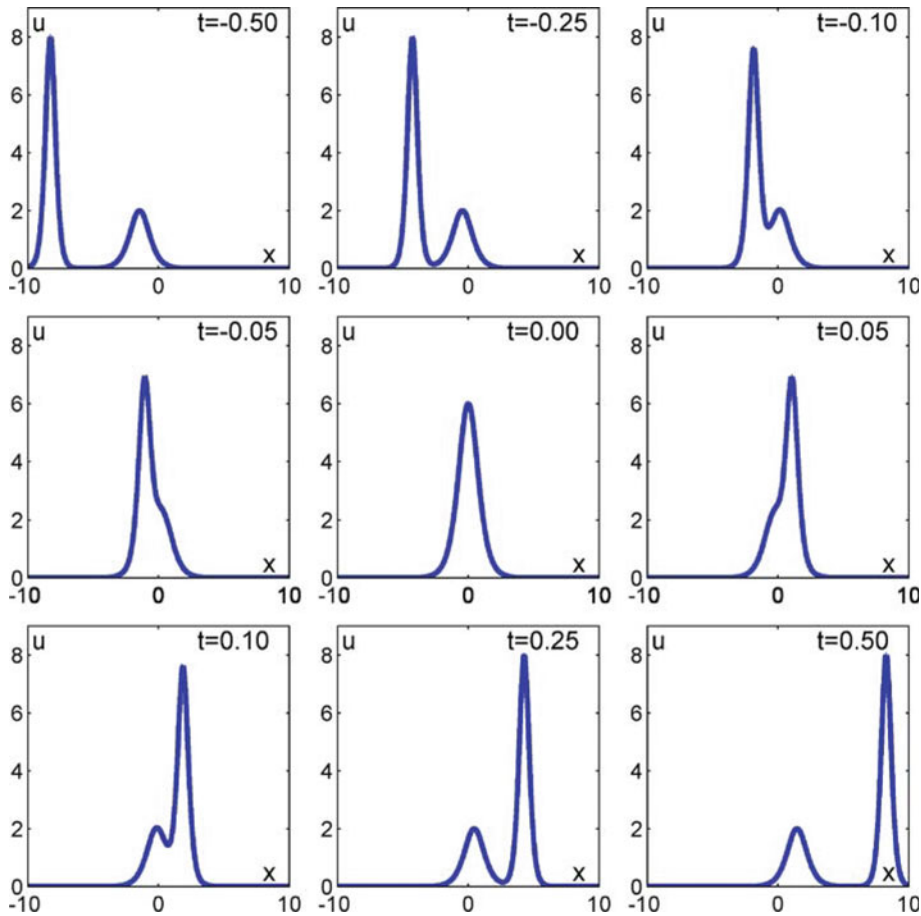
Elastic Interactions of One-Dimensional and Line Solitons

The term soliton was originally introduced for nonlinear disturbances, the interaction of which resembles the collision of particles and is fully elastic (Table 1), so that the number of solitons is always conserved and their amplitudes are fully restored afterwards. In low-dimensional cases such as those described by the KdV equation, the amplitudes, directions and propagation velocities of each soliton always recover their initial values (Fig. 2). The shape of each soliton is prescribed by the structure of (single-)soliton solutions to this equation and, as the simplest evidence of the recurrence of the system, it is not surprising that the shape is perfectly restored as well.

Three phases of the interactions of solitons can be distinguished either in time or in space (Fig. 2). First, well-defined, clearly separated solitons approach each other. In the second (interaction)

Solitons Interactions, Table 1 Properties of solitons in elastic interactions of 1D and line solitons. Analogous rules hold for angular momentum and spin in elastic collisions in higher dimensions

Property	1D interactions and nonresonant interactions of line solitons	Exceptions in resonant interactions
Number of solitons	Conserved except in a short phase of interaction	Changed
Energy	Restored after interactions for each soliton	Merging possible; total energy conserved.
Amplitude Shape (steepness)	Substantial local changes may occur in the interaction region; restored after interactions	Durable changes may occur
Phase or location	Finite phase shifts commonly occur; yet no phase shifts in certain environments. The total phase shift is exactly equal to the sum of partial shifts that would result from separate collisions with each of the solitons	Infinite phase shifts
Geometry	Substantial changes in the interaction region, restored after interaction	Durable changes may occur
Propagation direction, linear momentum	Conserved for each soliton	Conservation of bulk linear momentum
Topological charge	Conserved for each soliton	



Solitons Interactions, Fig. 2 Temporal evolution of KdV solitons described by the two-soliton solution $u(x, t) = 12 \times (3 + 4 \cosh(2x - 8t) + \cosh(4x - 64t)) / ([3 \cosh(x - 28t) + \cosh(3x - 36t)]^2)$ (Drazin and Johnson 1989). The amplitudes of the counterparts

are $u_{1\max} = 8$ and $u_{2\max} = 2$. Notice the gradual decrease of the taller soliton when it catches the shorter, and its gradual increase when it moves further ahead of the shorter one, and the relatively small amplitude $\bar{u} = 6$ of the composite structure

phase, they usually lose their identity and merge into a composite structure. After a while, the solitons emerge again. The appearance of the composite structure depends on the particular physical system; as a rule it is neither a linear superposition of (properly shifted) counterparts nor a new soliton. Therefore solitons survive the collision event, even though they completely merge for a while. The variety of the composites range from the complete vanishing of the counterparts (e.g. kink-antikink collision in the SG equation) over certain interactions of KdV solitons, in which the individual identities are almost conserved and two humps are always visible, to a fourfold elevation of the water surface in a resonant interaction of the KP solitons.

In some environments, one cannot say whether the solitons propagate through each other as waves do or collide as particles do. Both interpretations have a solid ground. The interaction of unidirectional KdV solitons resembles the collision of two particles whereas, e.g. in head-on interactions of Rossby dipole solitons (Larichev and Reznik 1983) the identity of both counterparts can be continuously tracked and the interaction process resembles a sort of complex overtaking.

Attraction and Repulsion

Another feature of soliton interactions resembling collisions of real massive and/or electrically charged particles that exert certain forces on

each other is that soliton interactions may show an *attractive* or *repulsive* nature. This feature can be demonstrated in the framework of the kink-antikink solution of the sine-Gordon equation

$$\theta_{SA} = 4 \arctan \frac{c_0 \sinh vt \zeta}{v \cosh x \zeta}, \quad \text{where } \zeta = \frac{\omega_0}{c_0} \frac{1}{\sqrt{1 - v^2/c_0^2}} \quad (6)$$

and that only exists if $v > 0$. The motion of the counterparts speeds up when they move towards each other, and slows down when they move apart after passing each other. If one examines the positions of the centers of the counterparts of this solution, the sequences of their positions in the vicinity of their meeting point are not the same as they would be if they had been alone in the system, or far from the other counterpart.

If the solitons were massive or charged particles, one would say that an attractive force exists between them which decreases as their separation increases. On the contrary, the speed of two kinks as well as two antikinks approaching each other slows down as their separation decreases, which is characteristic for repulsive interaction. Another classical reflection of “forces” between solitons is the effect of attraction or repulsion of (envelope) solitons in optical fibers (Gordon 1983).

In two dimensions, the attractive interactions of two obliquely interacting line solitons become evident as an attraction of the relevant wave crests. In the case of optical spatial solitons the attraction affects their centroids. For repulsive interactions

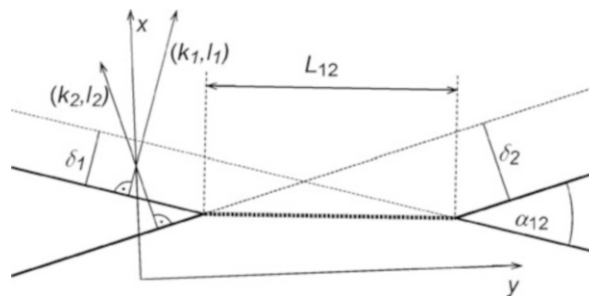
the opposite holds. The relevant effects are vividly demonstrated in the framework of the KP equation for the description of drift vortex solitons in environment, otherwise described by the Hasegawa–Mima equation (Tajiri and Maesono 1997). This environment is equivalent to the one described by the Charney–Obukhov equation for large-scale motions in geophysical hydrodynamics admitting Rossby waves and solitons.

Transient Amplitude Changes, Durable Phase Shifts and Recurrence Patterns

Already the first study of solitons (Zabusky and Kruskal 1965) revealed some other universal aspects of soliton interactions. First, certain durable phase shifts may occur, the magnitude of which depends on the interaction details. For example, during the collision presented in Fig. 2, the taller soliton is shifted forward by an amount $\Delta x_1 = \frac{1}{2} \log 3$ and the shorter one backwards by $\Delta x_2 = \log 3$ compared to the case when the solitons are alone in the system. The above-discussed “forces”, if present in the system, may be interpreted as the cause of the phase shifts.

The total phase shift of a soliton induced by elastic collisions with any number of solitons is exactly equal to the sum of (partial) shifts that would result from separate collisions with each of the solitons involved. This property is commonly referred to as the absence of many-particle effects.

In the case of line solitons the phase shifts become evident in the form of durable shifts of the counterparts during the interaction (Fig. 3). In



Solitons Interactions, Fig. 3 Idealized patterns of crests of interacting KP solitons (*bold lines*), their position in the absence of interaction (*dashed lines*) and the crest of the

residue s_{12} (*bold dashed line*, see Eq. (7)) corresponding to the negative phase shift case. (Adapted from (Peterson et al. 2003))

the negative phase shift case the solitons are delayed compared to their position in the absence of interactions, whereas in the positive phase shift case they seem to be accelerated.

Second, the amplitudes of the interacting solitons may gradually change as they approach to each other (Fig. 2). The amplitude of the composite structure is generally not equal to the sum of amplitudes of the interacting solitons. In interactions of a small number of KdV solitons running along an infinite medium, the behavior of each counterpart and its influence on the partners can always be identified from the shape of the elevation since after some time the solitons become separated enough to detect all of them and their perfectly restored properties in the limit of infinite separation.

More insight into the properties of the soliton interaction is provided by long-term simulations of ensembles of solitons excited from a periodical signal (Fig. 4). The total number of interacting parties is infinite then, the solitons are tightly packed and such a separation not necessarily occurs. In some cases the presence of several shorter solitons can only be identified through their contributions to the phase shifts of the partners since they are seldom visually identifiable; such solitons are called hidden solitons (Engelbrecht and Salupere 2005).

The trajectories of the (crests of the) counterparts may show quite a complex pattern and may exhibit both simple recurrence (at which the initial system is approximately restored) as well as super-recurrence (understood as a nearly perfect restoration of the initial system over longer times).

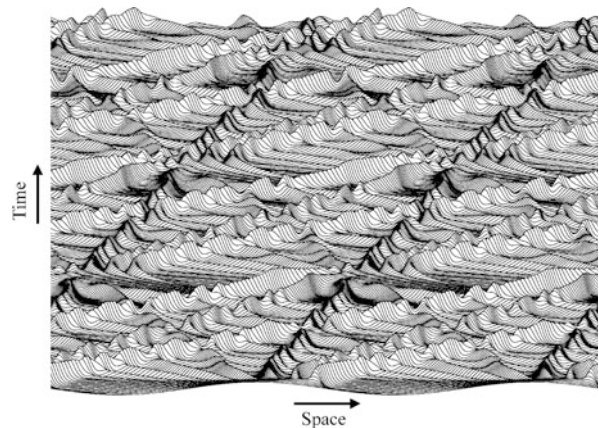
Practically periodical in time rhombus-like patterns (suggesting that interactions of quite a small number of solitons govern the properties of the system) optionally occur in cases when a super-recurrence is possible (Fig. 5), whereas in other cases these patterns show gradual variation also over extremely long times. There seems to exist a critical value d^* in the range $-1.8 < \log d^* < -1.9$ of the dispersion parameter d in the KdV equation (presented in the form $\eta_t + \eta\eta_x + d\eta_{xxx} = 0$), which distinguishes if a super-recurrence exist. If $d < d^*$, no super-recurrence can be detected even within extremely long calculation times (Engelbrecht and Salupere 2005; Salupere et al. 2003).

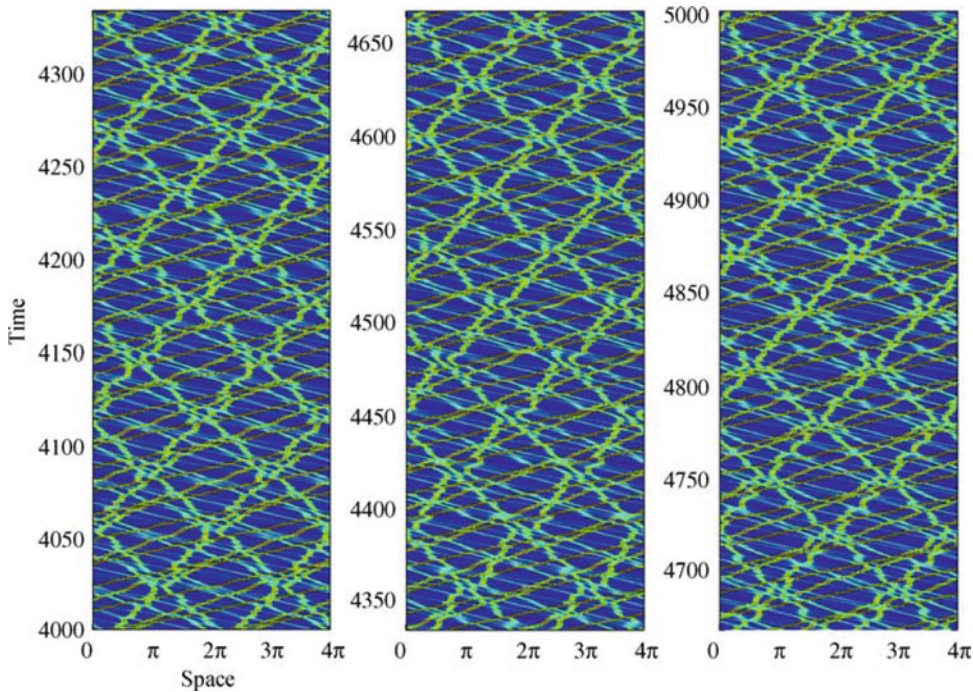
Durable Local Amplitude Changes in Oblique Interactions of Line Solitons

The interaction of unidirectional KdV solitons does not create any large changes in the wave amplitudes (Drazin and Johnson 1989, see also ► “History, Exact N-Soliton Solutions and Further Properties of the Korteweg-de Vries Equation (KdV)”). However, amplitude amplification may occur under certain conditions when line solitons propagating in slightly different directions meet each other. Extensive evidence comes from coastal engineering, where it was known already a long time ago that ocean waves approaching breakwaters and seawalls under a certain angle caused unexpectedly large overtopping. This phenomenon was systematically studied already in the 1950s (Perroud 1957), well before the word soliton was coined.

Solitons Interactions,

Fig. 4 Time-slice plot over two 2π periods of the evolution of the ensemble of the KdV solitons generated from a sinusoidal initial condition in space for $0 \leq t \leq 97$ at $\log d = -2.3$. (Reprinted from (Salupere et al. 2003))





Solitons Interactions, Fig. 5 Patterns of locations of crests of the KdV solitons generated from sinusoidal initial conditions at $\log d = -2.3209$. (Reused with permission from (Engelbrecht and Salupere 2005))

When a shallow-water wave is obliquely launched along an ideally reflecting wall, the reflection does not necessarily follow the rules of geometrical optics. For a certain range of incidence angles, the crests of the approaching and the reflected wave fuse together near the wall and the process resembles the Mach reflection. The common crest, an analogue of the Mach stem, is generally unsteady and gradually lengthens for sine waves (Berger and Kohlhase 1976) and Stokes waves (Yue and Mei 1980). This type of reflection also occurs during perfect reflection of random (Mase et al. 2002) and solitary waves of different nature (Funakoshi 1980; Miles 1977a, b; Peregrine 1983; Tanaka 1993).

For a certain set of parameters of the approaching KdV soliton, the resulting structure is equivalent to half of the pattern created by two interacting KP solitons of equal amplitude (Fig. 3). The prominent feature of both the (Mach) reflection and oblique interactions of shallow-water solitons is that the height of the common crest may considerably exceed the sum of the heights of the interacting

solitons. For interacting waves with equal amplitudes the hump can be up to four times as high as the incoming waves. Originally established in the context of the reflection of Boussinesq solitons (Miles 1977a, b; named Mach reflection by Melville 1980 (Melville 1980)), the amplitude amplification in a certain region occurs in all oblique attractive interactions of line KP solitons (that is, in interactions with a negative phase shift). Since the interaction pattern is stationary in an appropriately moving coordinate system for a range of the parameters of the interacting solitons, it may lead to durable local changes of the amplitude of the resulting pattern of elevation.

Resonance

While the theory of integrable systems requires that the 1D solitons retain their identity in interactions, collisions of solitons in multiple dimensions may lead to the formation of new solitons. The phenomenon called *resonant interaction* leads to the emergence of new structures that combine the energy of the counterparts and to

modifications of the number of solitons and of some of their geometrical features.

This possibility was first recognized by Newell and Redekopp (1977) in the context of the KP and NLS equations. Miles (1977b) gives an early answer to their question about the practical consequences of this phenomenon in terms of an essential increase of the wave amplitude occurring during the resonant Mach reflection. . Originally named “breakdown of the Zakharov–Shabat theory of integrable systems with more than one spatial dimension”, it highlights the more complex nature of soliton interactions in more than one dimension, where large phase shifts are possible.

Remarkably, the resonance conditions $\kappa_\infty = \kappa_1 + \kappa_2$, $\omega_\infty = \omega_1 + \omega_2$, where $\kappa_i = (k_i, l_i)$, $i = 1, 2, \infty$ are the wave vectors and ω_i are the frequencies of the solutions of the corresponding linearized KP equation, are precisely the same as those for the resonant interaction of three weakly nonlinear waves. Conceptually, however, such interactions more resemble so-called Double resonance, in which not only the above resonance conditions are met but also the group velocities of two or more counterparts are equal and otherwise sporadic energy exchange within the resonant triplets is replaced by much more intense interactions (e.g. Soomere (1992)).

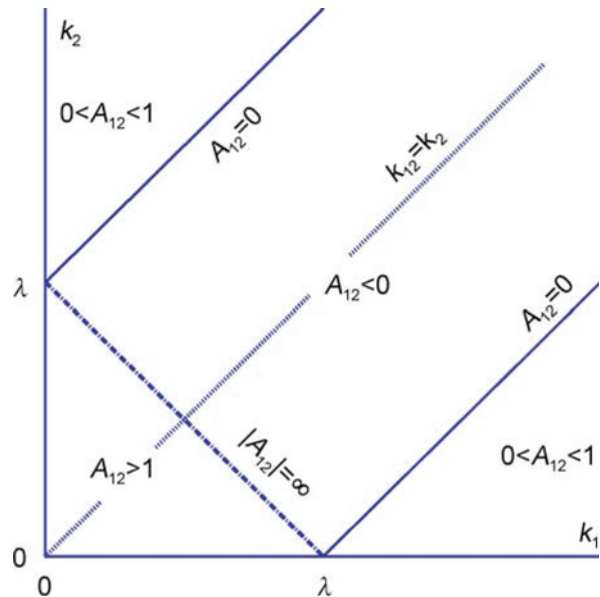
The patterns of line soliton crests (Fig. 3) suggest that only attractive interactions may result in resonance. The resonant interactions between line solitons correspond to infinitely large phase shifts and infinitely strong attraction, and lead to durable changes of the geometry of the system. In the situation depicted in Fig. 3, the resonance would lead to an infinitely long common crest and thus render the “four-arm” pattern to a “three-arm” one.

In the KP framework, the resonance phenomenon is connected with the fact that its multi-soliton solutions representing the interaction of line solutions only exist for a subset of the space of parameters of the interacting solitons (Fig. 6). Resonance occurs when the parameters of the interacting solitons lie at a certain part of the border of this subset.

A new soliton born at the exact resonance may resonantly interact with yet another soliton. This mechanism has been employed in (Biondini and Kodama 2003) to construct a family of exact solutions to the KP equation corresponding to resonance of several solitons. Such solutions consist of unequal numbers of incoming and outgoing line solitons, the parameters of which can be obtained from asymptotic analysis (Biondini and Chakravarty 2006). This class contains a variety of multisoliton solutions, many of which exhibit

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Fig. 6 Negative and positive phase shift areas for the two-soliton solution of the KP equation for fixed $l_{1,2}$. The phase shift $\Delta_{12} = -\ln A_{12}$ has a discontinuity along the line $k_1 + k_2 = \lambda$ at which $|A_{12}| = \infty$ and the resonant interactions occur. No two-soliton solution exists in the area where $A_{12} < 0$. An analogue of linear superposition occurs when $A_{12} = 0$. (Adapted from (Peterson et al. 2003))



nontrivial spatial interaction patterns that are generally characteristic to multisoliton solutions to the KP equation (Peterson and van Groesen 2000).

Geometry of Oblique Interactions of KP Line Solitons

Patterns

A relatively simple but instructive, easily visualizable and rich-in-content model demonstrating the discussed features of interactions of line solitons is the KP equation. It admits simple explicit formulae for multi-soliton solutions and extensive analytical treatment of their geometrical properties. The two-soliton solution to the KP equation can be written as a sum $\eta = s_1 + s_2 + s_{12}$ of terms, reflecting to some extent the two interacting solitons s_1, s_2 and residue s_{12}

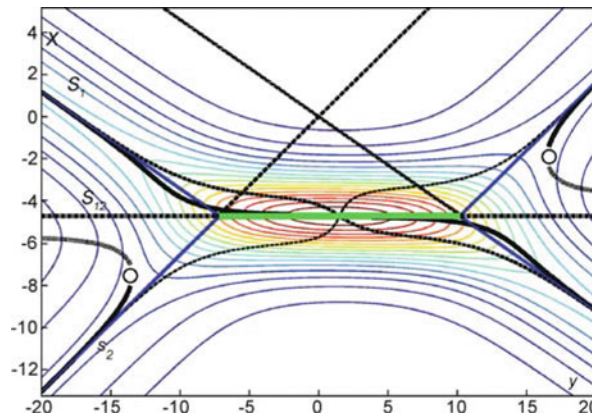
$$\begin{aligned} s_{1,2} &= \sqrt{A_{12}k_{1,2}^2} \Theta^{-2} \cosh \left(\varphi_{2,1}x + \ln \sqrt{A_{12}} \right), \\ s_{12} &= 2\Theta^{-2} \left[(k_1 - k_2)^2 + A_{12}(k_1 + k_2)^2 \right], \\ \Theta &= \cosh \frac{\varphi_1 - \varphi_2}{2} + \cosh \frac{\varphi_1 + \varphi_2 + \ln A_{12}}{2}. \end{aligned} \quad (7)$$

Here $\varphi_i = k_i x + l_i y + \omega_i t$, $a_i = \frac{1}{2}k_i^2$, $i = 1, 2$, are the phases and amplitudes of the interacting solitons, the ‘frequencies’ ω_i satisfy the dispersion relation $k_i \omega_i + k_i^4 + 3l_i^2 = 0$ of the linearized KP equation, $A_{12} = [\lambda^2 - (k_1 - k_2)^2]/[\lambda^2 - (k_1 + k_2)^2]$ is the phase shift parameter and $\lambda = l_1 k_1^{-1} - l_2 k_2^{-1}$.

The two-soliton solution only exists in a part of the parameter space $R^{(4)}(k_1, k_2, l_1, l_2)$. An interaction may result in either the positive or the negative phase shifts $\delta_{1,2} = -\ln A_{12}/|\mathbf{k}_{1,2}|$ of the counterparts (Fig. 6). The negative phase shift (cf. Figure 3) can be attributed to an attractive interaction and the positive phase shift to a repulsive interaction.

The residue s_{12} (with an amplitude a_{12}) is at times considered as a virtual 2D “interaction soliton” (Peterson and van Groesen 2000), which grows into the resonant soliton at the limit of exact resonance $A_{12} \rightarrow \infty$ (cf. Figure 7). This virtual structure can be heuristically interpreted as supported by both dispersive and diffractive effects, the latter being balanced by the oblique motion of the interacting solitons.

The patterns of both idealized (Fig. 3) and factual (Fig. 7) wave crests are symmetric with



Solitons Interactions, Fig. 7 Patterns of idealized (solid/dashed straight lines) and factual (dashed curves) crests of the incoming solitons s_1, s_2 , the crest of the residue s_{12} (green line), the visible wave crests and troughs (bold and bold dashed curves, respectively), and isolines of the two-soliton solution in the case $l = l_1 = -l_2 = 0.3$, $k_1 =$

0.6507 , $k_2 = 0.4599$ in normalized coordinates (x, y) . The interaction center is the crossing point of all factual crests. Contour interval is $\frac{1}{4} \max(a_1, a_2) = 0.0689$. Circles show the common points of the troughs and crests of the two-soliton solution. (Reprinted from (Soomere 2004))

respect to a particular point called interaction center, and are stationary in a properly moving coordinate frame. If the amplitudes of the solitons are equal, the pattern is equivalent to the reflection of the incoming soliton from the wall following the motion of the interaction center. For unequal amplitude solitons the equivalence is not complete because mass and energy flux occur through such a wall.

Amplitudes

The phase shifts $\delta_{1,2}$ of the counterparts only depend on the amplitudes of the incoming solitons and the angle $\alpha_{12} = 2 \arctan(\frac{1}{2}\lambda)$ between their crests. For the negative phase shift case $A_{12} > 1$ (that is typical in interactions of solitons with comparable amplitudes, Fig. 5), an interaction pattern emerges, the height of which exceeds the sum of the heights of two incoming solitons. The maximum surface elevation for equal amplitude solitons is $a_{\max} = 4a_{1,2}/(1 + A_{12}^{-1/2})$; thus, the nonlinear superposition of solitons may lead to a fourfold amplification of the surface elevation. In a highly idealized case of interactions of five solitons the maximum surface elevation may exceed the amplitude of the incoming solitons by more than an order. The largest elevation occurs if the interacting solitons are in exact resonance $A_{12} = \infty$, when solitons intersect under a physical angle $\tilde{\alpha}_{12} = 2 \arctan \sqrt{3\eta/h}$. For unequal amplitude solitons the maximum elevation $a_{\max}$ for finite A_{12} and the amplitude of the resonant

soliton a_{∞} at $A_{12} = \infty$ are (Gabl and Lonngren 1984; Soomere 2004).

$$a_{\max} = a_{12} + 2A_{12}^{1/2} \frac{a_1 + a_2}{(A_{12}^{1/2} + 1)^2},$$

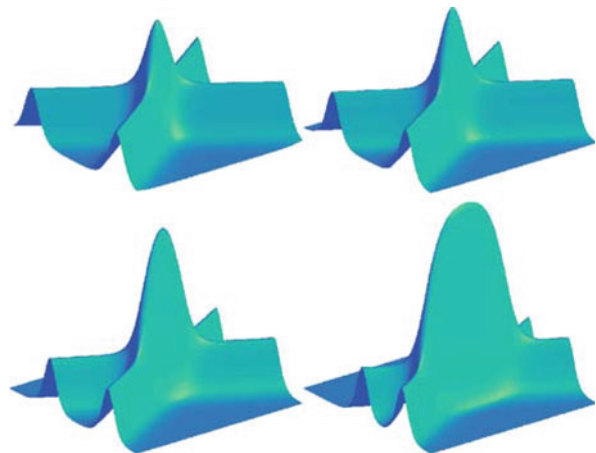
$$a_{\infty} = \frac{(k_1 + k_2)^2}{2}.$$
(8)

The near-resonant high hump is particularly narrow and its front is very steep. The maximum slope of the front of the two-soliton solution may be eight times as large as the slope of the incoming solitons (Soomere and Engelbrecht 2005). For unequal amplitude solitons, the amplification of slope of the front of the interaction pattern is roughly proportional to the amplitude amplification. The extraordinary steepness of the front of the near-resonant hump, although intriguing, is not unexpected, because the resonant soliton is higher and therefore narrower than the incoming solitons (Fig. 8).

The length L_{12} of the idealized common crest (Fig. 3) is $L_{12} \sim \ln A_{12}$ and therefore is modest unless the interacting solitons are near-resonant. The length of the area, where the total elevation exceeds the sum of the amplitudes of the counterparts, may be considerably longer; however it is also roughly proportional to $\ln A_{12}$ (Soomere and Engelbrecht 2005). For largely different amplitudes of the interacting solitons, the amplitude amplification remains modest but the spatial extent of substantial influence of nonlinear

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Fig. 8 Surface elevation in the vicinity of the interaction area, corresponding to the incoming solitons with equal amplitudes intersecting under different angles (Peterson et al. 2003)



interaction is roughly as large as if the amplitudes were equal (Soomere and Engelbrecht 2006). Their near-resonant interaction becomes evident as a wave with its crest oriented and propagating differently compared with the counterparts. The described features are universal for that part of the wave vector space, in which the two-soliton solution with a negative phase shift exists. In the case of a positive phase shift (that is typical for interactions of solitons with largely different amplitudes) the highest elevation does not exceed the amplitude of the larger soliton.

Numerical simulations of the process of formation of the composite structure show that a high wave hump is formed quite fast in the interaction region and its height soon considerably exceeds the sum of the heights of the counterparts (Haragus-Courcelle and Pego 2000; Porubov et al. 2005). A high and gradually lengthening wave hump is also formed in cases when no exact two-soliton solution of the KP equations exists. The interaction in such cases is inelastic: neither the orientation of the crests nor the height of the solitons before and after interaction match each other, and a certain amount of wave radiation takes place. The described features become evident in a number of simulations of soliton interactions in different environments (Oikawa and Tsuji 2006; Tsuji and Oikawa 2004).

Soliton Interactions in Laboratory and Nature

Ion-Acoustic Solitons

The most productive studies of soliton interactions in the 1970s and the 1980s were performed in the framework of ion-acoustic waves. Their behavior is approximately described by the KP equation. An analytical solution describing interactions and resonance of two plane ion-acoustic solitons of small amplitude in the 3D collisionless plasma was given in (Yajima et al. 1978). This solution is wider than the two-soliton solution of the KP equation in the sense that the nearly unidirectional approximation is not used.

The first successful experiment proving that the 2D interaction of solitons has a wider

spectrum of features compared with the 1D collisions is reported in (Ze et al. 1979). A collision of two equal amplitude spherical solitons led to significant changes and to the emergence of a new 2D object. This interaction was not perfectly elastic: the solitons were not only shifted in phase but also reduced in amplitude. Soon afterwards, a near-resonant interaction of two KP solitons, with a large-amplitude wave hump of the counterparts oriented across the principal propagation direction, was observed near the interaction center, whereas the counterparts experienced clear phase shifts (Folkes et al. 1980). The amplitude amplification was found to be close to twofold the sum of the amplitudes of the incoming solitons (Nishida and Nagasawa 1980).

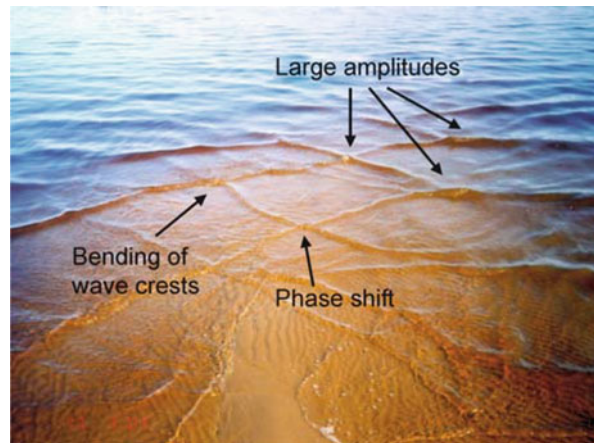
The first adequate evidence of the nonlinear increase in amplitude during collisions of unequal amplitude KP solitons was also obtained in this framework (Gabl and Lonngren 1984). For largely different amplitudes the measured maximum amplitudes were less than the predicted ones. A probable reason for the discrepancy is that the near-resonant wave hump occurs and covers a substantial area (equivalently, can be reliably estimated with the use of relatively large probes) only if the incoming solitons are very close to resonance. The common part of the crests of interacting solitons in the reported experiments was pretty short suggesting that the solitons were not exactly in resonance. Moreover, the complete identification of all the crests in the interactions of solitons with considerably different amplitudes is not possible because the crests are partially invisible (Fig. 7, Soomere (2004)). The generic reason for experimental difficulties in this and similar experiments is the above-discussed feature that large-amplitude structures tend to mask the presence of shorter ones.

Shallow-Water and Internal Solitons

Phase shifts, formation of a common crest of the interacting soliton-like waves, and accompanying amplitude and orientation changes are commonly observable in very shallow areas (Fig. 9). A famous photo by Terry Toedtemeier (1978), in which two solitonic shallow-water waves have significant phase shifts and quite a long common

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Fig. 9 Interaction patterns of solitonic surface waves in very shallow water near Kauksi resort on Lake Peipsi, Estonia. (Photo and copyright by Lauri Ilison, July 2003. First published in Soomere and Engelbrecht (2006))



crest on a beach in the US state of Oregon, has been reproduced in a number of sources (e.g. p. 24 in Dauxois and Peyrard (2006)).

Although (Miles 1977b) predicted up to four-fold amplification of the surface elevation, much weaker amplitude amplification was found in early experiments (Melville 1980). An amplification close to the theoretical one was detected as a by-product of studies of shallow-water channel superconductivity (Chen et al. 2003a).

Russell's "great wave of translation" (Russell 1844) probably had a practically straight crest, reflecting a well-known feature of the generation of solitons in relatively narrow channels: a nearly perfectly 1D upstream soliton is created also by a 2D disturbance. The KdV model, which is frequently used to describe this phenomenon, is intrinsically 1D and certainly results in straight wave crests. An analysis of the simplified problem of ship motion in the KP equation and the Boussinesq equation shows, in accordance with common experience, that the wave crests are curved (Katsis and Akylas 1987). In fact, the essentially 2D waves with paraboloidal crests generally emerge in front of the disturbance. Their curvature monotonously decreases as the waves propagate (Lee and Grimshaw 1990; Li and Sclavounos 2002).

Although the sidewall of the channel is not essential for the radiation of the upstream solitons, still it plays a crucial role in the transformation of already radiated waves. The straight-crested solitons are formed in the process of the (Mach) reflection of solitons with curved crests from the

sidewalls (Pedersen 1988). This feature can be distinguished in many numerical studies, where an initially curved soliton starts to straighten when it comes in touch with the sidewall (Soomere 2007). Since the reflection of solitons, which in this case is equivalent to soliton interactions, plays the key important role in creating straight-crested waves in channels, with a certain exaggeration it can be said that Russell's soliton exists due to soliton interactions.

Probably the most common solitonic phenomenon in nature next to the shallow water waves is represented by internal solitary waves in the oceans and in the atmosphere (Grimshaw 2001; Osborne and Burch 1980; Ostrovsky and Stepanyants 1989; Rottman and Grimshaw 2001, see also the entry ► "Nonlinear Internal Waves"). In many cases, the KdV equation is an appropriate model for internal waves and the effects occurring during their interactions are equivalent to those described above.

Large-scale internal solitary waves and their groups exhibit many features unique to soliton interactions and are often called simply internal solitons. Intersecting solitary wave packets on Synthetic Aperture Radar (SAR) images frequently exhibit phase shifts while crossing each other (Gasparovic et al. 1988), similarly to phase shifts occurring in the KP model, where the agreement between theory and experiments is quite good. Apel et al. (2007) summarize advances in the descriptions and observations of internal solitons and their interactions.

The other environment for internal solitons, which provides some instructive aspects and is eligible, e.g., for the description of small-scale internal solitons in a two-layer medium, is the Benjamin–Ono (BO) equation (Ablowitz and Segur 1981).

$$\eta_t + c\eta_x + \alpha\eta\eta_x + \frac{\beta}{\pi} \frac{\partial^2}{\partial x^2} P.V. \left[\int_{-\infty}^{\infty} \frac{\eta(x', t)}{x - x'} dx' \right] = 0. \quad (9)$$

First, the BO solitons do not acquire a phase shift after a collision. Second, this environment demonstrates that generalization of integrable 1D soliton-admitting equations to even a weakly 2D case may lead to the loss of integrability, thus the preservation of this feature in the KP equation is not universal. The weakly 2D analogues of the BO equation Ablowitz and Segur 1981 apparently are not integrable (Apel et al. 2007) and do not admit simple analytic multi-soliton solutions. Numerically simulated oblique interactions of weakly 2D BO solitons show that a phenomenon resembling the resonant interaction of the KP solitons and an accompanied amplitude amplification do occur but the newly generated wave in the zone of nonlinear interaction is far from the BO soliton (Oikawa and Tsuji 2006; Tsuji and Oikawa 2001).

Solitons at Planetary Scale

Large-scale internal waves and solitons in nature are affected by the joint influence of the Earth's rotation and sphericity, and perfect internal line solitons generally cannot exist in the ocean. The physical reason of this “Antisoliton theorem” is that due to rotational dispersion, there is always a phase synchronism between a source moving at an arbitrary speed and linear perturbations in such environments. This leads to unavoidable wave radiation from a large-scale solitary internal wave (Galkin and Stepanyants 1991); however this effect is negligible for relatively small-scale solitons, the typical time scales of which in the oceans are a few tens of minutes.

The existence of nonstationary solitary waves is still possible (Gilman et al. 1996). They exhibit a phenomenon similar to the recurrence of solitons – the repeating decay and reemergence process, and formation of a nearly localized wave packet consisting of a long-wave envelope and shorter, faster solitary-like waves that propagate through the envelope (Helfrich 2007). The robust, long-lived structure may contain as much as 50% of the energy in the initial solitary wave. Interacting packets may either pass through one another, or merge to form a longer packet. Their further studies may reveal interesting features concerning the functioning of oceans and other large stratified water bodies. Yet in the majority of practically interesting cases such long-term behavior is masked by topographic effects that modify the internal waves much more dramatically.

Another medium of particular practical importance as well as an instructive one from the viewpoint of soliton interactions is the large-scale geophysical flow in the oceans and the atmosphere. It is frequently treated as anisotropic quasi-2D turbulence, which has the property of energy concentration in large-scale structures. Eddies in such flows are stabilized by the background rotation and also affected by the joint effect of the Earth's rotation and sphericity. The latter becomes evident through the North-South variation of the Coriolis force, the so-called (planetary) beta-effect. In this environment, large-scale vortices often maintain their coherence for surprisingly long times and show an amazing variety of interactions; perhaps the most well-known example being the Jovian Great Red Spot.

Form-preserving, uniformly translating, horizontally localized solutions (of the equations of planetary dynamics) are called modons. In non-dissipative quasi-geostrophic dynamics they exhibit a variety of properties of solitons. They carry a certain amount of fluid with them, which distinguishes them from (solitary) waves that cause, if at all, a finite displacement of the medium. The word modon was coined a decade after soliton was first used (Stern 1975).

The standard quasi-geostrophic models predict that the accompanying Rossby-wave radiation should cause a decay of isolated eddies. Several

models explain the persistence of monopolar vortices using external features of the flow, or interactions with a background shear that suppresses the Rossby-wave radiation (Busse 1994; Read and Hide 1983; Sommeria et al. 1991). A number of generalizations of the quasi-geostrophic model predict that, within a certain range of parameters, a nonlinear anticyclonic vortex of a special form can exist for a long time (Petviashvili 1980; Stegner and Zeitlin 1996; Williams and Yamagata 1984) due to the mutual compensation of weak (scalar) nonlinearity and weak dispersion. Some authors call such structures (Petviashvili-type) Rossby solitons, although their collisions are not elastic.

Long-living coherent vortices and their interactions have been studied in many experiments with rapidly rotating parabolic vessels (Nezlin and Sneshkin 1993). The increase of the effective gravity due to the centrifugal acceleration towards the side walls modifies the governing equations, so that the (Petviashvili-type) Rossby soliton does not exist in the paraboloidal geometry (Nycander 1993); yet anticyclonic vortices have a very long lifetime (Stegner and Zeitlin 1998). Another option to study the evolution of such vortices is a tank with a sloping bottom to imitate the planetary beta-effect (e.g. (Firing and Beardsley 1976; Masuda et al. 1990), among others; for an overview of earlier studies see (Hopfinger and van Heijst 1993)).

The majority of interactions of solitary vortices in such an environment are inelastic. The 2D localized vortices of comparable size and intensity, and like sign attract each other and generally coalesce soon. Only practically equivalent vortices may form a quasi-stable pair rotating around each other. The vortices of different signs either repulse from each other or form a relatively stable dipole (Griffiths and Hopfinger 1986, 1987). The simultaneous evolution of a large number of soliton-like vortices has mostly been studied theoretically or numerically (e.g. in terms of a cluster of point vortices or hetons, (Gryanik et al. 2000)).

The simplest stable barotropic modon in flat-bottom beta-plane dynamics is the Larichev–Reznik dipole (Larichev and Reznik 1976). It is a traveling dipolar vortex with a characteristic north/south antisymmetry. It propagates eastward

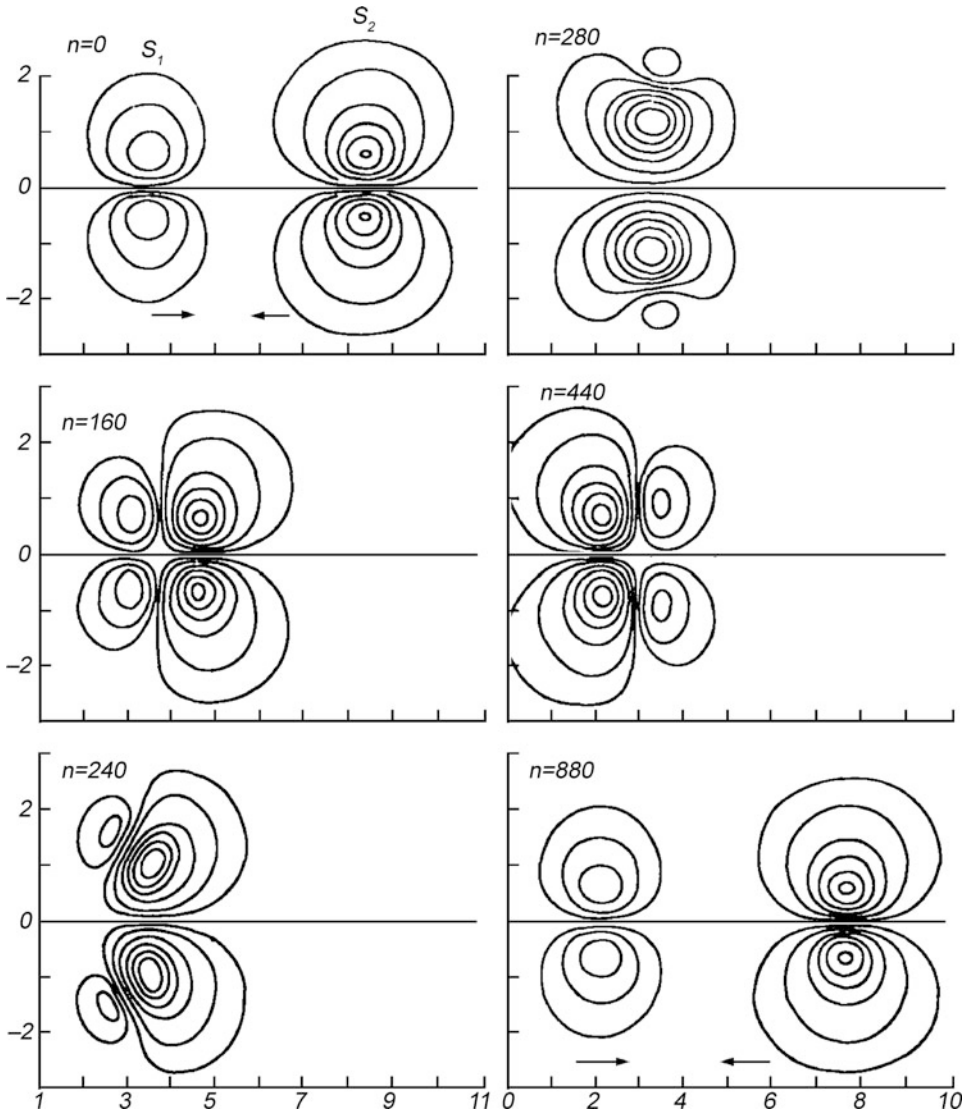
at any speed, or westward at speeds greater than the longest-wave speed. The speed of translation is therefore out of the range of the phase speed of Rossby waves, which is the reason why this modon does not radiate waves (unlike large-scale internal solitons). The significance of such structures in everyday life and climate dynamics (where they may be responsible, e.g., for certain unusual weather events since the vortex pair may behave completely differently compared with a single cyclone or anticyclone) is far from being well understood.

A part of the numerically simulated direct head-on and overtaking collisions of Larichev–Reznik modons are elastic (Larichev and Reznik 1983). An instructive feature of the head-on collisions is that the identity of all four vortices can be sometimes tracked during all the interaction phase (Fig. 10). The process resembles a collision of soft particle pairs connected by an elastic chain. Their oblique collisions generally are not elastic and may lead to the formation of new dipoles, tripoles, or to the destruction of the structures.

Effects in Higher Dimensions

Optical Spatial Solitons

Starting from the mid-1980s, various optical solitary entities have become a key arena for studies of the properties and interactions of solitary structures in multiple dimensions. The interest in these studies is continuously supported by a variety of existing and emerging applications of such solitons and their interactions in communication and computation technology (see also the entry “Optical Computing”). Another reason for the rapid progress in these studies is the relative ease of their generation in suitable materials, combined with the precision of contemporary optical experiments and the possibilities of precise control over almost every parameter (Segev and Stegeman 1998; Stegeman and Segev 1999). They offered a lot of the further conceptual progress of soliton interactions through making possible 3D propagation of solitary entities in laboratory conditions, albeit in many cases these entities did not match the classical nomenclature of solitons. They were



Solitons Interactions, Fig. 10 Head-on collision of Larichev–Reznik dipoles (Kamenkovich et al. 1987)

called solitary waves in the field for several decades, because the governing equations are not necessarily integrable and the interactions not necessarily elastic. Still their interaction at times shows amazingly elastic nature and they are commonly called optical spatial solitons (OSS) in the modern nonlinear optics nomenclature (Segev and Stegeman 1998).

A large number of various kinds of OSSs have been identified (Kivshar and Luther-Davies 1998; Segev 1998). Observations of spatial Kerr solitons became possible from the mid-1980s, when

so-called slab waveguides were built (Aitchison et al. 1990; Barthelemy et al. 1985). Their behavior substantially depends on the dimensions of the soliton and the waveguide. For example, the bright Kerr solitons are stable only in planar systems whereas the 2D solitons were found to collapse. Quasi-stable 2D solitons in Kerr media described by the cubic NLS in two spatial dimensions exist in the form of so-called necklace-ring beams (Soljacic et al. 1998).

Optical spatial solitons occurring due to the saturation of the nonlinear change in the refractive

index were first demonstrated in the 1970s (Bjorkholm and Ashkin 1974) and realized in a solid medium in the 1990s (Khitrova et al. 1993). The net effect of the multiple physical effects involved in photorefractive materials is that the underlying nonlinearities are also saturable. Such solitons were also discovered in the early 1990s (Segev et al. 1992) and were found to be stable in both slab waveguides (Kip et al. 1998) and in bulk media (Duree et al. 1993).

So-called quadratic solitons (that consist of multifrequency waves coupled by means of a second-order nonlinearity) were predicted in the mid-1970s (Karamzin and Sukhorukov 1976) and realized experimentally in the 1990s (Torruellas et al. 1995). They can be thought of as vector solitons because they involve mutual self-trapping of two or more components (Stegeman and Segev 1999). Their interactions are similar to those occurring in other saturable nonlinear media.

Coherent and Incoherent Collisions

Coherent interactions of OSSs occur when the nonlinear medium responds quickly to the (interference) effects in the overlapping area of the beams. The increase of the intensity of light in the overlapping region of two parallel launched equivalent in-phase solitons in the focusing medium leads to an increase in the refractive index in that region. This in turn forces the centroid of each soliton toward it. Hence the solitons attract each other. The spatial distribution of the intensity of light resembles the temporal distribution of the amplitude of attracting interacting line solitons. In the case of antiphase beams, an opposite process occurs and such solitons exert repulsion. Coherent collisions of Kerr solitons have been demonstrated and the attraction for in-phase and the repulsion for antiphase solitons were clearly observed at the beginning of the 1990s (Aitchison et al. 1991a, b; Shalaby et al. 1992). The collision of nonequivalent coherent OSSs results in an energy exchange between the beams and in a repulsive force that makes the beams diverge.

Some reactions (e.g. photorefractive and thermal) of the optical medium are relatively slow. Solitons within which the phase varies randomly

across the beam were first demonstrated in (Mitchell et al. 1996) and later extended to the beams of incoherent white light (Mitchell and Segev 1997). So-called incoherent soliton interactions occur when the response of the medium only follows the average intensity of light. Interactions of incoherent bright solitons are always attractive (Andersen and Lisak 1995).

The long-distance behavior of the counterparts in attractive interactions depends on the properties of the medium and the orientation of the beams. Pure Kerr solitons launched along nearly parallel directions result in periodic paths of the solitons' centroids. As in the case of the recurrence of the KdV solitons, Kerr solitons exactly return to their initial conditions after each cycle. Solitons launched under large enough converging angles exhibit a slight lateral deflection (which is equivalent to the phase shift of the KdV solitons). The solitons launched under large enough diverging angles never collide.

If the energy transfer were neglected, the force between the equivalent 1D Kerr solitons would vary from the maximum attractive between in-phase solitons to the maximum repulsive for antiphase solitons. This process leads to the (generally nonperiodic) energy transfer, where one soliton may gather net energy from the other. The details of the trajectories and other properties of the interacting solitons can be quite complex (Stegeman and Segev 1999).

Interactions of 2D OSSs (e.g. in media with saturating nonlinearities) have a much larger variety of scenarios (Stegeman and Segev 1999); for example, interactions of incoherent photorefractive solitons may be both attractive and repulsive (Stepken et al. 1998). A number of resonance-like inelastic phenomena have been observed in which the number of solitons is not conserved. The fusion of two or more solitons into one structure resembles the resonance phenomenon. Here it happens owing to the gradual change of the orientation of the waveguides of attracting solitons launched under small relative angles into a new waveguide (Gatz and Herrmann 1992). Differently from the resonance of line solitons, this happens for a certain range of initial parameters. The threshold is the maximum total internal

reflection angle in the induced waveguide (Shih and Segev 1996; Snyder and Sheppard 1993). For larger collision angles, the solitons interact as described above. The solitons can fuse together already on the first merging or after a certain number of oscillations of their trajectories with decreasing amplitudes and periods (Baek et al. 1997; Krolkowski and Holmstrom 1997; Tikhonenko et al. 1995, 1996). A sort of inverse of this process is the breakup (fission, Snyder and Sheppard (1993)) of optical solitons into new solitons upon their interaction (Torner et al. 1996). Annihilation of solitons upon collision also may occur, when three solitons collided and only two emerged from the collision process (Krolkowski et al. 1998) (Fig. 11).

Three-Dimensional Effects

In 3D space the trajectories of interacting beam solitons do not necessarily lie on a single plane and the system possesses nonzero initial angular momentum. The trajectories of solitons that are launched along skew lines passing close to each other may bend in three dimensions. The solitons spiraling around each other form a double helix orbit (Poladian et al. 1991), much like two celestial objects or two moving charged particles moving along nearly parallel trajectories do. The interaction generally leads to spiraling-fusion or spiraling-repulsion of the trajectories (Belic et al. 1999; Tikhonenko et al. 1995, 1996). Since the mutual rotation encounters a centrifugal force (which is always repulsive), a perfect double-helix DNA-like system, an interesting class of elastic interactions, requires soliton attraction. It may be formed from identical in-phase coherent solitons; yet such a double helix is unstable with respect to perturbations of both the relative phase and amplitude of one of the solitons. This effect resembles processes occurring during the interactions of vortices of different and like sign in quasi-2D rotating fluids.

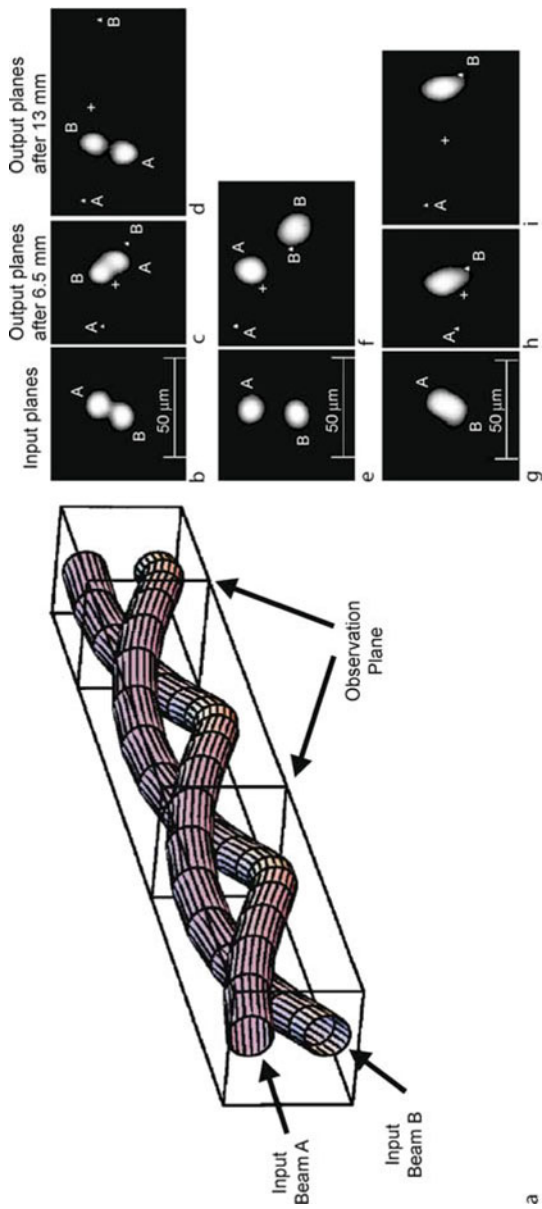
Attractive interactions of incoherent solitons have made possible the observation of spiraling solitons in saturable media (Shih et al. 1997). The two solitons of equal power orbit periodically about each other. Such a pair conserves angular momentum and is more or less stable for a certain

range of the initial orientation and distance between the beams. Although the underlying saturable nonlinearities are described by non-integrable equations, the spiraling process does not lead to measurable energy radiation (Buryak et al. 1999). When the initial distance between the solitons is increased, they do not form a long-living interacting system. The double spiral for beams located initially very close to each other converges fast and the counterparts eventually fuse. The pair apparently has a limited lifetime. A likely reason for its fusion or off-spiraling is the potential coherence of the light in the beams.

Interactions of Vector Solitons

Vector (composite) solitons consist of two or more components (also called modes) that mutually self-trap in a nonlinear medium. The simplest vector solitons, first suggested by Manakov (1974), consist of two orthogonally polarized components in a nonlinear Kerr medium in which self-phase modulation is identical to cross-phase modulation. They are described by an integrable system and interact elastically. An appropriate nonlinear material and experimental conditions for their demonstration were developed a long time after their discovery (Kang et al. 1996). Similar solitons may exist if each field component is at a different frequency, and the frequency difference between components is much larger than the inverse of the nonlinearity relaxation time (Shalaby and Barthelemy 1992; Trillo et al. 1988), or if the field components are incoherent with one another (Chen et al. 1996; Christodoulides et al. 1996). The energy exchange between the components of the interacting vector solitons takes place without radiative losses (Anastassiou et al. 1999).

Vector solitons can also consist of different modes of their jointly induced waveguide (Christodoulides and Joseph 1988; Tratnik and Sipe 1988). These multimode solitons may possess multiple humps, may contain both bright and dark components (Christodoulides 1988; Trillo et al. 1988), may have a certain internal dynamics, and were found to be stable (or weakly unstable) in large regions of their parameter space (Mitchell et al. 1998; Ostrovskaya et al. 1999a, b).



Solitons Interactions, Fig. 11 (a) The experimentally observed soliton spiraling process. The arrows indicate the initial trajectories. (b, e, and g) show different input conditions and (c, f, and h) are the outputs after 6.3 mm and d and i after 13 mm. The triangles indicate the centers of the corresponding diffracting beams. (From Stegeman and Segev (1999))

Interactions between vector solitons have, additionally to the above-discussed generic properties of solitons interactions (e.g. Leo and Assanto (1997)), also some unique features. The shape transformations of colliding multimode solitons can lead to two different multimode solitons emerging from the collision process (Akhmediev et al. 1998; Kanna and Lakshmanan 2003; Krolikowski et al. 1999). This “polarization switching” was predicted initially for Manakov solitons (Radhakrishnan et al. 1997).

Composite multimode multihump solitons may carry topological charge in one of the vector components (Carmon et al. 2000; Musslimani et al. 2000). This charge carried by a soliton of finite energy can be interpreted as spin of real particles. Elastic collisions of such 3D solitons should, ideally, conserve not only energy, and linear and angular momentum (Pigier et al. 2001), but also the equivalent of spin. Two colliding composite solitons carrying opposite spins may form a metastable state that later decays into two or three new solitons. If the solitons interact under a certain critical angle, angular momentum is transferred from spin to orbital angular momentum. Finally, the shape transformation of the vortex component occurs at large collision angles, for which scalar solitons of all types simply go through one another unaffected or only have phase shifts (Musslimani et al. 2001a; Musslimani et al. 2001b). The recent progress in optical vortices, vortex solitons, and their interactions is reviewed in (Desyatnikov et al. 2005).

Applications of Line Soliton Interactions

A generic use of soliton interactions in every case where solitary waves or topological solitons may exist, follows from the definition of a soliton. It consists of establishing if the structure is a solitonic one or not from the properties of their interactions.

The practical use of specific features of soliton interactions was started largely in parallel with the first technologies based on the use of solitons (e.g. optical soliton-based communications) more than a decade ago. The most promising is

the technology of “Optical Computing” (see the relevant entry), which may be partially based on the interactions of vector solitons (Steiglitz 2001) and which may open completely new horizons in the architecture of computers.

Soliton Interactions on a Water Surface

Several applications based on properties of elastic soliton interactions (or changes of certain fields during such interactions) have been proposed in the framework of water waves. From the purely geometrical properties of oblique interactions of plane solitons one may extract, e.g., the water wave height. The relations for the phase shifts and for the intersection angle can be reduced to the following transcendental equation:

$$\delta_1 \sqrt{2a_1(1 + \lambda^2/4)} = \pm \ln \frac{\delta_2^2 \lambda^2 - 2(\delta_2 - \delta_1)^2 a_1^2}{\delta_2^2 \lambda^2 - 2(\delta_2 + \delta_1)^2 a_1^2}. \quad (10)$$

If the intersection angle $\alpha_{1,2}$, the phase shifts $\delta_{1,2}$ and their signs (or the wave propagation directions) are known, Eq. (10) uniquely defines the heights of the interacting solitons (Peterson and van Groesen 2000, 2001).

The phenomenon of drastic increase of surface elevation and slope of wave fronts described in section “Geometry of Oblique Interactions of KP Line Solitons” can be attributed to formation of “freak” or “giant” waves in the ocean, the height and steepness of which are considerably larger than expected based on the classical wave statistics (Kharif and Pelinovsky 2003). Their sudden appearance is a feature of paramount importance to and a generic source of danger for navigation and in coastal and offshore engineering. Interactions of envelope (Schrödinger) solitons and breather solutions of the NLS (Osborne et al. 2000) in deep water are their potential source, although the model based on envelope soliton interactions underestimates the probability of large wave formation compared with the fully nonlinear model (Clamond and Grue 2002). Interacting solitary wave groups that emerge from a long wave packet can produce freak wave events and may lead to a threefold increase of the wave slope (Clamond et al. 2006).

The problem of extremely large-amplitude internal waves is by no means less important, as they can pose acute (currently neither well understood nor accounted for) danger for submarines, oil and gas platforms, oil risers and pipelines, and to other engineering constructions in relatively deep water. The problem of “freak” internal waves and their generation due to internal soliton interactions has only recently attracted the attention of researchers (Kurkin and Pelinovsky 2004).

There is a clear potential in the use of the advances concerning the geometry of the line soliton interactions in studies of both internal and shallow water solitons. While phase shifts usually have no dynamical significance, unexpectedly large elevations, extreme slopes, or changes in orientation of wave crests owing to the (Mach) reflection or oblique interactions of solitonic waves frequently cause an acute danger. The particularly high (stem) waves may lead to hits by high waves arriving from an unexpected direction. They may break during propagation into shallow water and attack armor blocks from an unprotected side (Mase et al. 2002) or overtop the breakwaters in unexpected locations (Yoon and Liu 1989). These effects may be particularly pronounced in the case of ship wakes that often approach seawalls or breakwaters from other directions than wind waves do (Soomere 2006). The possibility of drastic steepening of the front of the near-resonant structure is immaterial in many physical systems but is a crucial component of danger from freak waves (Kharif and Pelinovsky 2003), since specifically steep and high waves present an acute danger (e.g. Toffoli et al. (2005)). Even if the steepness of the wave front is not large in absolute terms, relatively steep and long, tsunami-like waves tend to become asymmetric and exhibit unexpectedly large run up heights (Didenkulova et al. 2006).

The amplitude amplification, however, evidently becomes effective relatively seldom, because two or more systems of long-crested solitonic waves must approach a certain area from different directions, and extreme elevations and slopes only occur if the amplitudes of the interacting solitons, the angle between their crests and the water depth are specifically balanced. The

long life-time of the resulting wave hump in favorable conditions (Kharif and Pelinovsky 2003) may drastically increase the probability of the occurrence of abnormally high waves.

Ship-Induced Solitons and Wave Resistance

An increasing source of solitonic waves is the fast ship traffic in relatively shallow areas. The generation of shallow-water solitons is most effective for large ships sailing at speeds roughly equal to the maximum phase speed $\sqrt{gh}$ of surface gravity waves (Li and Sclavounos 2002). The low decay rates and exceptional compactness of the solitonic ship wakes has led to a significant impact on the safety of people, property and craft, unusually high hydrodynamic loads in a part of the near-shore, and to a considerable remote impact of the ship traffic in shallow areas (Parnell and Kofoed-Hansen 2001; Soomere 2005). Such a wake has probably caused a fatal accident as far as about 10 km from the sailing line already in 1912 (Soomere 2007). The interactions of ship-induced solitons may lead to dangerous waves in the vicinity of coastal fairways and harbor entrances in otherwise sheltered areas (Peterson et al. 2003).

A intriguing use of soliton interactions, combining effects occurring during an elastic oblique interaction of shallow-water solitons, followed by an annihilation of solitons and forced antisolitons, has been made for reducing the wave resistance in channels (Chen et al. 2003a) and for catamaran design (Chen et al. 2003b). In a channel, the bow wave reflected from a side wall is used to cancel the stern wave, whereas the bow wave of one hull of the catamaran is used to cancel the stern wave of the other hull. Its practical use is possible for ships sailing at near-critical speeds, the wake of which may consist of a single soliton. The adequate calculation of the phase shift occurring during its Mach reflection (equivalently, during the interaction of two bow solitons) is a key component in achieving a perfect annihilation of properly timed bow solitons and the sterns' waves of depression.

The use of this effect by adjusting the channel width, the water depth and the ship's speed probably is the first intentional use of the features occurring during soliton interactions in the design of a

certain technology (Chen and Sharma 1994, 1997). At the exact design conditions the reflected bow wave completely cancels the stern wave and leads to a sort of channel superconductivity (Chen et al. 2003a). The same effect may occur between two ships moving in parallel (Jiankang et al. 2001).

Finally, it is interesting to note that already J.S. Russell may have been aware of this possibility. He describes efforts of a spirited horse which, pulling a boat in a canal, had drawn the boat up into its own wave leading to a significant reduction in resistance. The boat owner had noted this, and the event led to 'high-speed' service on some canals in the 1820s and 1830s (Russell 1837). We shall probably never know if the decrease of wave resistance occurred due to simple crossing of the critical speed or owing to the channel superconductivity; however, even the remote possibility that the practical use of soliton interactions may have been reported even before the first report about solitons, is remarkable.

Future Directions

Although the theory of soliton interactions has been extensively developed during four decades and its basic features for low-dimensional environments have been well understood, it is still in the stage of rapid development. Its first practical applications appeared only a decade ago and there is evidently room for relevant developments in (geo)physical applications. Analysis of long-term evolution of ensembles of solitons is still a challenge and reveals ever new interesting features even in the simplest soliton-admitting systems such as the KdV equation. Although several formal procedures of building explicit multi-soliton solutions to many equations have been known since the 1970s, studies of properties of (optionally resonant) interactions of several line solitons only started at the turn of the millennium. Many fascinating features of elastic interactions of solitons of different kinds in more complex systems will obviously become evident in the nearest future. The results in this direction may largely widen understanding of the integrability of the underlying equations.

Extremely rapid progress may be expected in studies into properties of long-lived soliton-like structures, in particular, of optical spatial solitons. The classification of their nomenclature and interactions is far from being completed. This environment is the key framework for identifying the new features of soliton interactions in higher dimensions, even though interactions of such structures are not necessarily elastic.

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Introduction to Solitons

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The concept of solitons reflects one of the most important developments in science at the second half of the twentieth century: the nonlinear description of the world. It is not easy to give a comprehensive and precise definition of a soliton. Frequently, a *soliton* is explained as a spatially localized wave in a medium that can interact strongly with other solitons but will afterward regain its original form.

It is a nonlinear pulse-like wave that can exist in some nonlinear systems. The soliton wave can propagate without dispersing its energy over a large region of space; collision of two solitons leads to unchanged forms; and solitons also exhibit particle-like properties.

The most remarkable property of solitons is that they do not disperse and thus conserve their form during propagation and collision (► [“Solitons Interactions”](#)).

The nonlinear science grows since approximately fifty years. However, numerous nonlinear processes had been previously identified, but the nonlinear mathematical tools were not developed. The available tools were linear, and nonlinearities were avoided or treated as perturbations of linear theories.

The first experimental observation of a solitary wave was made in August 1834 by a Scottish engineer named John Scott Russell (1808–1882). Scott-Russell reported his observation in the British Association Reports [71] as follows:

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving

it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

This hump-shape localized wave that propagates along one space-direction with undeformed shape has spectacular stability properties. John Scott Russell carried out many experiments to get the properties of this wave (► [“Solitons: Historical and Physical Introduction”](#)).

A model equation describing the unidirectional propagation of long waves in water of relatively shallow depth with a localized solution (representing a single hump as discovered by Russell) was obtained by Korteweg and de Vries. This equation has become very famous and is now known as the Korteweg-de Vries equation or KdV equation (► [“Water Waves and the Korteweg-de Vries Equation”](#)).

There exists certain class of nonlinear partial differential equations that leads to solitons. Korteweg de Vries (KdV) equation, Kadomtsev-Petviashvili (KP) equation, Klein-Gordon (KG) equation, Sine-Gordon (SG) equation, nonlinear Schrödinger (NLS) equation, Korteweg-de Vries Burger’s (KdVB) equation, etc... are some known equations that lie in this specific class of this NLPDE. This class of nonlinear equations has a special type of the traveling wave solutions which are either solitary waves or solitons (► [“Partial Differential Equations that Lead to Solitons”](#)).

The Korteweg-de Vries equation (KdV) is a canonical equation in nonlinear wave physics demonstrating the existence of solitons and their elastic interactions. This KdV equation attracted many authors to publish an enormous number of

publications. Different analytical methods for solving this well-known equation are presented in (► [“Different Analytical Methods for Solving the Korteweg-de Vries Equation \(KdV\)”](#)).

One of the most famous analytical methods for solving the KdV equation is the *Inverse Scattering Transformation*. This method was first introduced by the well-known scientists Gardner, Green, Kruskal, and Miura. In 1967, Gardner et al. published a magnificent and original paper on the analytical solution to the initial value problem due to a disturbance of finite amplitude in an infinite domain. This systematic method is not easy and needs several long steps to get the complete solution (► [“Inverse Scattering Transform and the Theory of Solitons”](#)).

There are various algebraic and geometric characteristics that the KdV equation possesses. More significantly, a lot of physically important solutions to the KdV equation can be presented explicitly through a simple, specific form, called *the Hirota bilinear form*. More complicated types of solutions leading to N-solitons are presented in (► [“History, Exact N-Soliton Solutions and Further Properties of the Korteweg-de Vries Equation \(KdV\)”](#)).

It was suitable to search an easier method to solve this famous and well-known nonlinear partial differential equation (KdV). Many semi-analytical methods have been developed and used to solve the KdV equation. The most popular and famous semianalytical method that was introduced to solve this nonlinear partial differential equation (KdV) was the *Adomian Decomposition Method* (“Adomian Decomposition Method Applied to Nonlinear Evolution Equations in Solitons Theory”).

Other semianalytical methods like variational iteration method (VIM), homotopy analysis method (HAM), and homotopy perturbation method (HPM) are usable to solve these types of equations – the NLPDEs Korteweg de Vries equation (KdV) and even the modified Korteweg-de Vries equation (mKdV) (► [“Semi-analytical Methods for Solving the KdV and mKdV Equations”](#)).

It is indispensable to introduce other methods to illustrate the solution of this NLPDE. The numerical technique is a powerful tool which is

always needed to present and compare the results in an easiest manner. There exist different numerical tools for solving the KdV and mKdV equations, and even for any evolution equations (► [“Some Numerical Methods for Solving the Korteweg-de Vries Equation \(KdV\)”](#)).

In hydrodynamical problems, the motion of incompressible inviscid fluid subject to a constant vertical gravitational force (where the fluid is bounded below by an impermeable bottom and above by a free surface) with some physical assumptions leads to the shallow water model. The shallow water wave equations are usually used in oceanography and atmospheric science. The shallow water approximation theory leads to the KdV equation. This means that the soliton solution as well as the solitary waves is the straightforward solution for many physical problems (► [“Shallow Water Waves and Solitary Waves”](#)).

The study of the water waves in the Pacific Ocean leads directly to a special type of long gravity waves named *tsunamis*. This word is a Japanese one, which means harbor’s wave.

Tsunami is essentially a long wavelength water wave train or a series of waves generated in a body of water (mostly in oceans) that vertically displaces the water column. Earthquakes, landslides, volcanic eruptions, nuclear explosions, and impact of cosmic bodies can generate tsunamis. Tsunamis, as they approach coastlines, can rise enormously and savagely attack and inundate to cause huge damage on properties and thousands of lives.

The theoretical study of waves in oceanography is very complicated. The simplest mathematical model includes the KdV equation. This KdV represents the governing equation that leads to tsunamis (► [“Applications of Solitons, Tsunamis and Oceanographical”](#)).

The rapid change in water density with depth is called *Pycnocline*. This happens in open or closed seas, as well as in oceans. In the frame of shallow water, approximation’s theory governs the oceans, and due to the pycnocline, the nonlinear internal gravity waves are presented. They arise from perturbations to hydrostatic equilibrium, where balance is maintained between the force of gravity and the buoyant restoring force. An illustrative study of nonlinear waves (that were generated

inside a stratified fluid occupying a semi-infinite channel of finite and constant depth by a wave maker situated in motion at the finite extremity of the channel) is presented in this section (► [“Nonlinear Internal Waves”](#)).

The soliton perturbation theory is used to study solitons that are governed by the various nonlinear equations in presence of the perturbation terms (► [“Soliton Perturbation”](#)).

A *compacton* is a special solitary traveling wave that, unlike a soliton, does not have exponential tails.

Another expression and terminology called *compacton-like soliton* is a special wave solution

which can be expressed by the squares of sinusoidal or cosinusoidal functions.

Soliton and compacton are two kinds of nonlinear waves. They play an indispensable and vital role in all branches of science and technology and are used as constructive elements to formulate the complex dynamical behavior of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasmas to shock waves, from tornados to the Great Red Spot of Jupiter, and from tsunamis to turbulence. More recently, soliton and compacton are of key importance in the quantum fields and nanotechnology especially in nanohydrodynamics (► [“Solitons and Compactons”](#)).

Applications of Solitons, Tsunamis and Oceanographical

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Glossary

Soliton A class of nonlinear dispersive wave equations in (1+1) dimensions having a delicate balance between dispersion and nonlinearity admit localized solitary waves which under interaction retain their shapes and speeds asymptotically. Such waves are called solitons because of their particle like elastic collision property. The systems include Korteweg–de Vries, nonlinear Schrödinger, sine-Gordon and other nonlinear evolution equations. Certain (2+1) dimensional generalizations of these systems also admit soliton solutions of different types (plane solitons, algebraically decaying lump solitons and exponentially decaying dromions).

Shallow and deep water waves Considering surface gravity waves in an ocean of depth h , they are called shallow-water waves if $h \ll \lambda$, where λ is the wavelength (or from a practical point of view if $h < 0.07\lambda$). In the linearized

case, for shallow water waves the phase speed $c = \sqrt{gh}$, where g is the acceleration due to gravity. Water waves are classified as deep (practically) if $h > 0.28\lambda$ and the corresponding wave speed is given by $c = \sqrt{g/k}$, $k = \frac{2\pi}{\lambda}$.

Tsunami Tsunami is essentially a long wavelength water wave train, or a series of waves, generated in a body of water (mostly in oceans) that vertically displaces the water column. Earthquakes, landslides, volcanic eruptions, nuclear explosions and impact of cosmic bodies can generate tsunamis. Propagation of tsunamis is in many cases in the form of shallow water waves and sometimes can be of the form of solitary waves/solitons. Tsunamis as they approach coastlines can rise enormously and savagely attack and inundate to cause devastating damage to life and property.

Internal solitons Gravity waves can exist not only as surface waves but also as waves at the interface between two fluids of different density. While solitons were first recognized on the surface of water, the commonest ones in oceans actually happen underneath, as internal oceanic waves propagating on the pycnocline (the interface between density layers). Such waves occur in many seas around the globe, prominent among them being the Andaman and Sulu seas.

Rossby solitons Rossby waves are typical examples of quasigeostrophic dynamical response of rotating fluid systems, where long waves between layers of the atmosphere as in the case of the Great Red Spot of Jupiter or in the barotropic atmosphere are formed and may be associated with solitonic structures.

Bore solitons The classic bore (also called mascaret, poroca and aeger) arises generally in funnel shaped estuaries that amplify incoming tides, tsunamis or storm surges, the rapid rise propagating upstream against the flow of the river feeding the estuary. The profile depends on the Froude number, a dimensionless ratio of inertial and gravitational effects. Slower bores can take on oscillatory profile

with a leading dispersive shockwave followed by a train of solitons.

Definition of the Subject

Surface and internal gravity waves arising in various oceanographic conditions are natural sources where one can identify/observe the generation, formation and propagation of solitary waves and solitons. Unlike the standard progressive waves of linear dispersive type, solitary waves are localized structures with long wavelengths and finite energies and propagate without change of speed or form and are patently nonlinear entities. The earliest scientifically recorded observation of a solitary wave was made by John Scott Russel in August 1834 in the Union Canal connecting the Scottish cities of Glasgow and Edinburgh. The theoretical formulation of the underlying phenomenon was provided by Korteweg and de Vries in 1895 who deduced the now famous Korteweg–de Vries (KdV) equation admitting solitary wave solutions. With the insightful numerical and analytical investigations of Martin Kruskal and coworkers in the 1960s the KdV solitary waves have been shown to possess the remarkable property that under collision they pass through each other without change of shape or speed except for a phase shift and so they are solitons. Since then a large class of soliton possessing nonlinear dispersive wave equations such as the sine-Gordon, modified KdV (MKdV) and nonlinear Schrödinger (NLS) equations occurring in a wide range of physical phenomena have been identified.

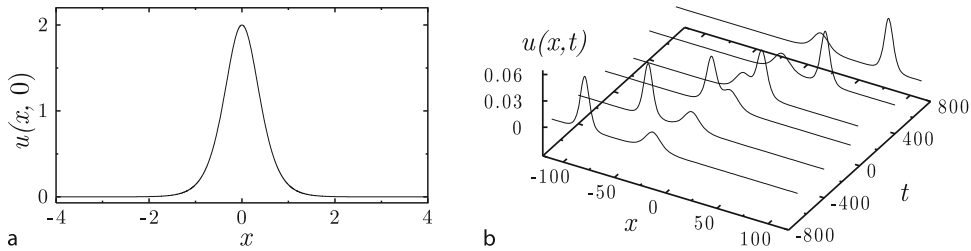
Several important oceanographic phenomena which correspond to nonlinear shallow water wave or deep water wave propagation have been identified/interpreted in terms of soliton propagation. These include tsunamis, especially earthquake induced ones like the 1960 Chilean or 2004 Indian Ocean earthquakes, internal solitary waves arising in stratified stable fluids such as the ones observed in Andaman or Sulu seas, Rossby waves including the Giant Red Spot of Jupiter and tidal bores occurring in estuaries of rivers. Detailed observations/laboratory experiments

and theoretical formulations based on water wave equations resulting in the nonlinear evolution equations including KdV, Benjamin–Ono, Intermediate Long Wave (ILW), Kadomtsev–Petviashvili (KP), NLS, Davey–Stewartson (DS) and other equations clearly establish the relevance of soliton description in such oceanographic events.

Introduction

Historically, the remarkable observation of John Scott Russel (1844; Bullough 1988) of the solitary wave in the Union Canal connecting the cities of Edinburgh and Glasgow in the month of August 1834 may be considered as the precursor to the realization of solitons in many oceanographic phenomena. While riding on a horse back and observing the motion of boat drawn by a pair of horses which suddenly stopped but not so the mass of water it had set in motion, the wave (which he called the ‘Great Wave of Translation’) in the form of large solitary heap of water surged forward and travelled a long distance without *change of form or diminution of speed*. The wave observed by Scott Russel is nothing but a solitary wave having a remarkable staying power and a patently nonlinear entity. Korteweg and de Vries in 1895, starting from the basic equations of hydrodynamics and considering unidirectional shallow water wave propagation in rectangular channels, deduced (Korteweg and de Vries 1895) the now ubiquitous KdV equation as the underlying nonlinear evolution equation. It is a third order nonlinear partial differential equation in (1+1) dimensions with a delicate balance between dispersion and nonlinearity. It admits elliptic function cnoidal wave solutions and in a limiting form exact solitary wave solution of the type observed by John Scott Russel thereby vindicating his observations and putting to rest all the controversies surrounding them.

It was the many faceted numerical and analytical study of Martin Kruskal and coworkers (Zabusky and Kruskal 1965; Gardner et al. 1967) which firmly established by 1967 that the KdV solitary waves have the further remarkable



Applications of Solitons, Tsunamis and Oceanographical, Fig. 1 KdV equation: (a) One soliton solution (solitary wave) at a fixed time, say $t = 0$, (b) Two soliton solution (depicting elastic collision)

feature that they are solitons having elastic collision property (Fig. 1).

It was proved decisively that the KdV solitary waves on collision pass through each other except for a finite phase shift, thereby retaining their forms and speeds asymptotically as in the case of particle like elastic collisions. The inverse scattering transform (IST) formalism developed for this purpose clearly shows that the KdV equation is a completely integrable infinite dimensional nonlinear Hamiltonian system and that it admits multisoliton solutions (Lakshmanan and Rajasekar 2003; Ablowitz and Clarkson 1991; Ablowitz and Segur 1981; Scott 1999). Since then a large class of nonlinear dispersive wave equations such as the sine-Gordon (s-G), modified Korteweg–de Vries (MKdV), NLS, etc. equations in (1+1) dimensions modeling varied physical phenomena have also been shown to be completely integrable soliton systems (Lakshmanan and Rajasekar 2003; Ablowitz and Clarkson 1991; Ablowitz and Segur 1981; Scott 1999). Interesting (2+1) dimensional versions of these systems such as Kadomtsev–Petviashvili (KP), Davey–Stewartson (DS) and Nizhnik–Novikov–Veselov (NNV) equations have also been shown to be integrable systems admitting basic nonlinear excitations such as line (plane) solitons, algebraically decaying lump solitons and exponentially localized dromion solutions (Ablowitz and Clarkson 1991; Ablowitz and Segur 1981).

It should be noted that not all solitary waves are solitons while the converse is always true. An example of a solitary wave which is not a soliton is the one which occurs in double-well $\lambda\phi^4$ wave equation which radiates energy on collision with another such wave. However even such solitary waves having finite energies are sometimes

referred to as solitons in the condensed matter, particle physics and fluid dynamics literature because of their localized structure.

As noted above solitary waves and solitons are abundant in oceanographic phenomena, especially involving shallow water wave and deep water wave propagations. However, these events are large scale phenomena very often difficult to measure experimentally, rely mostly on satellite and other indirect observations and so controversies and differing interpretations do exist in ascribing exact solitonic or solitary wave properties to these phenomena. Yet it is generally realized that many of these events are closely identifiable with solitonic structures. Some of these observations deserve special attention.

Tsunamis

When large scale earthquakes, especially of magnitude 8.0 in Richter scale and above, occur in seabeds at appropriate geological faults tsunamis are generated and can propagate over large distances as small amplitude and long wavelength structures when shallowness condition is satisfied. Though the tsunamis are hardly felt in the midsea, they take monstrous structures when they approach land masses due to conservation of energy. The powerful Chilean earthquake of 1960 (Dudley and Miu 1988) led to tsunamis which propagated for almost fifteen hours before striking Hawaii islands and a further seven hours later they struck the Japanese islands of Honshu and Hokkaido. More recently the devastating Sumatra–Andaman earthquake of 2004 in the Indian Ocean generated tsunamis which not only struck the coastlines of Asian countries including Indonesia, Thailand, India and Sri Lanka but

propagated as far as Somalia and Kenya in Africa, killing more than a quarter million people. There is very good sense in ascribing soliton description to such tsunamis.

Internal Solitons

Peculiar striations, visible on satellite photographs of the surface of the Andaman and Sulu seas in the far east (and in many other oceans around the globe), have been interpreted as secondary phenomena accompanying the passage of “internal solitons”. These are solitary wavelike distortions of the boundary layer between the warm upper layer of sea water and cold lower depths. These internal solitons are travelling ridges of warm water, extending hundreds of meters down below the thermal boundary, and carry enormous energy. Osborne and Burch (1980) investigated the underwater currents which were experienced by an oil rig in the Andaman sea, which was drilling at a depth of 3600 ft. One drilling rig was apparently spun through ninety degrees and moved one hundred feet by the passage of a soliton below.

Rossby Waves and Solitons

Rossby waves (Rossby 1939) are long waves between layers of the atmosphere, created by the rotation of the planet. In particular, in the atmosphere of a rotating planet, a fluid particle is endowed with a certain rotation rate, determined by its latitude. Consequently its motion in the north-south direction is constrained by the conservation of angular momentum as in the case of internal waves where gravity inhibits the vertical motion of a density stratified fluid. There is an analogy between internal waves and Rossby waves under suitable conditions. The KdV equation has been proposed as a model for the evolution of Rossby solitons (Redekopp 1977) and NLS equation for the evolution of Rossby wave packets. The Great Red Spot of the planet of Jupiter is often associated with a Rossby soliton.

Bore Solitons

Rivers which flow to the open oceans are very often affected by tidal effects, tsunamis or storm surges. Tidal motions generate intensive water

flows which can propagate upstream on tens of kilometers in the form of step-wise perturbation (hydraulic jumps) analogous to shock waves in acoustics. This phenomenon is known as a bore or mascaret (in French). Examples of such bores include the tidal bore of Seine river in France, Hooghly bore of the Ganges in India, the Amazon river bore in Brazil and Hangzhou bore in China. Bore disintegration into solitons is a possible phenomenon in such bores (Caputo and Stepanyants 2003).

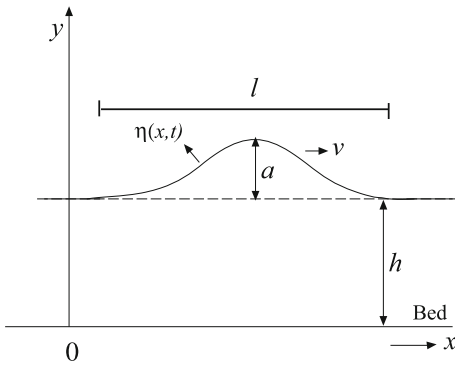
Besides these there are other possible oceanographic phenomena such as capillary wave solitons (Longuet-Higgins 1993), resonant three-wave or four wave interaction solitons (Philips 1974), etc., where also soliton picture is useful.

In this entry we will first point out how in shallow channels the long wavelength wave propagation is described by KdV equation and its generalizations (section “[Shallow Water Waves and KdV Type Equations](#)”). Then we will briefly point out how NLS family of equations arise naturally in the description of deep water waves (section “[Deep Water Waves and NLS Type Equations](#)”). Based on these details, we will point how the soliton picture plays a very important role in the understanding of tsunami propagation (section “[Tsunamis as Solitons](#)”), generation of internal solitons (section “[Internal Solitons](#)”), formation of Rossby solitons (section “[Rossby Solitons](#)”) and disintegration of bores into solitons (section “[Bore Solitons](#)”).

Shallow Water Waves and KdV Type Equations

Kortweg and de Vries (1895) considered the wave phenomenon underlying the observations of Scott Russel from first principles of fluid dynamics and deduced the KdV equation to describe the unidirectional shallow water wave propagation in one dimension.

Consider the one-dimensional (x -direction) wave motion of an incompressible and inviscid fluid (water) in a shallow channel of height h , and of sufficient width with uniform cross-section leading to the formation of a solitary wave propagating under gravity. The effect of surface



Applications of Solitons, Tsunamis and Oceanographical, Fig. 2 One-dimensional wave motion in a shallow channel

tension is assumed to be negligible. Let the length of the wave be l and the maximum value of its amplitude, $\eta(x, t)$, above the horizontal surface be a (see Fig. 2).

Then assuming $a \ll h$ (shallow water) and $h \ll l$ (long waves), one can introduce two natural small parameters into the problem $\epsilon = a/h$ and $\delta = h/l$. Then the analysis proceeds as follows (Korteweg and de Vries 1895; Lakshmanan and Rajasekar 2003).

Equation of Motion: KdV Equation

The fluid motion can be described by the velocity vector $\mathbf{V}(x, y, t) = u(x, y, t)\mathbf{i} + v(x, y, t)\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors along the horizontal and vertical directions, respectively. As the motion is irrotational, we have $\nabla \times \mathbf{V} = 0$. Consequently, we can introduce the velocity potential $\phi(x, y, t)$ by the relation $\mathbf{V} = \nabla \phi$.

Conservation of Density

The system obviously admits the conservation law for the mass density $\rho(x, y, t)$, of the fluid, $d\rho/dt = \rho_t + \nabla \cdot (\rho \mathbf{V}) = 0$. As ρ is a constant, we have $\nabla \cdot \mathbf{V} = 0$. Consequently ϕ obeys the Laplace equation

$$\nabla^2 \phi(x, y, t) = 0. \quad (1)$$

Euler's Equation

As the density of the fluid $\rho = \rho_0 = \text{constant}$, using Newton's law for the rate of change of

momentum, we can write $d\mathbf{V}/dt = \partial\mathbf{V}/\partial t + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho_0}\nabla p - g\mathbf{j}$, where $p = p(x, y, t)$ is the pressure at the point (x, y) and g is the acceleration due to gravity, which is acting vertically downwards (here $\mathbf{j}$ is the unit vector along the vertical direction). Since $\mathbf{V} = \nabla \phi$ we obtain (after one integration)

$$\phi_t + \frac{1}{2}(\nabla \phi)^2 + \frac{p}{\rho_0} + gy = 0. \quad (2)$$

Boundary Conditions

The above two Eqs. (1) and (2) for the velocity potential $\phi(x, y, t)$ of the fluid have to be supplemented by appropriate boundary conditions, by taking into account the fact (see Fig. 2) that (a) the horizontal bed at $y = 0$ is hard and (b) the upper boundary $y = y(x, t)$ is a free surface. As a result

- (a) the vertical velocity at $y = 0$ vanishes, $v(x, 0, t) = 0$, which implies

$$\phi_y(x, 0, t) = 0. \quad (3)$$

- (b) As the upper boundary is free, let us specify it by $y = h + \eta(x, t)$ (see Fig. 2). Then at the point $x = x_1, y = y_1 \equiv y(x, t)$, we can write $\frac{dy_1}{dt} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \cdot \frac{dx_1}{dt} = \eta_t + \eta_x u_1 = v_1$. Since $v_1 = \phi_{1y}$, $u_1 = \phi_{1x}$, we obtain

$$\phi_{1y} = \eta_t + \eta_x \phi_{1x}. \quad (4)$$

- (c) Similarly at $y = y_1$, the pressure $p_1 = 0$. Then from (2), it follows that

$$u_{1t} + u_1 u_{1x} + v_1 v_{1x} + g\eta_x = 0. \quad (5)$$

Thus the motion of the surface of water wave is essentially specified by the Laplace Eq. (1) and Euler Eq. (2) along with one fixed boundary condition (3) and two *variable nonlinear* boundary conditions (4) and (5). One has to then solve the Laplace equation subject to these boundary conditions.

Taylor Expansion of $\phi(x, y, t)$ in y

Making use of the fact $\delta = h/l \ll 1$, $h \ll l$, we assume $y(=h + \eta(x, t))$ to be small to introduce the Taylor expansion

$$\phi(x, y, t) = \sum_{n=0}^{\infty} y^n \phi_n(x, t). \quad (6)$$

Substituting the above series for ϕ into the Laplace Eq. (1), solving recursively for $\phi_n(x, t)$'s and making use of the boundary condition (4), $\phi_y(x, 0, t) = 0$, one can show that

$$\begin{aligned} u_1 &= \phi_{1x} \\ &= f - \frac{1}{2} y^2 f_{xx} + \text{higher order in } y_1, \end{aligned} \quad (7)$$

$$\begin{aligned} v_1 &= \phi_{1y} \\ &= -y_1 f_x + \frac{1}{6} y_1^3 f_{xxx} \\ &\quad + \text{higher order in } y_1, \end{aligned} \quad (8)$$

where $f = \partial\phi_0/\partial x$. We can then substitute these expressions into the nonlinear boundary conditions (4) and (5) to obtain equations for f and η .

Introduction of Small Parameters ϵ and δ

So far the analysis has not taken into account fully the shallow nature of the channel ($a/h = \epsilon \ll 1$) and the solitary nature of the wave ($a/l = a/h \cdot h/l = \epsilon\delta \ll 1$, $\epsilon \ll 1$, $\delta \ll 1$), which are essential to realize the Scott Russel phenomenon. For this purpose one can stretch the independent and dependent variables in the above equations through appropriate scale changes, but retaining the overall form of the equations. To realize this one can introduce the natural scale changes

$$x = lx', \eta = a\eta', t = \frac{l}{c_0} t', \quad (9)$$

where c_0 is a parameter to be determined. Then in order to retain the form of (7) and (8) we require

$$\begin{aligned} u_1 &= \epsilon c_0 u'_1, v_1 = \epsilon \delta c_0 v'_1, f = \epsilon c_0 f', \\ y_1 &= h + \eta(x, t) = h(1 + \epsilon \eta'(x', t')). \end{aligned} \quad (10)$$

Then

$$\begin{aligned} u'_1 &= f' - \frac{1}{2} \delta^2 (1 + \epsilon \eta')^2 f'_{x'x'} \\ &= f' - \frac{1}{2} \delta^2 f'_{x'x'}, \end{aligned} \quad (11)$$

where we have omitted terms proportional to $\delta^2 \epsilon$ as small compared to terms of the order δ^2 . Similarly from (8), we obtain

$$v'_1 = -(1 + \epsilon \eta') f'_{x'} + \frac{1}{6} \delta^2 f'_{x'x'x'}. \quad (12)$$

Now considering the nonlinear boundary condition (4) in the form $v_1 = \eta_t + \eta_x u_1$, it can be rewritten as

$$\eta'_{t'} + f'_{x'} + \epsilon \eta' f'_{x'} + \epsilon f' \eta'_{x'} - \frac{1}{6} \delta^2 f'_{x'x'x'} = 0. \quad (13)$$

Similarly considering the other boundary condition (5) and making use of the above transformations, it can be rewritten, after neglecting terms of the order $\epsilon^2 \delta^2$, as

$$f'_{t'} + \epsilon f' f'_{x'} + \frac{ga}{\epsilon c_0^2} \eta'_{x'} - \frac{1}{2} \delta^2 f'_{x'x'x'} = 0. \quad (14)$$

Now choosing the arbitrary parameter c_0 as $c_0^2 = gh$ so that $\eta'_{x'}$ term is of order unity, (14) becomes

$$f'_{t'} + \eta'_{x'} + \epsilon f' f'_{x'} - \frac{1}{2} \delta^2 f'_{x'x'x'} = 0. \quad (15)$$

(Note that $c_0 = \sqrt{gh}$ is nothing but the speed of the water wave in the linearized limit). Omitting the primes for convenience, the evolution equation for the amplitude of the wave and the function related to the velocity potential reads

$$\eta_t + f_x + \epsilon \eta f_x + \epsilon f \eta_x - \frac{1}{6} \delta^2 f_{xxx} = 0, \quad (16)$$

$$f_t + \eta_x + \epsilon f f_x - \frac{1}{2} \delta^2 f_{xxt} = 0. \quad (17)$$

Note that the small parameters ϵ and δ^2 have occurred in a natural way in (16) and (17).

Perturbation Analysis

Since the parameters ϵ and δ^2 are small in (16) and (17), we can make a perturbation expansion of f in these parameters:

$$f = f^{(0)} + \epsilon f^{(1)} + \delta^2 f^{(2)} + \text{higher order terms}, \quad (18)$$

where $f^{(i)}$, $i = 0, 1, 2, \dots$ are functions of η and its spatial derivatives. Note that the above perturbation expansion is an asymptotic expansion. Substituting this into Eqs. (16) and (17) and regrouping and comparing different powers proportional to (ϵ, δ^2) and solving them successively one can obtain (see for example (Lakshmanan and Rajasekar 2003) for further details) in a self consistent way,

$$f^{(0)} = \eta, \quad f^{(1)} = -\frac{1}{4} \eta^2, \quad f^{(2)} = \frac{1}{3} \eta_{xx}. \quad (19)$$

Using these expressions into (18) and substituting it in (16) and (17), we ultimately obtain the KdV equation in the form

$$\eta_t + \eta_x + \frac{3}{2} \epsilon \eta \eta_x + \frac{\delta^2}{6} \eta_{xxx} = 0, \quad (20)$$

describing the unidirectional propagation of long wavelength shallow water waves.

The Standard (Contemporary) Form of KdV Equation

Finally, changing to a moving frame of reference, $\xi = x - t$, $\tau = t$, and introducing the new variables $u = (3\epsilon/2\delta^2)\eta$, $\tau' = (6/\delta^2)\tau$ and redefining the variables τ' as t and ξ as x , we finally arrive at the ubiquitous form of the KdV equation as

$$u_t + 6uu_x + u_{xxx} = 0. \quad (21)$$

The Korteweg–de Vries Eq. (21) admits cnoidal wave solution and in the limiting case solitary wave solution as well. This form can be easily obtained (Lakshmanan and Rajasekar 2003; Helal and Molines 1981) by looking for a wave solution of the form $u = 2f(\xi)$, $\xi = (x - ct)$, and reducing the KdV equation into a third order nonlinear ordinary differential equation of the form

$$-c \frac{\partial f}{\partial \xi} + 12f \frac{\partial f}{\partial \xi} + \frac{\partial^3 f}{\partial \xi^3} = 0. \quad (22)$$

Integrating Eq. (22) twice and rearranging, we can obtain

$$\begin{aligned} \left(\frac{\partial f}{\partial \xi} \right)^2 &= -4f^3 + cf^2 - 2df - 2b \\ &\equiv P(f), \end{aligned} \quad (23)$$

where b and d are integration constants. Calling the three real roots of the cubic equation $P(f) = 0$ as α_1 , α_2 and α_3 such that

$$\left(\frac{\partial f}{\partial \xi} \right)^2 = -4(f - \alpha_1)(f - \alpha_2)(f - \alpha_3), \quad (24)$$

the solution can be expressed in terms of the Jacobian elliptic function as

$$f(\xi) = f(x - ct) = \alpha_3 - (\alpha_3 - \alpha_2)sn^2 \cdot [\sqrt{\alpha_3 - \alpha_1}(x - ct), m], \quad (25a)$$

where

$$\begin{aligned} (\alpha_1 + \alpha_2 + \alpha_3) &= \frac{c}{4}, \\ m^2 &= \frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1}, \\ \delta &= \text{constant}. \end{aligned} \quad (25b)$$

Here m is the modulus parameter of the Jacobian elliptic function. Eq. (25a) represents in fact the so called cnoidal wave (because of its elliptic function form). In the limiting case $m = 1$, the form (25a) reduces to

$$f = \alpha_2 + (\alpha_3 - \alpha_2) \operatorname{sech}^2 \left[\sqrt{\alpha_3 - \alpha_1} (x - ct) \right]. \quad (26)$$

Choosing now $\alpha_1 = 0$, $\alpha_2 = 0$, and using (25b) we have

$$f = \frac{c}{4} \operatorname{sech}^2 \left[\frac{\sqrt{c}}{2} (x - ct) \right]. \quad (27)$$

Then the solitary wave solution to the KdV Eq. (21) can be written in the form

$$u(x, t) = \frac{c}{2} \operatorname{sech}^2 \frac{\sqrt{c}}{2} (x - ct + \delta), \quad (28)$$

δ : constant

Note that the velocity of the solitary wave is directly proportional to the amplitude: larger the wave the higher is the speed. More importantly, the KdV solitary wave is a soliton: it retains its shape and speed upon collision with another solitary wave of different amplitude, except for a phase shift, see Fig. 1 (Lakshmanan and Rajasekar 2003; Ablowitz and Clarkson 1991; Ablowitz and Segur 1981). In fact for an arbitrary initial condition, the solution of the Cauchy initial value problem consists of N-number of solitons of different amplitudes in the background of small amplitude dispersive waves. All these results ultimately lead to the result that the KdV equation is a completely integrable, infinite dimensional, nonlinear Hamiltonian system. It possesses (Lakshmanan and Rajasekar 2003; Ablowitz and Clarkson 1991; Ablowitz and Segur 1981)

- (i) a Lax pair of linear differential operators and is solvable through the so called inverse scattering transform (IST) method,
- (ii) infinite number of conservation laws and associated infinite number of involutive integrals of motion,
- (iii) N-soliton solution,
- (iv) Hirota bilinear form,
- (v) Hamiltonian structure

and a host of other interesting properties (see for example (Lakshmanan and Rajasekar 2003; Ablowitz and Clarkson 1991; Ablowitz and Segur 1981; Scott 1999)).

KdV Related Integrable and Nonintegrable NLEEs
Depending on the actual physical situation, the derivation of the shallow water wave equation can be suitably modified to obtain other forms of nonlinear dispersive wave equations in (1+1) dimensions as well as in (2+1) dimensions relevant for the present context. Without going into the actual derivations, some of the important equations possessing solitary waves are listed below (Lakshmanan and Rajasekar 2003; Ablowitz and Clarkson 1991; Ablowitz and Segur 1981; Scott 1999).

1. Boussinesq equation (Ablowitz and Clarkson 1991)

$$u_t + uu_x + g\eta_x - \frac{1}{3}h^2u_{xxx} = 0, \quad (29a)$$

$$\eta_t + [u(h + \eta)]_x = 0 \quad (29b)$$

2. Benjamin–Bona–Mahoney (BBM) equation (Benjamin et al. 1972).

$$u_t + u_x + uu_x - u_{xxt} = 0 \quad (30)$$

3. Camassa–Holm equation (Camassa and Holm 1992)

$$u_t + 2\kappa u_x + 3uu_x - u_{xxt} = 3u_x u_{xx} + uu_{xxx} \quad (31)$$

4. Kadomtsev–Petviashvili (KP) equation (Ablowitz and Clarkson 1991)

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \quad (32)$$

($\sigma^2 = -1$: KP-I, $\sigma^2 = +1$: KP-II).

In the derivation of the above equations, generally the bottom of water column or fluid bed is assumed to be flat. However in realistic situations the water depth varies as a function of the horizontal coordinates. In this situation, one often encounters inhomogeneous forms of the above wave equations. Typical example is the variable coefficient KdV equation (Caputo and Stepanyants 2003):

$$u_t + f(x, t)uu_x + g(x, t)u_{xxx} = 0, \quad (33)$$

where f and g are functions of x, t . More general forms can also be deduced depending upon the actual situations, see for example (Caputo and Stepanyants 2003).

Deep Water Waves and NLS Type Equations

Deep water waves are strongly dispersive in contrast to the weakly dispersive nature of the shallow water waves (in the linear limit). Various authors (see for details (Ablowitz and Segur 1981)) have shown that nonlinear Schrödinger family of equations models the evolution of a packet of surface waves in this case. There are several oceanographic situations where such waves can arise (Ablowitz and Segur 1981):

- (i) A localized storm at sea can generate a wide spectrum of waves, which then propagates away from the source region in all horizontal directions. If the propagating waves have small amplitudes and encounter no wind away from the source region, these waves can eventually sort themselves into nearly one-dimensional packets of nearly monochromatic waves. For appropriately chosen scales, the underlying evolution of each of these packets can be shown to satisfy the nonlinear Schrödinger equation and its generalizations in (2+1) dimensions.
- (ii) Nearly monochromatic, nearly one-dimensional waves can cover a broad range of surface waves in the sea that results due to a steady wind of long duration and fetch.

Then the generalization of the NLS equation in (2+1) dimensions can describe the waves that result in when the wind stops.

In all the above situations one looks for the solution of the equations of motion (1–5) but generalized in three dimensions which consists mainly in the form of a small amplitude, nearly monochromatic, nearly one-dimensional wave train. Assuming that this wave train travels in the x -direction with a mean wave number $\kappa = (k, l)$ with a characteristic amplitude ‘ a ’ of the disturbance and a characteristic variation δk in k , one can deduce the NLS equation in (2+1) dimensions under the following conditions:

- (i) small amplitudes such that $\hat{\epsilon} = \epsilon \delta \equiv \kappa a \ll 1$
- (ii) slowly varying modulations, $\frac{\delta k}{\kappa} \ll 1$
- (iii) nearly one dimensional waves, $\frac{|l|}{k} \ll 1$
- (iv) balance of all three effects, $\frac{\delta k}{\kappa} = \frac{|l|}{k} \approx O(\hat{\epsilon})$
- (v) for finite and deep water waves, $(kh)^2 \gg \hat{\epsilon}$

In the lowest order approximation (linear approximation) of water waves, the prediction is that a localized initial state will generally evolve into wave packets with a dominant wave number κ and corresponding frequency ω , given by the dispersion relation $\omega = (g\kappa + \sigma\kappa^3) \tanh \kappa h$, $\kappa = |\kappa| = \sqrt{k^2 + l^2}$ within which each wave propagates with the phase speed $c = \omega/k$, while the envelope propagates with the group velocity $c_g = d\omega/d\kappa$. After a sufficiently long time the wave packet tends to disperse around the dominant wave number.

This tendency for dispersion can be offset by cumulative nonlinear effects. In the absence of surface tension, the outcome for unidirectional waves can be shown to be describable by the NLS equation. If the surface wave in the lowest order is $\phi \approx A \exp i(kx - \omega t) + \text{c.c.}$, where ϕ is the velocity potential, then to leading order the wave amplitude evolves as

$$i(A_t + c_g A_x) + \frac{1}{2} \lambda A_{xx} + \mu |A|^2 A = 0. \quad (34)$$

The coefficients here are given by

$$\lambda = \frac{\partial^2 \omega}{\partial k^2}$$

$$\mu = -\frac{\omega k^2}{16S^4} (8C^2 S^2 + g - 2T^2) \quad (35)$$

$$+ \frac{\omega}{8C^2 S^2} \frac{(2\omega C^2 + kc_g)^2}{(gh - c_g^2)}$$

where $C = \cosh(kh)$, $S = \sinh(kh)$, $T = \frac{\delta}{C}$. Equation (34) has been obtained originally by Zakharov in 1968 for deep water waves (Zakharov 1968) and by Hasimoto and Ono for waves of finite depth in 1972 (Hasimoto and Ono 1972).

The NLS equation is also a soliton possessing integrable system and is solvable by the IST method (Lakshmanan and Rajasekar 2003; Ablowitz and Clarkson 1991; Ablowitz and Segur 1981; Scott 1999). For $\lambda > 0$ (focusing case), the envelope solitary (soliton) wave solution (also called bright solitons in the optical physics context) is given by

$$A(x, t) = a \operatorname{sech} \gamma(x - c_g t) \exp(-i\Omega t), \quad (36)$$

where $\mu a^2 = \lambda \gamma^2$, $\Omega = +\frac{1}{2}\mu a^2$.

When the effects of modulation in the transverse y -direction are taken into account, so that the wave amplitude is now given by $A(x, y, t)$, the NLS equation is replaced by the Benney–Roskes system (Benney and Roskes 1969) also popularly known as the Davey–Stewartson equations (Davey and Stewartson 1974),

$$i(A_t + c_g A_x) + \frac{1}{2} \lambda A_{xx} + \frac{1}{2} \delta A_{yy} + \mu |A|^2 A + UA = 0, \quad (37a)$$

$$\alpha U_{xx} + U_{yy} + \beta \left(|A|^2 \right)_{yy} = 0, \quad (37b)$$

where $\delta = \frac{c_g}{k}$, $\alpha = 1 - \left(\frac{c_g}{gh} \right)$, $gh\beta = \frac{\omega}{8C^2 S^2} (2\omega C^2 + kc_g)^2$. Here $U(x, y, t)$ is the wave induced mean flow. In the deep water wave limit, $kh \rightarrow \infty$, and $U \rightarrow 0$, $\beta \rightarrow 0$ and one has the nonintegrable (2+1) dimensional NLS

equation. On the other hand in the shallow water limit, one has the integrable Davey–Stewartson (DS) equations. For details see (Ablowitz and Clarkson 1991; Ablowitz and Segur 1981). The DS-I equation admits algebraically decaying lump solitons and exponentially decaying dromions (Fokas and Santini 1990) besides the standard line solitons for appropriate choices of parameters. A typical dromion solution of DS-I equation is shown in Fig. 3.

Tsunamis as Solitons

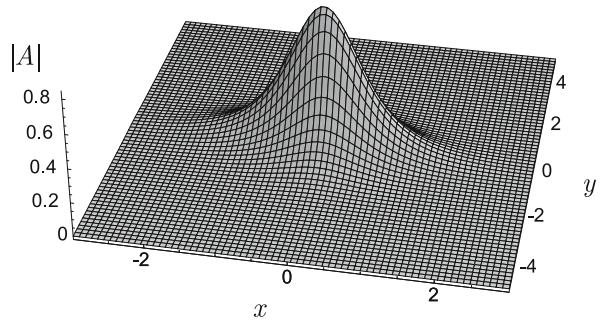
The term ‘tsunami’ (tsu:harbour, nami:wave in Japanese) which was perhaps an unknown word even for scientists in countries such as India, Srilanka, Thailand, etc. till recently has become a house-hold word since that fateful morning of December 26, 2004. When a powerful earthquake of magnitude 9.1–9.3 on the Richter scale, epic entered off the coast of Sumatra, Indonesia, struck at 07:58:53, local time, described as the 2004 or Sumatra–Andaman earthquake (Fig. 4), it triggered a series of devastating tsunamis as high as 30 meters that spread throughout the Indian Ocean killing about 275,000 people and inundating coastal communities across South and Southeast Asia, including parts of Indonesia, Srilanka, India and Thailand and even reaching as far as the east coast of Africa (Kundu 2007). The catastrophe is considered to be one of the deadliest disasters in modern history.

Since this earthquake and consequent tsunamis, several other earthquakes of smaller and larger magnitudes keep occurring off the coast of Indonesia. Even as late as July 17, 2006 an earthquake of magnitude 7.7 on the Richter scale struck off the town of Pandering at 15.19 local time and set off a tsunami of 2 m high which had killed more than 300 people.

These tsunamis, which can become monstrous tidal waves when they approach coastline, are essentially triggered due to the sudden vertical rise of the seabed by several meters (when earthquake occurs) which displaces massive volume of water. The tsunamis behave very differently in deep water than in shallow water as pointed out

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Fig. 3 Exponentially localized dromion solution of the Davey–Stewartson equation at a fixed time ($t = 0$) for suitable choice of parameters



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Fig. 4 26 December 2004 Indian Ocean tsunami. (Adapted from the website www.blogaid.org.uk with the courtesy of Andy Budd)



below. By no means the tsunami of 2004 and later ones are exceptional; More than two hundred tsunamis have been recorded in scientific literature since ancient times. The most notable earlier one is the tsunami triggered by the powerful earthquake (9.6 magnitude) off southern Chile on May 22, 1960 (Dudley and Miu 1988) which traveled almost 22 h before striking Japanese islands.

It is clear from the above events that the tsunami waves are fairly permanent and powerful ones, having the capacity to travel extraordinary distances without practically diminishing in size or speed. In this sense they seem to have considerable resemblance to shallow water nonlinear dispersive waves of KdV type, particularly solitary waves and solitons. It is then conceivable that tsunami dynamics has close connection with soliton dynamics.

Basics of Tsunami Waves

As noted above tsunami waves of the type described above are essentially triggered by

massive earthquakes which lead to vertical displacement of a large volume of water. Other possible reasons also exist for the formation and propagation of tsunami waves: underwater nuclear explosion, larger meteorites falling into the sea, volcano explosions, rockslides, etc. But the most predominant cause of tsunamis appear to be large earthquakes as in the case of the Sumatra–Andaman earthquake of 2004. Then there are three major aspects associated with the tsunami dynamics (Dias and Dutykh 2007):

1. Generation of tsunamis
2. Propagation of tsunamis
3. Tsunami run up and inundation

There exist rather successful models to approach the generation aspects of tsunamis when they occur due to the earthquakes (Dutykh and Dias 2007). Using the available seismic data it is possible to reconstruct the permanent deformation of the sea bottom due to earthquakes and

simple models have been developed (see for example, (Okada 1992)). Similarly the tsunami run up and inundation problems (Carrier et al. 2003) are extremely complex and they require detailed critical study from a practical point of view in order to save structures and lives when a tsunami strikes.

However, here we will be more concerned with the propagation of tsunami waves and their possible relation to wave propagation associated with nonlinear dispersive waves in shallow waters. In order to appreciate such a possible connection, we first look at the typical characteristic properties of tsunami waves as in the case of 2004 Indian Ocean tsunami waves or 1960 Chilean tsunamis.

The Indian Ocean Tsunami of 2004

Considering the Indian Ocean 2004 tsunami, satellite observations after a couple of hours after the earthquake establish an amplitude of approximately 60 cms in the open ocean for the waves. The estimated typical wavelength is about 200 kms (Banerjee et al. 2005). The maximum water depth h is between 1 and 4 kms. Consequently, one can identify in an average sense the following small parameters (ϵ and δ^2) of roughly equal magnitude:

$$\epsilon = \frac{a}{h} \approx 10^{-4} \ll 1, \quad \delta^2 = \frac{h^2}{l^2} \approx 10^{-4} \ll 1 \quad (38)$$

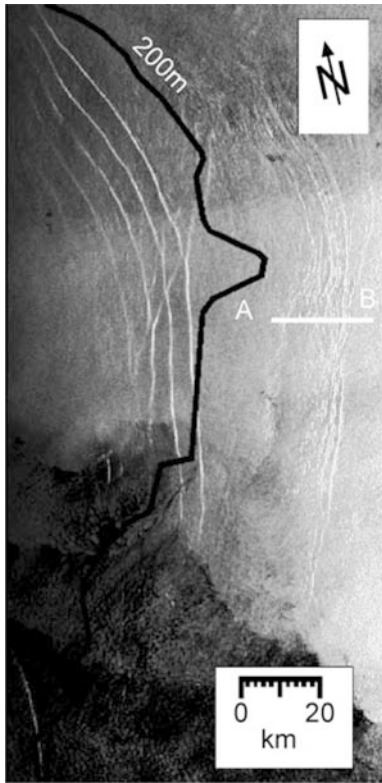
As a consequence, it is possible that a nonlinear shallow water wave theory where dispersion (KdV equation) also plays an important role (as discussed in section “Shallow Water Waves and KdV Type Equations”) has considerable relevance (Dias and Dutykh 2007). However, we also wish to point out here that there are other points of view: Constantin and Johnson (2006) estimate $\epsilon \approx 0.002$ and $\delta \approx 0.04$ and conclude that for both nonlinearity and dispersion to become significant the quantity $\delta\epsilon^{-3/2} \times \text{wavelength}$ estimated as 90,000 kms is too large and shallow water equations with variable depth (without dispersion) should be used. However, it appears that these estimates can vary over a rather wide range and with suitable estimates it is possible that the range of 10,000–20,000 kms could be

also possible ranges and hence taking into account the fact that both the Indian Ocean 2004 and Chilean 1960 tsunamis have traveled over 10 hours or more (in certain directions) before encountering land mass appears to allow for the possibility of nonlinear dispersive waves as relevant features for the phenomena. Segur (2007) has argued that in the 2004 tsunamis, the propagation distances from the epicenter of the earthquake to India, Srilanka, or Thailand were too short for KdV dynamics to develop. In the same way one can argue that the waves that hit Somalia and Kenya in the east coast of Africa (or as in the case of Chilean earthquake see also (Segur 2007)) have traveled sufficiently long distance for KdV dynamics to become important (Lakshmanan 2007). In any case one can conclude that at least for long distance tsunami propagation solitary wave and soliton picture of KdV like equations become very relevant.

Internal Solitons

For a long time seafarers passing through the Strait of Malacca on their journeys between India and the Far East have noticed that in the Andaman sea, between Nicobar islands and the north east coast of Sumatra, often bands of strongly increased surface roughness (rippings or bands of choppy water) occur (Osborne and Burch 1980; Alpers et al. 1997). Similar observations have been reported in other seas around the globe from time to time. In recent times there has been considerable progress in understanding these kind of observations in terms of internal solitons in the oceans (Ablowitz and Clarkson 1991). These studies have been greatly facilitated by photographs taken from satellites and space-crafts orbiting the earth, for example by synthetic aperture radar (SAR) images of ERS-1/2 satellites (Alpers et al. 1997; Hyder et al. 2005).

Peculiar striations of 100 km long, separated by 6 to 15 km and grouped in packets of 4 to 8, visible on satellite photographs (see Fig. 5) of the surface of the Andaman and Sulu seas in the Far East, have been interpreted as secondary phenomena accompanying the passage of ‘internal



Applications of Solitons, Tsunamis and Oceanographical, Fig. 5 SAR image of a 200 km $\times$ 200 km large section of the Andaman Sea acquired by the ERS-2 satellite on April 15, 1996 showing sea surface manifestations of two internal solitary wave packets, see (Alpers et al. 1997). (Figure reproduced from ESA website www.earth.esa.int/workshops/ers97/papers/alpers3 with the courtesy of European Space Agency and W. Alpers)

solitons', which are solitary wavelike distortions of the boundary layer between warm upper layer of sea water and cold lower depths. These internal solitons are traveling edges of warm water, extending hundreds of meters down below the thermal boundary (Ablowitz and Clarkson 1991). They carry enormous energy with them which is perhaps the reason for unusually strong underwater currents experienced by deep-sea drilling rigs. Thus these internal solitons are potentially hazardous to sub-sea oil and gas explorations. The ability to predict them can improve substantially the cost effectiveness and safety of offshore drilling.

A systematic study of the underwater currents experienced by an oil rig in the Andaman sea

which was drilling at a depth of 3600 ft was carried out by Osborne and Burch in 1980 (Osborne and Burch 1980). They spent four days measuring underwater currents and temperatures. The striations seen on satellite photographs turned out to be kilometer-wide bands of extremely choppy water, stretching from horizon to horizon, followed by about two kilometers of water "as smooth as a millpond". These bands of agitated water are called "tide rips", they arose in packets of 4 to 8, spaced about 5 to 10 km apart (they reached the research vessel at approximately hourly intervals) and this pattern was repeated with the regularity of tidal phenomenon.

As described in (Ablowitz and Clarkson 1991), Osborne and Burch found that the amplitude of each succeeding soliton was less than the previous one, is precisely what is expected for solitons (note that the velocity of a solitary wave solution of KdV equation increases with amplitude, vide Eq. (22)). Thus if a number of solitons are generated together, then we expect them eventually to be arranged in an ordered sequence of decreasing amplitude. From the spacing between successive waves in a packet and the rate of separation calculated from the KdV equation, Osborne and Burch were able to estimate the distance the packet had traveled from its source and thus identify possible source regions (Ablowitz and Clarkson 1991). They concluded that the solitons are generated by tidal currents off northern Sumatra or between the islands of the Nicobar chain that extends beyond it and that their observations have good general agreement with the predictions for internal solitons as given by the KdV equation. Numerous recent observations and predictions of solitons in the Andaman sea have clearly established that it is a site where extraordinarily large internal solitons are encountered (Alpers et al. 1997; Hyder et al. 2005).

Further, Apel and Holbrook (1980) undertook a detailed study of internal waves in the Sulu sea. Satellite photographs had suggested that the source of these waves was near the southern end of the Sulusea and their research ship followed one wave packet for more than 250 miles over a period of two days – an extraordinary coherent phenomenon (Ablowitz and Clarkson 1991).

These internal solitons travel at speeds of about 8 km/h (5 miles per hour), with amplitude of about 100 m and wavelength of about 1700 m.

Similar observations elsewhere have confirmed the presence of internal solitons in oceans including the strait of Messina, the strait of Gibraltar, off the western side of Baja California, the Gulf of California, the Archipelago of La Maddalena and the Georgia strait (Ablowitz and Clarkson 1991). There has also been a number of experimental studies of internal solitons in laboratory tanks in the last few decades (Ostrovsky and Stepanyants 2005). These experiments provide detailed quantitative information usually unavailable in the field conditions, and are also an efficient tool for verifying various theoretical models.

As a theoretical formulation of internal solitons (Ablowitz and Clarkson 1991), consider two incompressible, immiscible fluids, with densities ρ_1 and ρ_2 and depths h_1 and h_2 respectively such that the total depth $h = h_1 + h_2$. Let the lighter fluid of height h_1 be lying over a heavier fluid of height h_2 , in a constant gravitational field (Fig. 6). The lower fluid is assumed to rest on a horizontal impermeable bed, and the upper fluid is bounded by a free surface.

Then as in section “Shallow Water Waves and KdV Type Equations”, we denote the characteristic amplitude of wave by ‘ a ’ and the characteristic wavelength $l = k^{-1}$. Then the various nonlinear wave equations to describe the formation of

internal solitons follow by suitable modification of the formulation in section “Shallow Water Waves and KdV Type Equations”, assuming viscous effects to be negligible. Each of these equations is completely integrable and admits soliton solutions (Ablowitz and Clarkson 1991).

(a) **KdV equation** (Eq. (21)) follows when

- (i) the waves are of long wavelength $\delta = \frac{h}{l} \ll 1$,
- (ii) the amplitude of the waves are small, $\epsilon = \frac{a}{h} \ll 1$, and
- (iii) the two effects are comparable $\delta^2 = O(\epsilon)$

(b) **Intermediate-Long-Wave (ILW) equation** (Joseph 1977)

$$u_t + u_x + 2uu_x + Tu_{xx} = 0, \quad (39)$$

where Tu is the singular integral operator

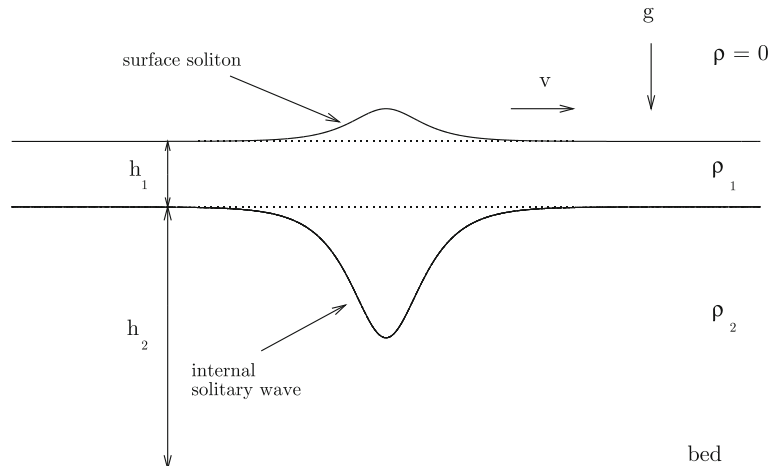
$$(Tf)(x) = \frac{1}{2l} \int_{-\infty}^{\infty} \coth\left\{\frac{\pi}{2l}(y-x)\right\} f(y) dy \quad (40)$$

with $\int_{-\infty}^{\infty}$ the Cauchy principal value integral is obtained under the assumption that

- (a) there is a thin (upper) layer, $\epsilon = \frac{h_1}{h_2} \ll 1$,
- (b) the amplitude of the waves is small, $a \ll h_1$,
- (c) the above two effects balance, $\frac{a}{h_1} = O(\epsilon)$,
- (d) the characteristic wavelength is comparable to the total depth of the fluid, $l = kh = O(1)$ and

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Fig. 6 Formation of internal soliton (note that under suitable conditions small amplitude surface soliton can also be formed)



- (e) the waves are long waves in comparison with the thin layer, $kh_1 \ll 1$.

(c) **Benjamin-Ono equation** (Benjamin 1967)

$$u_t + 2uu_x + Hu_{xx} = 0, \quad (41)$$

where Hu is the Hilbert transform

$$(Hf)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{y-x} dy \quad (42)$$

is obtained under the assumption

- (a) there is a thin (upper) layer $h_1 \ll h_2$,
- (b) the waves are long waves in comparison with the thin layer, $kh_1 \ll 1$,
- (c) the waves are short in comparison with the total depth of the fluid, $kh \gg 1$ and
- (d) the amplitude of the waves is small, $a \ll h_1$.

It may be noted that in the shallow water limit, as $\delta \rightarrow 0$, the ILW equation reduces to the KdV equation, while the Benjamin-Ono equation reduces to it in the deep water wave limit as $\delta \rightarrow \infty$. Each of these equations have their own ranges of validity and admit solitary wave and soliton solutions to represent internal solitons of the oceans.

Rossby Solitons

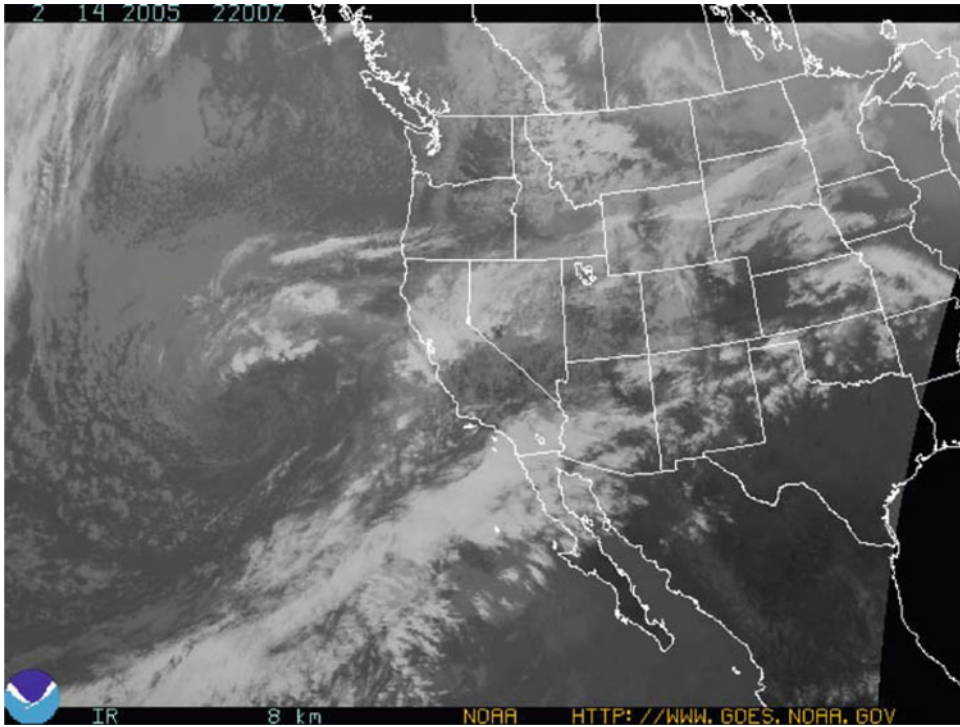
The atmospheric dynamics is an important subject of human concern as we live within the atmosphere and are continuously affected by the weather and its rather complex behavior. The motion of the atmosphere is intimately connected with that of the ocean with which it exchanges fluxes of momentum, heat and moisture (Kundu and Cohen 2002). Then the atmospheric dynamics is dominated by the rotation of the earth and the vertical density stratification of the surrounding medium, leading to newer effects. Similar effects are also present in other planetary dynamics as well.

In the atmosphere of a rotating planet, a fluid particle is endowed with a certain rotation rate

(Coriolis frequency), determined by its latitude. Consequently its motion in the north-south direction is inhibited by conservation of angular momentum. The large scale atmospheric waves caused by the variation of the Coriolis frequency with latitude are called *Rossby waves*. In section “[Internal Solitons](#)” we saw that KdV equation and its modifications model internal waves. Since there is a resemblance between internal waves and Rossby waves, it is expected that KdV like equations can model Rossby waves as well (Ablowitz and Segur 1981). Under the simplest assumptions like long waves, incompressible fluid, β -plane approximations, etc. Benney (1966) had derived the KdV equation as a model for Rossby waves in the presence of east-west zonal flow. Boyd (1980) had shown that long, weakly nonlinear, equatorial Rossby waves are governed either by KdV or MKdV equation. Recently it has been shown by using the eight years of Topex/Poseidon altimeter observations (Susanto et al. 1998) that a detailed characterization of major components of Pacific dynamics confirms the presence of equatorial Rossby solitons.

Also observational and numerical studies of propagation of nonlinear Rossby wave packets in the barotropic atmosphere by Lee and Held (1993; Tan 1996) have established their presence, notably in the Northern Hemisphere as storm tracks, but more clearly in the Southern Hemisphere (see Fig. 7). They also found that the wave packets both in the real atmosphere and in the numerical models behave like the envelope solitons of the nonlinear Schrödinger equation (Fig. 7).

An interesting application of KdV equation to describe Rossby waves is the conjecture of Maxworthy and Redekopp (1976) that the planet Jupiter’s Great Red Spot might be a solitary Rossby wave. Photographs taken by the spacecraft Voyager of the cloud pattern show that the atmospheric motion on Jupiter is dominated by a number of east-west zonal currents, corresponding to the jet streams of the earth’s atmosphere (Ablowitz and Segur 1981). Several oval-shaped spots are also seen, including the prominent Great Red Spot in the southern hemisphere. The latter one has been seen approximately this latitude for hundreds of years and maintains its form despite interactions with other atmospheric objects. One possible



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Fig. 7 A coupled high/low pressure systems in the form of a Rossby soliton formed off the coast of California/Washington on Valentine's Day 2005. The low pressure front hovers over Los Angeles dumping 30 inches of rain on the city in two weeks. The high pressure system lingers off the coast of Washington state providing unseasonably warm and sunny weather. This particular Rossby soliton proved exceptionally stable because of its

remarkable symmetry. In an accompanying animated gif in this website one can watch the very interesting phenomenon as the jet stream splits around the soliton suctioning warm wet air from the off the coast of Mexico to Arizona, leaving behind a welcome drenching of rain. (The figure and caption have been adapted from the website http://mathpost.la.asu.edu/%7Erubio/rossby_soliton/rs.html with the courtesy of the National Oceanic and Atmospheric Administration (NOAA) and Antonio Rubio)

explanation is that the Red Spot is a solitary wave/soliton of the KdV equation, deduced from the quasi geostrophic form of potential vorticity equation for an incompressible fluid. A second test of the model is the combined effect of interaction of the Red Spot and Hollow of the Jupiter atmosphere on the South Tropical Disturbance which occurred early in the 20th century and lasted several decades. Maxworthy and Redekopp (1976) have interpreted this interaction as that of two soliton collision of KdV equation, with the required phase shift (Lakshmanan and Rajasekar 2003; Ablowitz and Clarkson 1991; Ablowitz and Segur 1981).

The above connection between the KdV solitary wave and the rotating planet may be seen more concretely by the following argument as

detailed in (Ablowitz and Segur 1981). One can start with the quasi geostrophic form of the potential vorticity equation for an incompressible fluid in the form

$$\left\{ \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) + \epsilon \left(\frac{\partial \psi}{\partial y} \right) \left(\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \right\} \left\{ \mu^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z} \left(K^2 \frac{\partial}{\partial z} \right) \right\} \psi + (\beta - U'') \frac{\partial \psi}{\partial x} = 0 \quad (43)$$

where x , y and z represent the east, north and vertical directions, while the function U is

related to the horizontal stream function Φ as $\Phi(x, y, z, t) = \int_y^y U(\eta) d\eta + \epsilon \psi(x, y, z, t)$ in terms of a zonal shear flow and a perturbation. In (43), in the β -plane approximation, the Coriolis parameter is given as $f = 2\Omega \sin \theta_0 + \beta y$ and the function $K(z)$ is given by $K(z) = 2\Omega \sin \theta_0 l_2 / N(z) d$, where $N(z)$ is the Brunt–Väisälä frequency and l_2 is the characteristic length scales in the north-south direction while d is the length scale in the vertical direction. Note that K compares the effects of rotation and density of variation. Then $\mu = l_2 / l_1$ represents the ratio of the length scales in the north-south and east-west directions.

In the linear ($\epsilon \rightarrow 0$), long wave ($\mu \rightarrow 0$) limit, the potential function ψ has the form $\psi = \sum_n A_n$

$(x - c_n t) \phi_n(y) p_n(z)$, where c_n is deduced by solving two related eigenvalue problems (Ablowitz and Segur 1981), $(K^2 p_n')' + k_n^2 p_n = 0, p_n(0) = p_n(1) = 0$ and $\phi_n'' - k_n^2 \phi_n + [(\beta - U'') / (U - c_n)] \phi_n = 0, \phi_n(y_s) = \phi_n(y_N) = 0$. If the various modes are separable on a short time scale, one can then deduce an evolution equation for the individual modes, by eliminating secular terms at higher order in the expansion. Depending on the nature and existence of a stable density of stratification, which is characterized by the function $N(z)$ mentioned above, either KdV or mKdV equation can be deduced for a given mode and thereby establishing the soliton connection in the present problem.

In spite of the complexity of the phenomenon underlying Rossby solitons there is clear evidence of the significance of solitonic picture.

Bore Solitons

Rivers which flow into the open oceans are usually affected by tidal flows, tsunami or storm surge. For a typical estuary as one moves towards the mouth of the river, the depth increases and width decreases. When a tidal wave, tsunami or storm surge hits such an estuary, it can be seen as a hydraulic jump (step-wise perturbations like a shock-wave) in the water height and speed which will propagate upstream (Caputo and Stepanyants 2007). Far less dangerous but very similar is the bore (mascaret in French), a tidal wave which can propagate in a river for considerable distances.

Typical examples of bores occur in Seine river in France and the Hooghli river in West Bengal in India. A bore existed in the Seine river upto 1960 and disappeared when the river was dredged. The tide amplitude here is one of the largest in the world. The Hughli river is a branch of the Ganges that flows through Kolkata where a bore of 1 m is present, essentially due to the shallowness of the river. Another interesting example is that at the time of the 1983 Japan sea tsunami, waves in the form of a bore ascended many rivers (Tsuji et al. 1991). In some cases, bores had the form of one initial wave with a train of smaller waves and in other cases only a step with flat water surface behind was observed (Fig. 8). The Hangzhou bore in China is a tourist attraction. Other well known bores occur in the Amazon in Brazil and in Australia.



Applications of Solitons, Tsunamis and Oceanographical, Fig. 8 Tidal bore at the mouth of the Araguari River in Brazil. The bore is undular with over 20 waves visible. (Adapted from the book “Gravity Currents in the

Environment and the Laboratory” by John E. Simpson, Cambridge University Press with the courtesy of D.K. Lynch, John E. Simpson and Cambridge University Press)

Another interesting situation where bore solitons were observed was in the International H₂O Project (IHOP), as a density current (such as cold air from thunderstorm) intrudes into a fluid of lesser density that occurs beneath a low level inversion in the atmosphere. A spectacular bore and its evolution into a beautiful amplitude-ordered train of solitary waves were observed and sampled during the early morning of 20 June 2002 by the Leandre-II aboard the P-3 aircraft. The origin of this bore was traceable to a propagating cold outflow boundary from a meso-scale convective system in extreme western Kansas (Koch et al. 2005).

In the process of propagation the bore undergoes dissipation, dispersive disintegration, enhancement due to decrease of the river width and depth, influence of nonlinear effects and so on. The profile depends on the Froude number which is a dimensionless ratio of inertial and gravitational effects. Theoretical models have been developed to study these effects based on KdV and its generalizations (Caputo and Stepanyants 2003). For example, bore disintegration into solitons in channel of constant parameters can be studied in signal coordinates in terms of the KdV like equation for the perturbation of water surface,

$$\eta_x + \frac{1}{c_0}(1 - \alpha\eta)\eta_t - \beta\eta_{ttt} = 0, \quad (44)$$

where $c_0 = \sqrt{gh}$, $\alpha = 3/2h$, $\beta = h^2/6c_0^3$, h being the depth of the river, with the bore represented by a Heaviside step function as the boundary condition at $x = 0$. Other effects then can be incorporated into a variable KdV equation of the form (26).

Future Directions

We have indicated a few of the most important oceanographical applications of solitons including tsunamis, internal solitons, Rossby solitons and bore solitons. There are other interesting phenomena like capillary wave solitons (Longuet-Higgins 1993), resonant three and four wave interaction solitons (Philips 1974), etc. which are

also of considerable interest depending on whether the wave propagation corresponds to shallow, intermediate or deep waters. Whatever be the situation, it is clear that experimental observations as well as their theoretical formulation and understandings are highly challenging complex nonlinear evolutionary problems. The main reason is that the phenomena are essentially large scale events and detailed experimental observations require considerable planning, funding, technology and manpower. Often satellite remote sensing measurements need to be carried out as in the case of internal solitons and Rossby solitons. Events like tsunami propagation are rare and time available for making careful measurements are limited and heavily dependent on satellite imaging, warning systems and after event measurements. Consequently developing and testing theoretical models are extremely hazardous and difficult. Yet the basic modeling in terms of solitary wave/soliton possessing nonlinear dispersive wave equations such as the KdV and NLS family of equations and their generalizations present fascinating possibilities to understand these large scale complex phenomena and opens up possibilities of prediction. Further understanding of such nonlinear evolution equations, both integrable and nonintegrable systems particularly in higher dimensions, can help to understand the various phenomena clearly and provide means of predicting events like tsunamis and damages which occur due to internal solitons and bores. Detailed experimental observations can also help in this regard. It is clear that what has been understood so far is only the qualitative aspects of these phenomena and much more intensive work is needed to understand the quantitative aspects to predict them.

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Water Waves and the Korteweg-de Vries Equation

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Glossary

Axisymmetric (concentric) KdV equation This is a partial differential equation for the free surface $\eta(R, t)$ in the form $(2\eta_R + \frac{1}{R}\eta + 3\eta\eta_\xi) + \frac{1}{3}\eta_{\xi\xi\xi} = 0$.

Benjamin-Feir instability of water waves This describes the instability of nonlinear water waves.

Bernoulli's equation This is a partial differential equation which determines the pressure in terms of the velocity potential in the form $\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{P}{\rho} + gz = 0$, where ϕ is the velocity potential, P is the pressure, ρ is the density and g is the acceleration due to gravity.

Boussinesq equation This is a nonlinear partial differential equation in shallow water of depth h given by $u_{tt} - c^2 u_{xx} + \frac{1}{2}(u^2)_{xx} = \frac{1}{3}h^2 u_{xxtt}$. Where $c^2 = \sqrt{gh}$.

Cnoidal waves Waves are represented by the Jacobian elliptic function $cn(z, m)$.

Continuity equation This is an equation describing the conservation of mass of a fluid. More precisely, this equation for an incompressible fluid is $\text{div } \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, where $\mathbf{u} = (u, v, w)$ is the velocity field, and $\mathbf{x} = (x, y, z)$.

Continuum hypothesis It requires that the velocity $\mathbf{u} = (u, v, w)$, pressure p and density ρ are continuous functions of position $\mathbf{x} = (x, y, z)$ and time t .

Crapper's nonlinear capillary waves Pure progressive capillary waves of arbitrary amplitude.

Dispersion relation A mathematical relation between the wavenumber, frequency and/or the amplitude of a wave.

Euler equations This is a nonlinear partial differential equation for an inviscid incompressible

ible fluid flow governed by the velocity field $\mathbf{u} = (u, v, w)$ and pressure $P(\mathbf{x}, t)$ under the external force $\mathbf{F}$. More precisely, $\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P + \mathbf{F}$, where ρ is the constant density of the fluid.

Group velocity The velocity defined by the derivative of the frequency with respect to the wavenumber ($c_g = d\omega/dk$).

Johnson's equation This is a nearly concentric KdV equation for $\eta(R, \xi, \theta)$ in cylindrical polar coordinates in the form $\left(2\eta_R + \frac{1}{R}\eta + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi} + \frac{1}{R^2}\eta_{\theta\theta}\right) = 0$.

Kadomtsev–Petviashvili (KP) equation This is a two-dimensional KdV equation in the form $\left(2\eta_t + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi}\right)_\xi + \eta_{yy} = 0$.

KdV–Burgers equation This is a nonlinear partial differential equation in the form $\eta_t + c_0\eta_x + d\eta\eta_x + \mu\eta_{xxx} - \nu\eta_{xx} = 0$, where $\mu = \frac{1}{6}c_0h_0^2$.

Korteweg–de Vries (KdV) equation This is a nonlinear partial differential equation for a solitary wave (or soliton). This equation in a shallow water of depth h is governed by the free surface elevation $\eta(x, t)$ in the form $\frac{\partial \eta}{\partial t} + c\left(1 + \frac{3}{2h}\eta\right)\eta_x + \left(\frac{ch^2}{6}\right)\eta_{xxx} = 0$, where $c = \sqrt{gh}$ is the shallow water speed.

Laplace equation This is partial differential equation of the form $\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz}$ where the $\phi = \phi(x, y, z)$ is the potential.

Linear dispersion relation A mathematical relation between the wavenumber k and the frequency ω of waves ($\omega = \omega(k)$).

Linear dispersive waves Waves with the given dispersion relation between the wavenumber and the frequency.

Linear Schrödinger equation This can be written in the form $ia_t + \frac{1}{2}\omega''(k)\frac{\partial^2 a}{\partial x^2} = 0$, where $a = a(x, t)$ is the amplitude and $\omega = \omega(k)$.

Linear wave equation in Cartesian coordinates In three dimensions this can be written in the form $u_{tt} = c^2\nabla^2 u$, where c is a constant and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three-dimensional Laplacian.

Linear wave equation in cylindrical polar coordinates This can be written in the form $u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$.

Navier–Stokes equations This is a nonlinear partial differential equation for an incompressible and viscous fluid flow governed by the velocity field $\mathbf{u} = (u, v, w)$ and pressure $P(\mathbf{x}, t)$ under the action of external force $\mathbf{F}$. More precisely, $\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P + \nu^2\mathbf{u} + \mathbf{F}$, where ρ is the density and $\nu = (\mu/\rho)$ is the kinematic viscosity.

Nonlinear dispersion relation A mathematical relation between the wavenumber k , frequency ω and the amplitude a that is $D(\omega, k, a) = 0$.

Nonlinear dispersive waves Waves with the given dispersion relation between the wavenumber, frequency and the amplitude.

Non-linear Schrödinger (NLS) equation This is a nonlinear partial differential equation for the nonlinear modulation of a monochromatic wave. The amplitude, $a(x, t)$ of the modulation satisfies the equation $i\left(\frac{\partial a}{\partial t} + \omega'_0\frac{\partial a}{\partial x}\right) + \frac{1}{2}\omega''_0\frac{\partial^2 a}{\partial x^2} + \gamma|a|^2a = 0$, where $\omega_0 = \omega_0(k)$, and γ is a constant.

Ocean waves Waves observed on the surface or inside the ocean.

Phase velocity The velocity defined by the ratio of the frequency ω and the wavenumber k , ($c_p = \frac{\omega}{k}$).

Resonant or critical phenomenon Waves with unbounded amplitude.

Sinusoidal (or exponential) wave A wave is of the form $u(x, t) = a \operatorname{Re} \exp [i(kx - \omega t)] = a \cos(kx - \omega t)$, where a is amplitude, k is the wavenumber ($k = \frac{2\pi}{\lambda}$), λ is the wavelength and ω is the frequency.

Solitary waves (or soliton) Waves describing a single hump of given height travel in a medium without change of shape.

Stokes expansion This is an expansion of the frequency in terms of the wavenumber k and the amplitude a , that is, $\omega(k) = \omega_0(k) + \omega_2(k)a^2 + \dots$

Stokes wave Water waves with dispersion relation involving the wavenumber, frequency and amplitude.

Surface-capillary gravity waves Waves under the joint action of the gravitational field and surface tension.

Surface gravity waves Water waves under the action of the gravitational field.

Variational principle For three-dimensional water waves, it is of the form $\delta I = \delta \int \int_D L \, dx dt = 0$, where L is called the Lagrangian.

Velocity potential A single valued function $\phi = \phi(\mathbf{x}, t)$ defined by $\mathbf{u} = \nabla \phi$.

Water waves Waves observed on the surface or inside of a body of water.

Waves on a running stream Waves observed on the surface or inside of a body of fluid which is moving with a given velocity.

Whitham averaged variational principle This can be formulated in the form $\delta \int \int \mathcal{L} \, dx dt = 0$, where $\mathcal{L}$ is called the Whitham average Lagrangian over the phase of the integral of the Lagrangian L defined by $\mathcal{L}(\omega, \mathbf{k}, a, \mathbf{x}, t) = \frac{1}{2\pi} \int_0^{2\pi} L \, d\theta$, where L is the Lagrangian.

Whitham's conservation equations These are first order nonlinear partial differential equations in the form $\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0$, $\frac{\partial}{\partial t} \times \{f(k)A^2\} + \frac{\partial}{\partial x} \{f(k)C(k)A^2\} = 0$, where $k = k(x, t)$ is the density of waves, $\omega = \omega(x, t)$ is the flux of waves, $A = A(x, t)$ is the amplitude and $f(k)$ is an arbitrary function.

Whitham's equation This first order nonlinear partial differential equation represents the conservation of waves. Mathematically, $(\partial k / \partial t) + (\partial \omega / \partial x) = 0$ where $k = k(x, t)$ is the density of waves and $\omega = \omega(x, t)$ is the flux of waves.

Whitham's equation for slowly varying wavetrain This is written in the form $\frac{\partial}{\partial t} \mathcal{L}_\omega - \frac{\partial}{\partial x_i} \mathcal{L}_{k_i} = 0$, where $\mathcal{L}$ is the Whitham averaged Lagrangian.

Whitham's nonlinear nonlocal equations It is in the form $u_t + duu_x + \int_{-\infty}^{\infty} K(x-s)u_s(s, t)ds = 0$, where $K(x) = \mathcal{F}^{-1}\{c(k) = \omega/k\}$ and $\mathcal{F}^{-1}$ is the inverse Fourier transformation.

Definition of the Subject

A *wave* is usually defined as the propagation of a disturbance in a medium.

The simplest example is the exponential or sinusoidal wave which has the form

$$\begin{aligned} u(x, t) &= a \operatorname{Re} \exp [i(kx - \omega t)] \\ &= a \cos (kx - \omega t), \end{aligned} \quad (1)$$

where a is called the *amplitude*, Re stands for the real part, $k (= \frac{2\pi}{\lambda})$ is called the *wavenumber* and λ is called the *wavelength* of the wave, and ω is called the *frequency* and it is a definite function of the wavenumber k and hence, $\omega = \omega(k)$ is determined by the particular equation of the problem. The quantity $\theta = kx - \omega t$ is called the *phase* of the wave so that a wave of a constant phase propagate with $kx - \omega t = \text{constant}$.

The mathematical relation

$$\begin{aligned} \omega &= \omega(k) \quad \text{or} \\ D(\omega, k) &= 0, \end{aligned} \quad (2)$$

is called the *dispersion relation*.

The *phase(or wave) velocity* is defined by

$$c(k) = \frac{\omega(k)}{k}. \quad (3)$$

This shows that the phase velocity, in general, depends on the wavenumber k (or wavelength λ) so that waves with different wavelength propagate with different phase velocity. The waves are called *dispersive* if the phase velocity is not a constant, but depends on the wavenumber k . On the other hand, waves are called *nondispersive* if $c(k)$ is constant, that is, independent of k .

In general, the dispersion relation (2) can be written in the complex form

$$\begin{aligned} \omega &= \omega(k) \\ &= \sigma(k) + i\nu(k), \end{aligned} \quad (4)$$

where $\nu(k) < 0$, $\nu(k) = 0$, or $\nu(k) > 0$.

In this case, the sinusoidal wave takes the form

$$u(x, t) = a \exp[v(k)t] \exp[i(kx - \sigma t)]. \quad (5)$$

If $v(k) < 0$, the amplitude of the wave decays to zero as $v(k) > 0$ and the waves are called *dissipative*. When $v(k) = 0$, waves are dispersive. On the other hand, if $v(k) > 0$ for some or all k , the solution grows exponentially as $t \rightarrow \infty$. This case corresponds to instability.

The group velocity of the wave (1) is defined by

$$C(k) = \frac{d\omega}{dk}. \quad (6)$$

So, in general, $c(k) \neq C(k)$. It is convenient to modify the definition slightly. Waves are called dispersive if $\omega'(k)$ is not a constant, that is, $\omega''(k) \neq 0$.

In higher dimensions, all the above ideas can be generalized without any difficulty. The sinusoidal waves in three space dimensions are defined by

$$u(\mathbf{x}, t) = a \operatorname{Re} \exp[i(\boldsymbol{\kappa} \cdot \mathbf{x} - \omega t)], \quad (7)$$

Where a is the amplitude, $\mathbf{x} = (x, y, z)$ is the displacement vector, $\boldsymbol{\kappa} = (k, l, m)$ is the wavenumber vector and ω is the frequency which is related to $\boldsymbol{\kappa}$ by the dispersion relation

$$\omega = \omega(\boldsymbol{\kappa}) \quad \text{or} \quad D(\omega, \boldsymbol{\kappa}) = 0. \quad (8)$$

This function D is also determined by the particular equation of the problem. In this case, $\theta = \boldsymbol{\kappa} \cdot \mathbf{x} - \omega t$ is the phase function.

Similarly, the phase velocity of the waves are defined as follows:

$$c(k) = \frac{\omega(\boldsymbol{\kappa})}{\boldsymbol{\kappa}} \hat{\boldsymbol{\kappa}}, \quad (9)$$

where $\hat{\boldsymbol{\kappa}} = (\hat{k}, \hat{l}, \hat{m})$ is the unit vector in the $\boldsymbol{\kappa}$ direction and the group velocity is defined by

$$C(\boldsymbol{\kappa}) = \nabla_{\boldsymbol{\kappa}} \omega(\boldsymbol{\kappa}) = \hat{\mathbf{k}} \frac{\partial \omega}{\partial k} + \hat{\mathbf{l}} \frac{\partial \omega}{\partial l} + \hat{\mathbf{m}} \frac{\partial \omega}{\partial m}. \quad (10)$$

If the dispersion relation (2) depends on the amplitude a so that the dispersion relation becomes

$$\omega = \omega(k, a) \quad \text{or} \quad D(\omega, k, a) = 0. \quad (11)$$

In general, the *linear plane waves* are recognized by the existence of periodic wavetrains in the form

$$u(x, t) = f(\theta) = f(kx - \omega t), \quad (12)$$

where f is a periodic function of the phase θ .

For linear problems, solutions more general than (1) can be obtained by the principle of superposition to form Fourier integrals as

$$u(x, t) = \int_{-\infty}^{\infty} U(k) \exp[i\{kx - \omega(k)t\}] dk, \quad (13)$$

where the arbitrary function $U(k)$ may be chosen to fit arbitrary initial or boundary conditions, provided the data are reasonable enough to admit Fourier transform $U(k) = \mathcal{F}\{u(x, 0)\}$, and $\omega = \omega(k)$ is the dispersion relation (2) appropriate to the problem.

On the other hand, the *nonlinear dispersive waves* are also recognized by the existence of periodic wavetrains (12) and the solution must include the amplitude parameter a and it also requires a nonlinear dispersion relation of the form (11).

In shallow water of constant depth h , the free surface elevation $\eta(x, t)$ satisfies the Korteweg and de Vries (KdV) equation

$$\eta_t + c \left(1 + \frac{3}{2h} \eta\right) \eta_x + \left(\frac{ch^2}{6}\right) \eta_{xxx} = 0, \quad (14)$$

where $c = \sqrt{gh}$ is the shallow water velocity, g is the acceleration due to gravity, and the total depth $H = h + \eta(x, t)$. The first two terms ($\eta_t + c\eta_x$) describe the wave evolution at speed c , the third term with the coefficient $(3c/2h)$ represents the nonlinear wave steepening and the last term with the coefficient $(ch^2/6)$ describes linear dispersion.

The KdV equation admits an exact solution in the form

$$\eta(x, t) = a \operatorname{sech}^2 \left[\left(\frac{3a}{4h^3} \right)^{\frac{1}{2}} X \right], \quad (15)$$

where $X = x - Ut$, and U is the wave velocity given by

$$U = C \left(1 + \frac{a}{2h} \right). \quad (16)$$

The solution (15) is called the *solitary wave* (or *soliton*) describing a single hump of height a above the undisturbed depth h and tending rapidly to zero away from $X = 0$. The solitary wave propagates to the right with velocity $U(>c)$ which is directly proportional to the amplitude a and has width $b^{-1} = (3a/4h^3)^{-\frac{1}{2}}$, that is, b^{-1} is inversely proportional to the square root of the amplitude a . Another significant feature of the solitary wave is that it travels in the medium *without* change of shape.

In general, the solution for $\eta(x, t)$ can be expressed in terms of the Jacobian elliptic function $cn(X, m)$

$$\eta(X) = acn^2 \left[\left(\frac{3b}{4h^3} \right)^{\frac{1}{2}} X, m \right], \quad (17)$$

where $m = (a/b)^{\frac{1}{2}}$ is the modulus of the cn function. Consequently, the solution (17) is called the *cnoidal wave*.

The *nonlinear Schrödinger (NLS) equation* can be written in the standard form

$$i\psi_t + \psi_{xx} + \gamma|\psi|^2\psi = 0, \quad (18)$$

$$-\infty < x < \infty, \quad t > 0,$$

and γ is a constant.

With $X = x - Ut$, we seek the solution in the form

$$\psi = \exp[i(mX - nt)]f(X), \quad (19)$$

where $f(X)$ can be expressed in terms of the Jacobian elliptic function in the form

$$f(X) = (\alpha_1/\alpha_2)^{\frac{1}{2}} sn(\sigma X, \kappa), \quad (20)$$

where $\alpha_1, \alpha_2, \sigma = (\alpha_2\beta_2/\beta_1\alpha_2)$ and $\kappa = (\alpha_1\beta_2)/(\beta_1\alpha_2)$ are constants.

The limiting case of the solitary wave is possible and has the form

$$f(X) = \left(\frac{2\alpha}{\gamma} \right)^{\frac{1}{2}} \operatorname{sech}[\sqrt{\alpha}(x - Ut)], \quad (21)$$

where α and γ are positive constants. This solution represents a solitary wave solution which propagates without change of shape with constant velocity. However, unlike the KdV solitary waves, the amplitude and the velocity are independent parameters. It is important to note that the solution (21) is possible only in the unstable case $\gamma = 0$. This suggests that the end result of an unstable wavetrain subject to small modulation is a series of solitary waves.

Introduction

Water waves are the most common observable phenomena in Nature. The subject of water waves is most fascinating and highly mathematical, and varied of all areas in the study of wave motions in the physical world. The mathematical as well as physical problems deal with water waves and their breaking on beaches, with flood waves in rivers, with ocean waves from storms, with ship waves on water, with free oscillations of enclosed waters such as lakes and harbors. The study of water waves and their various ramifications remain central to fluid dynamics in general, and to the dynamics of oceans in particular. The mathematical theory of water waves is quite interesting in its own merit and intrinsically beautiful. It has provided the solid background and impetus for the development of the theory of nonlinear dispersive waves. Indeed, most of the fundamental ideas and results for nonlinear dispersive waves and solitons originated in the investigation of water waves.

Historically, the problems of water waves in oceans originated from the classic work of Leonhard Euler (1707–1783), A.G. Cauchy (1789–1857), S.D. Poisson (1781–1840), Joseph Boussinesq (1842–1929), George Airy (1801–1892), Lord Kelvin (1824–1907), Lord Rayleigh (1842–1919), George Stokes (1819–1903), Scott Russell (1808–1882) and many others. Indeed, Euler formulated the boundary value problem to

understand water wave phenomena and provided successful mathematical and physical description of inviscid and incompressible fluid motion in general. Based on Newton's second law of motion, Euler first formulated his celebration equation of motion of an inviscid fluid about 250 years ago. Euler's equation is still considered the basis of all inviscid fluid flows. In 1821, Claude Navier (1785–1836) included the effect of viscosity to the Euler equation, and first developed the equations of motion of viscous fluid. In 1845, Sir George Stokes provided a sound mathematical foundation of viscous fluid flows and rederived the equations of motion for an incompressible viscous fluid that is universally known as the *Navier–Stokes equations* in the form

$$\begin{aligned}\frac{D\mathbf{u}}{Dt} &= \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \\ &= -\frac{1}{\rho} \nabla P + \mathbf{F} + \nu \nabla^2 \mathbf{u},\end{aligned}\quad (22)$$

where (D/Dt) is the *total* (or *convective*) derivative given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla), \quad (23)$$

$\mathbf{u}(\mathbf{x}, t) = (u, v, w)$ is the fluid velocity, $P(\mathbf{x}, t)$ is the pressure field at a point $\mathbf{x} = (x, y, z)$, t is time, ρ is a constant density, $\mathbf{F}(\mathbf{x}, t)$ is a general body force per unit mass, $\nu = (\mu/\rho)$ is known as kinematic viscosity, μ is called the dynamic viscosity, $\nabla^2 = \nabla \cdot \nabla$ is the *Laplace operator* and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ is the familiar gradient operator.

In particular, when $\nu = 0$ ($\mu = 0$), the Navier–Stokes Eq. (22) reduces to the celebrated *Euler equation* of motion for an inviscid fluid

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P + \mathbf{F}. \quad (24)$$

Both the Euler Eq. (24) and the Navier–Stokes Eq. (22) form a closed set when the *equation of mass conservation* (or the *continuity equation*)

$$\text{div } \mathbf{u} = \nabla \cdot \mathbf{u} = 0 \quad (25)$$

is added to (22) or (24) so that there are four equations for four unknown quantities $u, v, w,$

and P . The study of these equations is based on the *continuum hypothesis* which requires that $\mathbf{u}, p$ and ρ are continuous function of $\mathbf{x} = (x, y, z)$ and t .

Remarkably, this 150-year old system of the Navier–Stokes Eqs. (22) and (25) provided the fundamental basis of modern fluid mechanics. However, there are certain major difficulties associated with the Navier–Stokes equations. First, there are no general results for the Navier–Stokes equations on existence of solutions, uniqueness, regularity, and continuous dependence on the initial conditions. Second, another difficulty arises from the strong nonlinear convective term, $(\mathbf{u} \cdot \nabla) \mathbf{u}$ in Eq. (22).

The Euler Equation of Motion in Rectangular Cartesian and Cylindrical Polar Coordinates

With $\mathbf{F} = (0, 0, -g)$, where g is the acceleration due to gravity and constant density ρ , the three components of the Euler's equation of motion in Cartesian coordinates and the continuity equation are

$$\begin{aligned}\frac{Du}{Dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial y}, \quad \frac{Dw}{Dt} \\ &= -\frac{1}{\rho} \frac{\partial P}{\partial z} - g,\end{aligned}\quad (26)$$

where the total derivative is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}, \quad (27)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (28)$$

The basic assumption in continuum mechanics is that the motion of the fluid can be described mathematically as a topological deformation that depends continuously on the time t . Consequently, we assume the fluid under consideration to have a boundary surface S , fixed or moving, which separated it from other media. We consider the case of

a body of water with air above it so that S is the interface between them. We represent that surface S by an equation $S(x, y, z, t) = 0$. The kinematic condition is derived from the fact that the normal fluid velocity of the surface $(-S_t/|\nabla S|)$ is equal to the normal velocity $(\mathbf{u} \cdot \mathbf{n} = \mathbf{u} \cdot \nabla S/|\nabla S|)$, that is,

$$\frac{DS}{Dt} = S_t + (\mathbf{u} \cdot \nabla)S = 0. \quad (29)$$

This means that any fluid particle originally on the boundary surface, S will remain on it.

It is often convenient to represent the free surface by the equation $z = h(x, y, t)$ so that the equation for the boundary surface S is

$$S = z - h(x, y, t) = 0, \quad (30)$$

where z is independent of other variables.

Thus, the kinematic free surface condition follows from (29) in the form

$$\begin{aligned} w - (h_t + uh_x + vh_y) &= 0 \\ \text{on } z &= h(x, y, t), \quad t > 0. \end{aligned} \quad (31)$$

Since the gravitational force is only body force which acts in the negative z -direction, the continuity Eq. (28) and the Euler equation in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P - g\hat{\mathbf{m}}, \quad (32)$$

where $\hat{\mathbf{m}}$ is the unit vector in the positive z -direction, represent the fundamental equations for water wave motion.

In general, the water wave motion is unsteady, and irrotational which physically means that the individual fluid particle do not rotate. Mathematically, this implies that vorticity $\boldsymbol{\omega} = \text{curl } \mathbf{u} = \mathbf{0}$. So, there exists a single-valued velocity potential ϕ so that $\mathbf{u} = \nabla \phi$, where $\phi = \phi(\mathbf{x}, t)$. Consequently, the continuity equation reduces to the Laplace equation

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (33)$$

So, ϕ is a harmonic function. This is, indeed, a great advantage because the velocity field $\mathbf{u}$ can be

obtained from a single potential function ϕ which satisfies a linear partial differential equation. This equation with prescribed boundary conditions can readily be solved in many simple cases without difficulty.

In view of a vector identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla u^2 - \mathbf{u} \times \boldsymbol{\omega} \quad (34)$$

combined with $\boldsymbol{\omega} = \text{curl } \mathbf{u} = \mathbf{0}$ and $\mathbf{u} = \nabla \phi$, the Euler Eq. (32) may be written as

$$\nabla \left[\phi_t + \frac{1}{2}(\nabla \phi)^2 + \frac{P}{\rho} + gz \right] = 0. \quad (35)$$

This can be integrated with respect to the space variables to give the equation for pressure P at every point of the fluid

$$\phi_t + \frac{1}{2}(\nabla \phi)^2 + \frac{P}{\rho} + gz = C(t), \quad t \geq 0, \quad (36)$$

where $C(t)$ is an arbitrary function of time only ($\nabla C = 0$). Since the pressure gradient affects the flow, a function of t alone added to the pressure field P has no effect on the motion. So, without loss of generality, we can set $C(t) = 0$ in Eq. (36). Thus, the pressure Eq. (36) becomes

$$\phi_t + \frac{1}{2}(\nabla \phi)^2 + \frac{P}{\rho} + gz = 0. \quad (37)$$

This is the so-called the *Bernoulli's equation* which determines the pressure in terms of the velocity potential ϕ . Thus, Eqs. (33) and (37) are used to determine the potential ϕ (hence, three velocity components u , v , and w) and the pressure field P .

We consider a body of inviscid, incompressible fluid occupying the region $b(x, y) \leq z \leq h(x, y, t) = h_0 + a\eta(x, y, t)$, where $z = b(x, y)$ is the bottom boundary surface and $z = h_0$ is the undisturbed (initial) typical constant depth, a is the typical amplitude and $\eta(x, y, t)$ is nondimensional the free surface elevation that tends to zero as $t \rightarrow 0$. We suppose that the bottom boundary $z = b(x, y)$ is a rigid solid surface.

Since the upper boundary is the surface exposed to a constant atmospheric pressure P_a , we have $P = P_a$ on this surface, S . Thus, Eq. (35) assumes the form

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{\rho}P_a + gz = C(t) \quad (38)$$

on S , $t \geq 0$.

Absorbing $\frac{1}{\rho}P_a$ and $C(t)$ into ϕ_t , this equation may be rewritten as

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz = 0, \quad (39)$$

on S , $t \geq 0$.

Since S is a upper boundary surface of the fluid, it contains the same fluid particles for all time t , that is, S is a material surface. Hence, it follows from (39) that

$$\frac{D}{Dt} \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz \right] = 0, \quad (40)$$

on S , $t \geq 0$.

Or, equivalently,

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + (\nabla\phi \cdot \nabla) \right] \left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz \right] \\ &= \phi_{tt} + 2\nabla\phi \cdot \nabla(\phi_t) + \frac{1}{2}\nabla\phi \cdot \nabla(\nabla\phi)^2 + g\phi_z \\ &= 0, \quad \text{on } S, \quad t \geq 0. \end{aligned} \quad (41)$$

We next include the effects of *surface tension force* per unit length which does support a pressure difference across a curved surface so that $P - P_a = - (T/R)$, where $R^{-1} = (R_1^{-1} + R_2^{-1})$ is called the *Gaussian curvature* expressed as the sum of two principal radii of curvatures R_1^{-1} and R_2^{-1} given by

$$\begin{aligned} R_1^{-1} &= \frac{\partial}{\partial x} \left[h_x \left(1 + h_x^2 + h_y^2 \right)^{-\frac{1}{2}} \right], \\ R_2^{-1} &= \frac{\partial}{\partial y} \left[h_y \left(1 + h_x^2 + h_y^2 \right)^{-\frac{1}{2}} \right]. \end{aligned} \quad (42ab)$$

Explicitly, R^{-1} can be written in terms of $h(x, y, t)$ and its partial derivatives as

$$\begin{aligned} R^{-1} &= (R_1^{-1} + R_2^{-1}) \\ &= \frac{h_{xx}(1 + h_y^2) - 2h_xh_yh_{xy} + h_{yy}(1 + h_x^2)}{(1 + h_x^2 + h_y^2)^{3/2}}. \end{aligned} \quad (43)$$

For small deviation of the free surface $z = h(x, y, t)$, its first partial derivatives h_x and h_y are small so that (43) takes the linearized form

$$R^{-1} \approx (h_{xx} + h_{yy}). \quad (44)$$

Consequently, the linearized dynamic condition at the free surface $z = h$ becomes

$$\phi_t + gz - \frac{T}{\rho}(h_{xx} + h_{yy}) = 0. \quad (45)$$

For an inviscid fluid, like the free surface condition (29), there is a bottom boundary condition which follows from the fact that

$$\frac{D}{Dt}[z - b(x, y, t)] = 0, \quad (46)$$

where $z = b(x, y, t)$ is the equation of the fixed solid bottom boundary surface. Thus, Eq. (46) assumes the form

$$\begin{aligned} w &= b_t + (\mathbf{u} \cdot \nabla)b = b_t + ub_x + vb_y \\ &\quad \text{on } z = b(x, y, t), \end{aligned} \quad (47)$$

where $z = b(x, y, t)$ is given. However, there is a class of problems including sediment movement, where b is not known. For stationary bottom, b is independent of time so that (47) becomes

$$w = ub_x + vb_y \quad \text{on } z = b(x, y). \quad (48)$$

For one-dimensional problem, $h = b(x)$ with $\mathbf{u} = (u, 0)$ so that (48) reduces to the simple bottom condition

$$w = ub'(x) \quad \text{on } z = b(x). \quad (49)$$

It is convenient to introduce a typical amplitude parameter a by writing the free surface $z = h(x, y, t)$ in the form

$$h = h_0 + a\eta(x, y, t), \quad (50)$$

where h_0 is the undisturbed depth of water. The pressure field can be rewritten as

$$P = P_a + g\rho(h_0 - z) + (g\rho h_0)p, \quad (51)$$

where P_a is the constant atmospheric pressure, p is the pressure variable which measures the deviation from the hydrostatic pressure, $g\rho(h_0 - z)$, and $g\rho h_0$ is the typical pressure scale based on the pressure at depth $h = h_0$.

We next introduce two fundamental parameters ε and δ as

$$\varepsilon = \frac{a}{h_0} \quad \text{and} \quad \delta = \frac{h_0^2}{\lambda^2}, \quad (52)$$

where λ is the typical wavelength of the surface gravity wave, ε and δ are called the *amplitude* and *long wavelength* parameters respectively.

In terms of typical depth scale h_0 , λ is the typical wavelength, $c_0 = \sqrt{gh_0}$ is the typical horizontal velocity scale, (λ/c_0) is the typical time scale, it is also convenient to introduce non-dimensional flow variables denoted by asterisks

$$(x^*, y^*) = \frac{1}{\lambda}(x, y), \quad (z^*, b^*) = \frac{1}{h_0}(z, b), \quad t^* = \left(\frac{c_0}{\lambda}\right)t, \quad (53)$$

$$(u^*, v^*) = \frac{1}{c_0}(u, v), \quad w^* = \left(\frac{\lambda c_0}{h_0}\right)w \quad \text{and} \quad p^* = \frac{p}{g\rho h_0}. \quad (54)$$

In terms of these nondimensional flow variables and parameters, the Euler Eq. (26) and the continuity Eq. (28) can be rewritten, dropping the asterisks, in the form

$$\frac{D}{Dt}(u, v) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)p, \quad \delta \frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \quad (55)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z},$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (56)$$

The most general dynamic condition is $P = P_a - \frac{T}{R}$ on the free surface $z = h = h_0 + a\eta$. Without surface tension ($T = 0$), it becomes $P = P_a$ on $z = h_0 + a\eta$. Using the pressure field (51), this free surface dynamic condition reduces to the nondimensional form $p = \varepsilon\eta$ on $z = 1 + \varepsilon\eta$. Thus, the nondimensional kinematic condition (31) and the dynamic condition at the free surface are given by

$$w - \varepsilon(\eta_t + u\eta_x + v\eta_y) = 0, \quad (57a)$$

$$p = \varepsilon\eta \quad \text{on } z = 1 + \varepsilon\eta.$$

The nondimensional form of the bottom boundary condition (48) remains the unchanged form.

Consistent with the governing equations and free surface boundary conditions, we introduce a set of scaled flow variables

$$(u, v, w, p) \rightarrow \varepsilon(u, v, w, p) \quad (58)$$

so that the Euler Eqs. (55) and the continuity Eq. (56) reduce to the form

$$\frac{D}{Dt}(u, v) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)p, \quad \delta \frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \quad (59)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (60)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \varepsilon \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right). \quad (61)$$

The free surface boundary conditions (57ab) remain the same. The horizontal bottom ($b = 0$) boundary condition is

$$w = 0, \quad \text{on } z = 0, \quad (62)$$

In cylindrical polar coordinates (r, θ, z) , the Euler equations and the continuity equation are given by

$$\begin{aligned} \frac{Du}{Dt} - \frac{v^2}{r} &= -\frac{1}{\rho} P_r, \\ \frac{Dv}{Dt} + \frac{uv}{r} &= -\frac{1}{\rho} \frac{1}{r} P_\theta, \\ \frac{Dw}{Dt} &= -\frac{1}{\rho} P_z - g, \end{aligned} \quad (63)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z}, \quad (64)$$

and

$$\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \quad (65)$$

In terms of the above nondimensional flow variables except for x and y that are replaced by $r^* = \frac{1}{\lambda} r$, the above Eqs. (63)–(65) assume the following nondimensional form, dropping the asterisks,

$$\begin{aligned} \frac{Du}{Dt} - \frac{v^2}{r} &= -\frac{\partial p}{\partial r}: \quad \frac{Dv}{Dt} + \frac{uv}{r} \\ &= -\frac{1}{r} \frac{\partial p}{\partial \theta}, \quad \delta \frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \end{aligned} \quad (66)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z}, \quad (67)$$

and

$$\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \quad (68)$$

Basic Equations of Water Waves with Effects of Surface Tension

We consider an irrotational unsteady motion of an inviscid and incompressible water occupying the region $b(x, y) \leq z \leq h_0 + a\eta(x, y, t)$ in a constant gravitational field g which is in the negative z -direction. The equation of uneven bottom boundary is $z = b(x, y)$. Including the effects of surface tension T , the basic equations, two free surface boundary conditions and the bottom boundary condition for the velocity potential $\phi = \phi(x, y, z, t)$ and the free surface elevation $\eta = \eta(x, y, t)$ are

$$\begin{aligned} \nabla^2 \phi &= \phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad b(x, y) \leq z \\ &\leq h_0 + a\eta, \quad t > 0, \end{aligned} \quad (69)$$

$$\begin{aligned} \eta_t + (\phi_x \eta_x + \phi_y \eta_y) - \phi_z &= 0, \\ \text{on } z &= h_0 + a\eta, \quad t > 0, \end{aligned} \quad (70)$$

$$\begin{aligned} \phi_t + \frac{1}{2} (\nabla \phi)^2 + g\eta - \frac{T}{\rho} (R_1^{-1} + R_2^{-1}) \\ = 0, \text{ on } z &= h_0 + a\eta, \quad t > 0, \end{aligned} \quad (71)$$

$$\phi_z - (\phi_x b_x + \phi_y b_y) = 0, \quad \text{on } z = b(x, y), \quad (72)$$

where R_1^{-1} and R_2^{-1} are given by (42ab), h_0 is the typical depth of water and a is the typical amplitude of the surface gravity wave.

In water of infinite depth with the origin at the free surface, the bottom boundary condition (72) is replaced by $(\nabla \phi) \rightarrow 0$ as $z \rightarrow -\infty$.

Because of the presence of nonlinear terms in the free surface boundary conditions (70)–(71), the determination of ϕ and η in the general case is a difficult task.

Similarly, we can write the basic equations for the water wave problems in cylindrical polar coordinates.

In terms of the nondimensional flow variables and the parameters ε and δ stated above, the basic water wave Eqs. (69)–(72) without surface tension can be written in the nondimensional form

$$\begin{aligned} \delta(\phi_{xx} + \phi_{yy}) + \phi_{zz} &= 0, \text{ on } b \leq z \\ &\leq 1 + \varepsilon\eta, \quad t > 0, \end{aligned} \quad (73)$$

$$\begin{aligned} \delta\left[\eta_t + \varepsilon(\phi_x\eta_x + \phi_y\eta_y)\right] - \phi_z &= 0, \\ \text{on } z &= 1 + \varepsilon\eta, \end{aligned} \quad (74)$$

$$\begin{aligned} \phi_t + \eta + \frac{\varepsilon}{2}(\phi_x^2 + \phi_y^2) + \frac{\varepsilon}{2\delta}\phi_z^2 &= 0, \\ \text{on } z &= 1 + \varepsilon\eta, \end{aligned} \quad (75)$$

$$\begin{aligned} \phi_z - \delta(\phi_x b_x + \phi_y b_y) &= 0, \\ \text{on } z &= b(x, y), \end{aligned} \quad (76)$$

where $(\varepsilon/\delta) = (a\lambda^2/h_0^3)$ is another fundamental parameter in water-wave theory.

Luke (1967) first explicitly formulated a variational principle for two-dimensional water waves and proved that the basic Laplace equation, free surface and bottom boundary conditions can be derived from the Hamilton principle. We formulate the variational principle for three-dimensional water waves in the form

$$\delta I = \delta \iint_D L d\mathbf{x} \, dt = 0, \quad (77)$$

where the Lagrangian L is assumed to be equal to the pressure so that

$$L = -\rho \int_{-h(x, y)}^{\eta(x, t)} \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz \right] dz, \quad (78)$$

where D is an arbitrary region in the $(\mathbf{x}, t)$ space, and $\phi(\mathbf{x}, z, t)$ is the velocity potential of an unbounded fluid lying between the rigid bottom $z = -h(x, y)$ and the free boundary surface $z = \eta(x, y, t)$. Using the standard procedure in the calculus of variations (see Debnath (1994)), the following nonlinear system of equations for the classical water waves can be derived:

$$\begin{aligned} \nabla^2 \phi &= 0, \quad -h < z < \eta, \quad -\infty < (x, y) < \infty, \\ & \quad (79) \end{aligned}$$

$$\begin{aligned} \eta_t + (\phi_x\eta_x + \phi_y\eta_y) - \phi_z &= 0, \\ \text{on } z &= \eta, \end{aligned} \quad (80)$$

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz = 0, \quad \text{on } z = \eta, \quad (81)$$

$$\phi_z + \phi_x h_x + \phi_y h_y = 0, \quad \text{on } z = -h. \quad (82)$$

These results are in perfect agreement with those of Luke for the two-dimensional waves on water of arbitrary but uniform depth h . In his pioneering work, Whitham (1965a, b) first developed a general approach to linear and nonlinear dispersive waves using a Lagrangian. It is now well known that most of the general ideas about dispersive waves have originated from the classical problems of water waves.

Making reference to Debnath (1994), we first state the solution of the linearized two-dimensional problem of the classical water waves on water of uniform depth h governed by the following equation, free surface and boundary conditions

$$\phi_{xx} + \phi_{zz} = 0, \quad -h \leq z \leq 0, \quad t > 0, \quad (83)$$

$$\begin{aligned} \eta_t = \phi_z, \quad \phi_t + g\eta &= 0, \quad \text{on } z = 0, \quad t > 0, \\ & \quad (84a) \end{aligned}$$

$$\phi_z = 0, \quad \text{on } z = -h, \quad t \geq 0. \quad (85)$$

The solutions for $\phi(x, z, t)$ and $\eta(x, t)$ representing a sinusoidal wave propagating in the x -direction are given by

$$\begin{aligned} \phi(x, z, t) &= \text{Re } a \left(\frac{ig}{\omega} \right) \frac{\cosh k(z+h)}{\cosh kh} \\ & \quad \exp[i(\omega t - kx)], \end{aligned} \quad (86)$$

$$\eta(x, t) = \text{Re } a \exp[i(\omega t - kx)], \quad (87)$$

where $a = (C\omega/ig) \cosh kh = \max |\eta|$ is the amplitude and C is an arbitrary constant.

Using (83), we obtain the celebrated *dispersion relation* between the frequency ω and the wavenumber k in the form:

$$\omega^2 = gk \tanh kh. \quad (88)$$

Physically, this relation describes the interaction between inertial and gravitational forces. This can also be rewritten in terms of the wave (or phase) velocity, $c(k)$ as

$$\begin{aligned} c(k) &= \frac{\omega}{k} = (gk^{-1} \tanh kh)^{\frac{1}{2}} \\ &= \left[\left(\frac{g\lambda}{2\pi} \right) \tanh \left(\frac{2\pi h}{\lambda} \right) \right]^{\frac{1}{2}}. \end{aligned} \quad (89)$$

This formula shows that the phase velocity $c(k)$ depends on the gravity g and depth h as well as the wavenumber k or wavelength $\lambda = (2\pi/k)$. Thus, water waves of different wavelengths travel with different wave (or phase) velocities. Such waves are called *dispersive*, as time passes, these waves disperse (or spread out) into various group of waves such that each group of waves such that each group would consist of waves having approximately the same wavelength. Figure 1 showing a plot of the phase velocity c given by (88) against the wavelength λ reveals a transition between the deep water limit $c \sim (g\lambda/2\pi)^{\frac{1}{2}}$ (parabolic form) when $\lambda < (3.5)h$ and the shallow water limit $c \sim \sqrt{gh}$ for $\lambda > (14)h$.

The quantity $(d\omega/dk)$ represents the velocity of such a group in the direction of propagation and is called the *group velocity*, denoted by $C(k)$ so that

$$C(k) = \frac{d\omega}{dk} = \left(\frac{g}{2\omega} \right) (\tanh kh + kh \operatorname{sech}^2 kh), \quad (90)$$

which is, using (88)

$$= \frac{1}{2} c(k) [1 + 2kh \operatorname{cosech}(2kh)]. \quad (91)$$

Evidently, the group velocity is, in general, different from the phase velocity.

Two limiting cases are of special interest: (i) Shallow water waves and (ii) Deep water waves.

For shallow water, the wavelength; $\lambda = (2\pi/k)$ is large compared with the depth h so that $kh \ll 1$, and hence, $\tanh kh \approx kh$ and $\sin 2kh \approx 2kh$. In such a case, (87)–(91) reduce to

$$\omega^2 = gkh^2, \quad c(k) = \sqrt{gh} = C(k). \quad (92)$$

Both phase and group velocities are independent of the wavenumber k . Thus, shallow water waves are nondispersive, and their phase velocity is equal to group velocity. Both vary as the square root of the depth h .

In the other limiting case dealing with deep water waves, the wavelength is very small compared with the depth so that $kh \gg 1$. In the limit as $kh \rightarrow \infty$, $\tanh kh \rightarrow 1$, $[\cosh k(z+h)/\cosh kh] \rightarrow \exp(kz)$, and the corresponding solutions for ϕ and η become

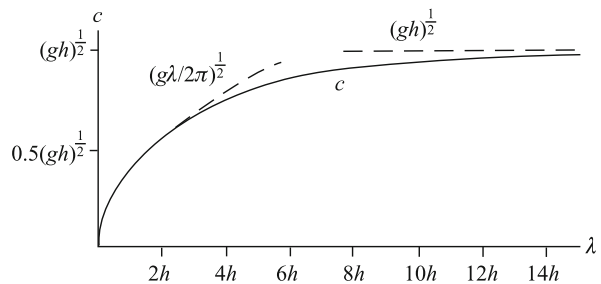
$$\begin{aligned} \phi &= \operatorname{Re} \left(\frac{ia g}{\omega} \right) e^{kz} \exp [i(\omega t - kx)] \\ &= \left(\frac{ag}{\omega} \right) e^{kz} \sin (kx - \omega t), \end{aligned} \quad (93)$$

$$\begin{aligned} \eta &= \operatorname{Re} a \exp [i(\omega t - kx)] \\ &= a \cos (kx - \omega t). \end{aligned} \quad (94)$$

Evidently, for deep water waves, the dispersion relation (88), the phase velocity (88) and the group velocity (90) become

Water Waves and the Korteweg-de Vries Equation, Fig. 1

The phase velocity $c(\lambda)$ against the wavelength λ (from (Lighthill 1978), Figure 52, page 217)



$$\omega^2 = gk = \left(\frac{2\pi g}{\lambda}\right), \quad c(k) = (g/k)^{\frac{1}{2}} = \left(\frac{g\lambda}{2\pi}\right)^{\frac{1}{2}},$$

$$C(k) = \frac{1}{2}c(k). \quad (95abc)$$

Thus, deep water waves are dispersive and their phase velocity is proportional to the square root of their wavelengths. Also, the group velocity is equal to one-half of the phase velocity.

Another equivalent form of the phase velocity $c(k)$ follows from the dispersion relation (88) in the form

$$c(k) = \frac{\omega}{k} = \left(\frac{\omega^2}{\omega k}\right) = \frac{g}{\omega} \tanh\left(\frac{\omega h}{c}\right). \quad (96)$$

The phase velocity $c(k)$ for water waves of frequency ω against depth h is shown in Fig. 2. This figure shows a transition between the deep water limit $c \sim (g/\omega)$ when $(\omega h/c) \gg 2$ or $h \geq 2g/\omega^2$ and the shallow water limit $c \sim \sqrt{gh}$ when $(\omega h/c) \ll 1$ or $h \ll g/\omega^2$.

Similarly, an alternative form of the group velocity follows from the dispersion relation (88) as

$$C(k) = \frac{d\omega}{dk} = \frac{d}{dk}(ck) = c + k \frac{dc}{dk} = c - \lambda \frac{dc}{d\lambda}. \quad (97)$$

Finally, we close the section by adding surface capillary-gravity waves on water of constant depth h with the free surface at $z = 0$. For such waves, the linearized free surface conditions (70)–(71) with the effect of surface tension are modified as follows:

$$\eta_t = \phi_z, \quad \phi_t + g\eta - \frac{T}{\rho}\eta_{xx} = 0 \quad \text{on } z = 0. \quad (98)$$

These conditions can be combined to obtain

$$(\phi_{tt} + g\phi_z) - \frac{T}{\rho}\phi_{xxz} = 0 \quad \text{on } z = 0. \quad (99)$$

In this case, the solutions for the velocity potential $\phi(x, z, t)$ and the free surface elevation $\eta(x, t)$ are exactly the same as (85) and (86) on water of constant depth h or as (93)–(94) for deep water. Evidently, the dispersion relation for capillary-gravity waves on water of constant depth h is given by

$$\omega^2 = gk \left(1 + \frac{Tk^2}{\rho g}\right) \tanh kh. \quad (100)$$

The phase velocity is

$$c(k) = \left[\frac{g}{k} \left(1 + \frac{Tk^2}{\rho g}\right) \tanh kh\right]^{\frac{1}{2}}. \quad (101)$$

The corresponding dispersion relation for deep water ($kh \gg 1$) is

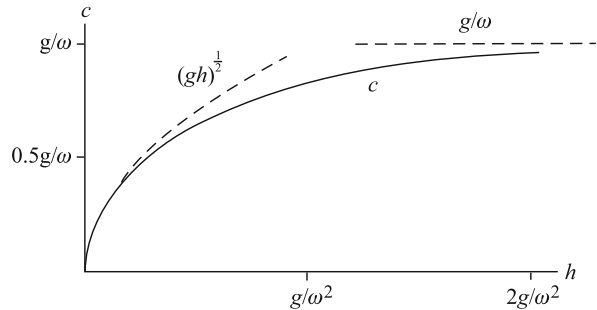
$$\omega^2 = gk(1 + T^*), \quad (102)$$

where $T^* = (Tk^2/\rho g) = (4\pi^2 T/\rho g \lambda^2)$ is a parameter giving the relative importance of surface tension and gravity. The phase velocity is

$$c(k) = \left[\frac{g}{k}(1 + T^*)\right]^{\frac{1}{2}} = \left[\left(\frac{g\lambda}{2\pi}\right)(1 + T^*)\right]^{\frac{1}{2}}. \quad (103)$$

Water Waves and the Korteweg-de Vries Equation, Fig. 2

The phase velocity c for waves of frequency ω on water of depth h (from (Lighthill 1978), Figure 53, page 218)



The phase velocity $c(k)$ has a minimum value at $k_m = \sqrt{g\rho/T}$ (or $T^* = 1$). The corresponding minimum value for $c = c_m$ attained at $k = k_m = \sqrt{g\rho/T}$ (or $T^* = 1$) is

$$c = c_m = \left(\frac{4gT}{\rho} \right)^{\frac{1}{4}} \quad (104)$$

at wavelength $\lambda = \lambda_m = 2\pi(T/g\rho)^{\frac{1}{2}}$.

The inequality $k \ll k_m$ holds for waves to be effectively pure gravity waves, since the surface tension is negligible by comparison. This inequality is equivalent to the wavelength λ to be large compared with $\lambda_m = (2\pi/k_m) = 2\pi(T/g\rho)^{\frac{1}{2}}$. The phase velocity $c(\lambda)$ for capillary-gravity waves against λ in deep water is shown in Fig. 3. This figure shows the transition between the pure capillary-wave value $(2\pi T/\lambda\rho)^{\frac{1}{2}}$ and the pure gravity-wave value $(g\lambda/2\pi)^{\frac{1}{2}}$. Also shown in this figure are (i) the minimum phase velocity c_m attained at $\lambda = \lambda_m$, (ii) the gravity-wave curve ($T^* = 0$) for $\lambda > \lambda_m$, and (iii) the capillary-wave curve with no gravitational effect $g \rightarrow 0$ or $T^* \rightarrow \infty$. Figure 3 also shows that for $\lambda > 4\lambda_m$ the capillary-gravity wave curve is rapidly running closer with the gravity-wave curve. On the other hand, when $\lambda < \lambda_m$, the rapid tendency is for wave speeds to increase again so that the capillary-gravity wave curve approaches the capillary wave curve as $T^* \rightarrow \infty$, which corresponds to very short waves called *capillary waves* (or *ripples*). In fact, when $\lambda < \frac{1}{4}\lambda_m$, results (102) and (103) for $T^* \rightarrow \infty$ give

$$\begin{aligned} \omega^2 &= \rho^{-1}Tk^3 \quad \text{and} \quad c(k) \\ &= (\rho^{-1}Tk)^{\frac{1}{2}}. \end{aligned} \quad (105ab)$$

For such capillary waves, surface tension is the only significant restoring force.

The Stokes Waves and Nonlinear Dispersion Relation

In his 1847 classic paper, Stokes (1847) first established the nonlinear solutions for periodic plane waves on deep water. We consider here some of the nonlinear effects neglected in the linearized theory. A more simple approach is to recall the exact free surface dynamic condition (41) with constant atmospheric pressure and the negligible surface tension so that (41) becomes

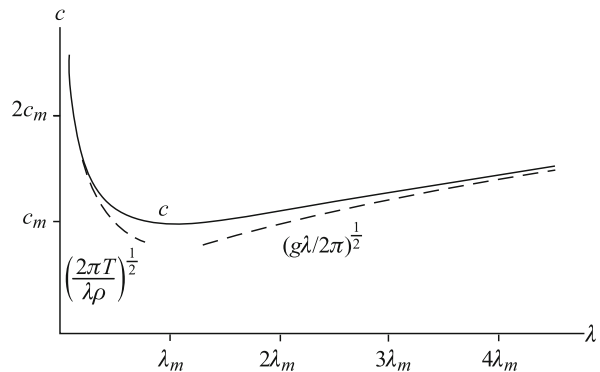
$$\begin{aligned} (\phi_{tt} + g\phi_z) + 2\nabla\phi \cdot \nabla\phi_t + \frac{1}{2}\nabla\phi \cdot \nabla(\nabla\phi)^2 \\ = 0, \text{ on } z = \eta, \quad \text{for } t \geq 0. \end{aligned} \quad (106)$$

This condition is applied on the unknown free surface $z = \eta$ given by (39), that is,

$$\eta = -\frac{1}{g} \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 \right]. \quad (107)$$

In the absence of nonlinear terms, results (106) and (107) reduce to (83) and (84ab), respectively.

Water Waves and the Korteweg-de Vries Equation, Fig. 3 The phase velocity $c(\lambda)$ given by (103) against λ (from (Lighthill 1978), Figure 56, page 224)



We use Taylor series expansions of ϕ and its derivatives from $z = \eta$ to $z = 0$ in the form

$$\begin{aligned}\phi(x, y, \eta, t) &= \phi(x, y, 0, t) \\ &+ \eta \left(\frac{\partial \phi}{\partial z} \right)_{z=0} \\ &+ \frac{1}{2} \eta^2 \left(\frac{\partial^2 \phi}{\partial z^2} \right)_{z=0} + \dots\end{aligned}\quad (108)$$

Using this expansion procedure for each of the derivatives in (106) and (107), we can generate a series of boundary conditions on the surface plane $z = 0$. Thus, we obtain first three conditions:

$$L\phi = \phi_{tt} + g\phi_z = 0 + O(\phi^2), \quad (109)$$

$$\begin{aligned}L\phi + 2\nabla\phi \cdot \nabla\phi_t - \frac{1}{g}\phi_t \frac{\partial}{\partial z} \\ (L\phi) = 0 + O(\phi^3),\end{aligned}\quad (110)$$

$$\begin{aligned}L\phi + 2\nabla\phi \cdot \nabla\phi_t \\ + \frac{1}{2}\nabla\phi \cdot \nabla(\nabla\phi)^2 \\ - \frac{1}{g}\phi_t \frac{\partial}{\partial z} (L\phi + 2\nabla\phi \cdot \nabla\phi_t) \\ - \frac{1}{g} \left[-\frac{1}{g}\phi_t \phi_{zt} + \frac{1}{2}(\nabla\phi)^2 \right] \frac{\partial}{\partial z} (L\phi) \\ + \frac{1}{2g^2}\phi_t^2 \frac{\partial^2}{\partial z^2} (L\phi) \\ = 0 + O(\phi^4),\end{aligned}\quad (111)$$

where the symbol $O()$ is used to indicate the magnitude of the neglected terms.

If we substitute the first-order velocity potential (93) for deep water in the second-order boundary condition (110), the second-order terms in (110) vanish. Thus, the first-order potential is a solution of the second-order boundary value problem, and we can write

$$\phi = \left(\frac{ga}{\omega} \right) e^{kz} \sin(kx - \omega t) + O(a^3). \quad (112)$$

Similarly, using (108), we can incorporate the second-order effects in $\eta(x, z, t)$ so that

$$\begin{aligned}\eta(x, z, t) &= -\frac{1}{g} \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 \right]_{z=\eta} \\ &= -\frac{1}{g} \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 \right]_{z=0} \\ &\quad + \eta \frac{\partial}{\partial z} \left[-\frac{1}{g} \left\{ \phi_t + \frac{1}{2}(\nabla\phi)^2 \right\} \right]_{z=0} + \dots \\ &= -\frac{1}{g} \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t \phi_{zt} \right]_{z=0} + \dots\end{aligned}\quad (113)$$

Direct substitution of (112) in (113) gives the following second-order result:

$$\begin{aligned}\eta &= a \cos(kx - \omega t) + \frac{1}{2}ka^2 \{ 2 \cos^2(kx - \omega t) - 1 \} \\ &\quad + \dots \\ &= a \cos \theta + \frac{1}{2}ka^2 \cos 2\theta + \dots,\end{aligned}\quad (114)$$

where the phase $\theta = kx - \omega t$. Clearly, the second term in (114) represents the nonlinear effects on the free-surface profile, and it is positive both at the crests $\theta = 0, 2\pi, 4\pi, \dots$ and at the troughs $\theta = \pi, 3\pi, 5\pi, \dots$. The notable feature of this solution (114) is that the wave profile is no longer sinusoidal as shown in Fig. 2.7 of Debnath (1994).

The actual shape of this wave profile is a curve known as a *trochoid*: The crests are steeper and the troughs are flatter. This feature becomes accentuated as the wave amplitude is increased.

For the third-order free-surface condition (111), direct substitution of the plane wave potential (112) in the nonlinear terms in (111) eliminates all but one term. Therefore, the free-surface boundary condition for the third-order plane wave solution is

$$L\phi + \frac{1}{2}\nabla\phi \cdot \nabla(\nabla\phi)^2 = 0 + O(\phi^4). \quad (115)$$

The first-order solution (112) satisfies this third-order boundary condition so that the dispersion relation (95abc) includes a second-order effect of the form

$$\omega^2 = gk(1 + a^2k^2) + O(a^3k^3). \quad (116)$$

This remarkable dependence of frequency on wave amplitude is usually known as *amplitude* (or *nonlinear*) *dispersion*. The modified phase velocity expression is

$$c(k) = \frac{\omega}{k} = \left(\frac{g}{k}\right)^{\frac{1}{2}} (1 + k^2 a^2)^{\frac{1}{2}} \approx \left(\frac{g}{k}\right)^{\frac{1}{2}} \left(1 + \frac{1}{2} a^2 k^2\right). \quad (117)$$

The significant change from the linearized theory is (116) or (117), which confirms that the phase velocity now depends on amplitude a as well as on wavelength; steep waves travel faster than less steep waves of the same wavelength. Surface gravity waves thus acquire amplitude dispersion as a second-order correction. The dependence of c on amplitude is known as the *amplitude dispersion* in contrast to the *frequency dispersion* as given by (95abc). It may be noted that Stokes' results (114), (116), and (117) can easily be approximated further to obtain solutions for long waves (or shallow water) and for short waves (or deep water).

We next discuss the phenomenon of breaking of water waves which is one of the most common observable phenomena on an ocean beach. A wave coming from deep ocean changes shape as it moves across a shallow beach. Its amplitude and wavelength also are modified. The wavetrain is very smooth some distance offshore, but as it moves inshore, the front of the wave steepens noticeably until, finally, it breaks. After breaking, waves continue to move inshore as a series of bores or hydraulic jumps, whose energy is gradually dissipated by means of the water turbulence. Of the phenomena common to waves on beaches, breaking is the physically most significant and mathematically least known. In fact, it is one of the most intriguing longstanding problems of water waves.

For waves of small amplitude in deep water, the maximum particle velocity is $v = a\omega = ack$. But the basic assumption of small amplitude theory implies that $\frac{v}{c} = ak \ll 1$. Therefore, wave breaking can never be predicted by the small amplitude wave theory. That possibility arises only in the theory of finite amplitude waves. It is to be noted that the Stokes expansions are limited to relatively small amplitude and cannot predict the wavetrain of maximum height at which the crests are found to be very sharp. For a wave profile of constant shape moving at a uniform velocity, it can be shown that the maximum total crest angle as the wave begins to break is 120° as shown in Fig. 4.

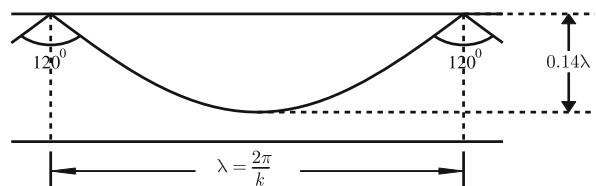
The upshot of the Stokes analysis reveals that the inclusion of higher-order terms in the representation of the surface wave profile distorts its shape away from the linear sinusoidal curve. The effects of nonlinearity are likely to make crests narrower (sharper) and the troughs flatter as depicted in Figure 2.7 of Debnath (1994). The resulting wave profile more accurately portrays the water waves that are observed in nature. Finally, the sharp crest angle of 120° was first found by Stokes.

On the other hand, in 1865, Rankine conjectured that there exists a wave of extreme height. In a moving reference frame, the Euler equations are Galilean invariant, and the Bernoulli Eq. (36) on the free surface of water with $\rho = 1$ becomes

$$\frac{1}{2} |\nabla\phi|^2 + gz = E. \quad (118)$$

Thus, this equation represents the conservation of local energy, where the first term is the kinetic energy of the fluid and the second term is the potential energy due to gravity. For the wave of maximum height, $E = gz_{\max}$, where $z_{\max}$ is the maximum height of the fluid. Thus, the velocity is

Water Waves and the Korteweg-de Vries Equation, Fig. 4 The steepest wave profile



zero at the maximum height so that there will be a stagnation point in the fluid flow. Rankine conjectured that a cusp is developed at the peak of the free surface with a vertical slope so that the angle subtended at the peak is 120° as also conjectured by Stokes (1847). Toland (1978) and Amick et al. (1982) have proved rigorously the existence of a wave of greatest height and the Stokes conjecture for the wave of extreme form. However, Toland (1978) also proved that if the singularity at the peak is *not* a cusp, that is, if there is no vertical slope at the peak of the free surface, then the Stokes remarkable conjecture of the crest angle of 120° is true. Subsequently, Amick et al. (1982) confirmed that the singularity at the peak is *not* a cusp. Therefore, the full Euler equations exhibit singularities, and there is a limiting amplitude to the periodic waves.

The nonlinear solutions for plane waves based on systematic power series in the wave amplitude are known as *Stokes expansions*. Stokes (1847) showed that the free surface elevation η of a plane wavetrain on deep water can be expanded in powers of the amplitude a as

$$\eta = a \cos \theta + \frac{1}{2}ka^2 \cos 2\theta + \frac{3}{8}k^2a^3 \cos 3\theta + \dots, \quad (119)$$

where $\theta = kx - \omega t$ and the square of the frequency is

$$\omega^2 = gk \left(1 + a^2k^2 + \frac{5}{4}a^4k^4 + \dots \right). \quad (120)$$

A question was raised about the convergence of the Stokes expansion in order to prove the existence of solution representing periodic water waves of permanent form. Considerable attention had been given to this problem by several authors. The problem was eventually resolved by Levi-Civita (1925), who proved formally that the Stokes expansion for deep water converges provided the wave steepness (ak) is very small. Struik (1926) extended the proof of Levi-Civita to small-amplitude waves on water of arbitrary, but constant depth. Subsequently, Kraskovskii (1960, 1961) finally established the existence of permanent

periodic waves for all amplitudes less than the extreme value at which the waves assume a sharp-crested form. Although the success of the perturbation expansion as a means of representing waves of finite amplitude depends on the convergence of Stokes' expansion, convergence *does not* at all imply stability. In spite of the preceding attempts to establish the possibility of finite-amplitude water waves of permanent form, the question of their stability was altogether ignored until the 1960s. The independent work of Lighthill (1965, 1967), Benjamin (1967) and Whitham (1967a, b) finally established that Stokes waves on deep water are definitely *unstable*! This is one of the most remarkable discoveries in the history of the subject of the theory of water waves.

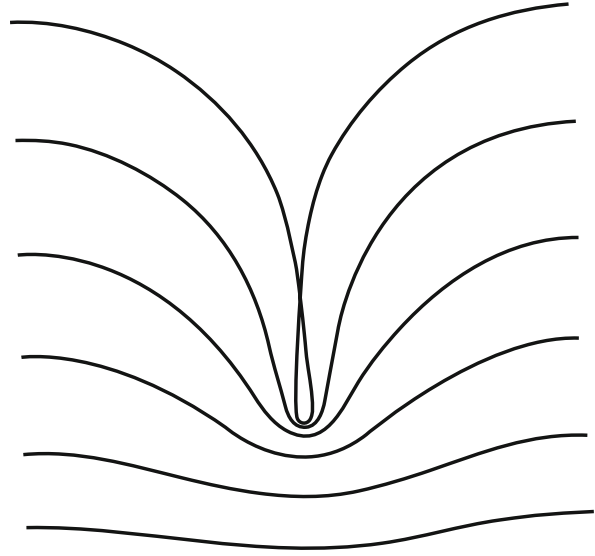
As a complement to the Stokes theory of pure gravity waves, Crapper (1957) first discovered the *exact nonlinear* solution for pure progressive capillary waves of arbitrary amplitude by using a complex variable method. He obtained a remarkably new result for the phase velocity

$$c(k) = \left(\frac{kT}{\rho} \right)^{\frac{1}{2}} \left(1 + \frac{\pi^2 H^2}{4\lambda} \right)^{-1/4}, \quad (121)$$

where $H = 2a$ is the wave height representing the vertical distance between crest and trough. This result clearly shows that the phase velocity decreases as the wave amplitude increases for a fixed wavelength. Result (121) is now known as *Crapper's nonlinear capillary wave solution*. It has also been shown by Crapper that the capillary wave of greatest height occurs when $H = 0.73\lambda$, which has a striking contrast with the corresponding result due to Michell (1893) for pure gravity waves, $H = 0.142\lambda = \frac{1}{7}\lambda$. Figure 5 exhibits Crapper's nonlinear capillary wave profiles.

One of the important features of Crapper's solution is that there is a maximum possible steepness for pure capillary waves, $(H/\lambda) = 0.73$, at which the waves hit each other. In the case of pure gravity waves, Stokes suggested that the wave steepness would possibly be greatest when the crest actually becomes a point with the maximum included crest angle of 120° . Miche (1944) obtained the maximum wave height in water of finite depth h as

Water Waves and the Korteweg-de Vries Equation, Fig. 5 Surface profile for nonlinear capillary waves up to maximum possible wave, $(H/\lambda) = 0.73$ (from (Crapper 1957))



$$\left(\frac{H}{\lambda}\right)_{\max} = (0.14) \tanh kh, \quad (122)$$

which, in very shallow water ($kh \ll 1$), gives

$$\left(\frac{H}{h}\right)_{\max} = 0.88. \quad (123)$$

This result is in excellent agreement with experimental observations. As the ratio of wave amplitude to local water depth increases, waves are gradually deformed and seem to behave more and more like a series of solitary waves or a train of cnoidal waves. As revealed by reliable observations, soon they become unstable and then break, forming a whitecap. However, little is known about the detailed nature of wave breaking except for the fact that it is a typical complicated nonlinear phenomenon. It has also been predicted that swell in shallow water tends to break when the crest-to-trough height is about 0.88 times the local water depth.

Surface Gravity Waves on a Running Stream in Water

Debnath and Rosenblat (1969) solved the *initial value problem* for the generation and propagation of two-dimensional surface waves at the free

surface of a running stream on water of finite depth. With the aid of generalized functions, it is proved that the asymptotic solution of the initial value problem leads to the ultimate steady-state solution and the transient solution *without* the need to resort to the use of a radiation condition at infinity or equivalent device.

The fundamental two-dimensional water wave equations on a running stream with velocity U in the x -direction in water of depth h (see Debnath (1994), page 115) are given by

$$\phi_{xx} + \phi_{zz} = 0, \quad -h \leq z \leq 0, \quad -\infty < x < \infty, \quad t > 0, \quad (124)$$

$$\eta_t + U\eta_x - \phi_z = 0, \quad \text{on } z = 0, \quad t > 0, \quad (125)$$

$$\phi_t + U\phi_x + g\eta = -\frac{P}{\rho}p(x)e^{i\omega t}, \quad z = 0, \quad t > 0, \quad (126)$$

$$\phi_z = 0, \quad \text{on } z = -h, \quad (127)$$

where the term on the right hand side of (126) represents the arbitrary applied pressure with frequency ω , P is a constant and $p(x)$ is arbitrary function of x .

Making reference to Debnath (1994) for a detailed method of solution and asymptotic analysis of the solution of the initial value problem, we discuss only the *inherentresonant(critical) phenomenon* involved in this problem. For certain limiting values of the three quantities U , ω , and h , there exists a critical speed U_c such that

$$(a) \quad U = U_c = \frac{1}{4} \left(\frac{g}{\omega} \right) \quad \text{for } h \rightarrow \infty, \quad \omega \text{ and } U \text{ are finite.} \quad (128)$$

and

$$(b) \quad U = U_c = \sqrt{gh} \quad \text{for } \omega = 0, \quad h \text{ and } U \text{ are finite.} \quad (129)$$

A special property of (a) and (b) is that in both situations there exists a critical value of the speed U , above and below which the steady-state wave system has respectively quite a different character; and that when U assumes its critical value, U_c the solution is singular due presumably to a breakdown of the linearized theory. More precisely, when $U > U_c$, these are two surface waves downstream of the origin traveling with speeds (ω/s_1) and (ω/s_2) , respectively, in the position x -direction; there are none upstream, where $-s_1$ and $-s_2$ are two roots (see Debnath and Rosenblat (1969)) of the equations

$$(\omega + kU) \mp (gk \tanh kh)^{\frac{1}{2}} = 0. \quad (130ab)$$

On the other hand, when $U < U_c$, these are two more surface waves which move with speeds (ω/σ_1) and (ω/σ_2) , both in the negative x -direction, where σ_1 and σ_2 are arising from (130ab) at positive values of $k = \sigma_1$ and $k = \sigma_2$. The former of these exists only on the upstream side of the origin, and the latter only on the downstream side. This latter wave thus appears to originate at infinity, but this is only when it is viewed from the moving coordinate system. It is easily verified from equation (130ab) that the speed (ω/σ_1) is always less than U , the speed of the frame, so

that relative to axes at rest this wave travels in a positive direction. Depending on U , ω and h , there is a delimiting case when roots σ_1 and σ_2 coalesce into a double root. This is the critical case which demand a certain relationship between physical quantities involved, which can be combined into velocities U , $\sqrt{gh}$ and (g/ω) . Indeed, when $h \rightarrow \infty$, we have the case (a), and when $\omega = 0$, the case (b) occurs.

When $\omega = 0$, the wave system is degenerate. In this case, $s_1 = s_2 = \sigma_1 = 0$ and the only wave that occurs is the one associated with the root σ_2 . As before, it exists on the downstream side, when $U > U_c$, only, but is now a steady motion relative to the moving frame. This is the case discussed by Stoker (1957, p. 214) who found a similar behavior.

Combining the results (128) and (129) together, we see that there are three quantities involved having the dimensions of velocity, namely U , $(\frac{g}{\omega})$ and $\sqrt{gh}$. In general, all these three quantities are finite, and hence the critical situation can be expressed as a functional relationship between them of the form

$$F\left(U, \quad g/\omega, \quad \sqrt{gh}\right) = 0. \quad (131)$$

This function F is found and is seen to yield U as a single-valued function of (g/ω) and $\sqrt{gh}$, with formulas (128) and (129) emerging as limiting cases.

Debnath and Rosenblat (1969) also showed that the free surface elevation $\eta(x, t)$ becomes singular when roots σ_1 and σ_2 coalesce into a double root, σ . Their asymptotic evaluation of $\eta(x, t)$ for large $|x|$ and t reveals that it represents two waves where the amplitude of one wave increases like x , while the amplitude of other wave is of $\frac{1}{t^2}$ as $t \rightarrow \infty$. This singular behavior is not unexpected and is in accord with the findings of Stoker (1957) and Kaplan (1957). Mathematically, this critical situation corresponds to the coalescence of two roots ($\sigma_1 = \sigma_2 = \sigma$), which is in turn is equivalent to the reinforcement by superposition of two like waves leading to a *resonance-type effect*. Physically, this situation would reveal itself through wave motions of large amplitude, which cannot

come within the scope of linearized theory. Consequently, it would be necessary to include nonlinear terms in the original formulation of the problem in order to achieve a mathematically valid and physically reasonable solution.

Akylas (1984a) developed a nonlinear theory near the critical speed $U_c = \sqrt{gh}$ under the action of surface pressure $p(x)$ traveling at a constant speed U . With slow time scale $T = \varepsilon t$ and the slow space variable $X = \varepsilon^{\frac{1}{3}}x$, where $\varepsilon = \frac{a}{h}$, his asymptotic analysis reveals that the nonlinear response is of bounded amplitude $A(X, t)$ and is governed by a *forced Korteweg and de Vries (KdV)* equation in the form

$$A_T + \gamma A_X - 2AA_X - \frac{1}{6}A_{XXX} = \pi \bar{p}(0)\delta'(X), \quad (132)$$

where $(U/\sqrt{gh}) = 1 + \gamma\varepsilon^{\frac{2}{3}}$, $\gamma = O(1)$, $\bar{p}(k)$ is the Fourier transform of $p(X)$ and $\delta(X)$ is the Dirac delta function.

The main conclusion of this asymptotic analysis is that the far field disturbance is of relatively large amplitude, but it remains bounded. The main question whether the nonlinear response evolves to solitons, or disperses out or even gives a cnoidal wave, still remains open. However, it was shown by numerical study that Eq. (132) admits a series of solitons that are generated in front of the pressure field. For $\gamma < 0$, the linearized solution consists of a periodic sinusoidal wave in $X < 0$. The nonlinear evolution of the wave disturbance at the resonance condition ($\gamma = 0$) remains bounded. The major conclusion of this study is that a series of solitary waves (or solitons) is successively generated when $X < 0$ and propagates in front of the pressure field. This prediction is in theoretical agreement with the nonlinear study of Wu and Wu (1982), and also in good agreement with experimental findings of Huang et al. (1982) who observed solitons in their experiments involving a ship moving in shallow water.

Akylas (1984b) also developed a nonlinear theory to extend Debnath and Rosenblat's linearized study for the evolution of surface waves generated by a moving oscillatory pressure near the *critical*(or *resonant*) speed. Based on the slow time scale $T = \varepsilon t$ and the slow spatial variable

$X = \sqrt{\varepsilon}x$, his nonlinear asymptotic analysis reveals that the evolution for the wave envelope $A(X, T)$ is governed by the *forced nonlinear Schrödinger equation*. For details, the reader is referred to Akylas (1984b) or Debnath (1994).

History of Russell's Solitary Waves and Their Interactions

Historically, John Scott Russell first experimentally observed the solitary wave, a long water wave without change in shape, on the Edinburgh–Glasgow Canal in 1834. He called it the “*great wave of translation*” and then reported his observations at the British Association in his 1844 paper “Report on Waves.” Thus, the solitary wave represents, not a periodic wave, but the propagation of a single isolated symmetrical hump of unchanged form. His discovery of this remarkable phenomenon inspired him further to conduct a series of extensive laboratory experiments on the generation and propagation of such waves. Based on his experimental findings, Russell discovered, empirically, one of the most important relations between the speed U of a solitary wave and its maximum amplitude a above the free surface of liquid of finite depth h in the form

$$U^2 = g(h + a), \quad (133)$$

where g is the acceleration due to gravity. His experiments stimulated great interest in the subject of water waves and his findings received a strong criticism from G.B. Airy (1845) and G.G. Stokes (1847). In spite of his remarkable work on the existence of periodic wavetrains representing a typical feature of nonlinear dispersive wave systems, Stokes' conclusion on the existence of the solitary wave was erroneous.

However, Stokes (1847) proposed that the free surface elevation of the plane wavetrains on deep water can be expanded in powers of the wave amplitude. His original result for the dispersion relation in deep water is given by (120). Despite these serious attempts to prove the existence of finite-amplitude water waves of permanent form, the independent question of their stability remained unanswered until the 1960s except for an isolated

study by Korteweg and de Vries in 1895 on long surface waves in water of finite depth. But one of the most remarkable discoveries made in the 1960s was that the periodic Stokes waves on sufficiently deep water are *definitely* unstable. This result seems revolutionary in view of the sustained attempts to prove the existence of Stokes waves of finite amplitude and permanent form. Scott Russell's discovery of solitary waves contradicted the theories of water waves due to Airy and Stokes; they raised questions on the existence of Russell's solitary waves and conjectured that such waves cannot propagate in a liquid medium without change of form. It was not until the 1870s that Russell's prediction was finally and independently confirmed by both J. Boussinesq (1871a, b, 1872, 1877) and Lord Rayleigh (1876). From the equations of motion for an inviscid incompressible liquid, they derived formula (133). In fact, they also showed that the Russell's solitary wave profile (see Fig. 6) $z = \eta(x, t)$ is given by

$$\eta(x, t) = a \operatorname{sech}^2[\beta(x - Ut)], \quad (134)$$

where $\beta^2 = 3a \div \{4h^2(h + a)\}$ for any $a > 0$.

Although these authors found the $a > 0$ solution, which is valid only if $a \ll h$, they did not write any equation for η that admits (134) as a solution. However, Boussinesq made much more progress and discovered several new ideas, including a nonlinear evolution equation for such long water waves in the form

$$\eta_{tt} = c^2 \left[\eta_{xx} + \frac{3}{2} \left(\frac{\eta^2}{h} \right)_{xx} + \frac{1}{3} h^2 \eta_{xxx} \right], \quad (135)$$

where $c = \sqrt{gh}$ is the speed of the shallow water waves. This is known as the *Boussinesq(bidirectional) equation*, which admits the solution

$$\eta(x, t) = a \operatorname{sech}^2 \left[(3a/h^3)^{1/2} (x \pm Ut) \right]. \quad (136)$$

This represents solitary waves traveling in both, the positive and negative x -directions.

More than 60 years later, in 1895, two Dutchmen, D.J. Korteweg (1848–1941) and G. de Vries (Korteweg and de Vries 1895), formulated a mathematical model equation to provide an explanation of the phenomenon observed by Scott Russell. They derived the now-famous equation for the propagation of waves in one direction on the surface water of density ρ in the form

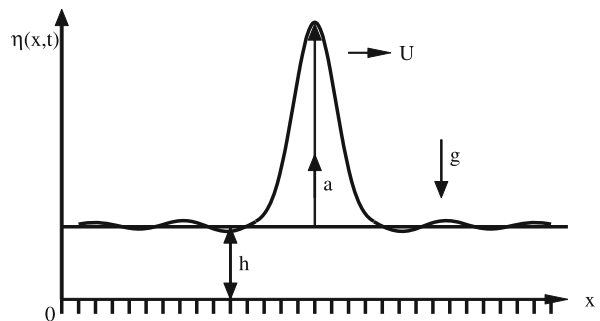
$$\eta_t = \frac{c}{h} \left[\left(\varepsilon + \frac{3}{2} \eta \right) \eta_X + \frac{1}{2} \sigma \eta_{XXX} \right], \quad (137)$$

where X is a coordinate chosen to be moving (almost) with the wave, $c = \sqrt{gh}$, ε is a small parameter, and

$$\sigma = h \left(\frac{h^2}{3} - \frac{T}{g\rho} \right) \sim \frac{1}{3} h^3, \quad (138)$$

when the surface tension $T (\ll \frac{1}{3} g\rho h^2)$ is negligible. Equation (137) is known as the *Korteweg–de Vries (KdV) equation*. This is one of the simplest and most useful nonlinear model equations for solitary waves, and it represents the longtime evolution of wave phenomena in which the steepening effect of the nonlinear term is counterbalanced by the smoothing effect of the linear dispersion.

Water Waves and the Korteweg-de Vries Equation,
Fig. 6 A solitary wave



It is convenient to introduce the change of variables $\eta = \eta(X^*, t)$ and $X^* = X + (\varepsilon/h)ct$ which, dropping the asterisks, allows us to rewrite Eq. (137) in the form

$$\eta_t = \frac{c}{h} \left(\frac{3}{2} \eta \eta_x + \frac{1}{2} \sigma \eta_{xxx} \right). \quad (139)$$

Modern developments in the theory and applications of the KdV solitary waves began with the seminal work published as a Los Alamos Scientific Laboratory Report in 1955 by Fermi, Pasta, and Ulam (1955) on a numerical model of a discrete nonlinear mass-spring system. In 1914, Debye suggested that the finite thermal conductivity of an anharmonic lattice is due to the nonlinear forces in the springs. This suggestion led Fermi, Pasta, and Ulam to believe that a smooth initial state would eventually relax to an equipartition of energy among all modes because of nonlinearity. But their study led to the striking conclusion that there is no equipartition of energy among the modes. Although all the energy was initially in the lowest modes, after flowing back and forth among various low-order modes, it eventually returns to the lowest mode, and the end state is a series of recurring states. This remarkable fact has become known as the *Fermi–Pasta–Ulam (FPU) recurrence phenomenon*. Cercignani (1977) and later on Palais (1997) described the FPU experiment and its relationship to the KdV equation in some detail.

This curious result of the FPU experiment inspired Martin Kruskal and Norman Zabusky (1965) to formulate a continuum model for the nonlinear mass-spring system to understand why recurrence occurred. In fact, they considered the initial-value problem for the KdV equation,

$$u_t + uu_x + \delta u_{xxx} = 0, \quad (140)$$

where $\delta = (\frac{h}{\ell})^2$, ℓ is a typical horizontal length scale, with the initial condition

$$u(x, 0) = \cos \pi x, \quad 0 \leq x \leq 2, \quad (141)$$

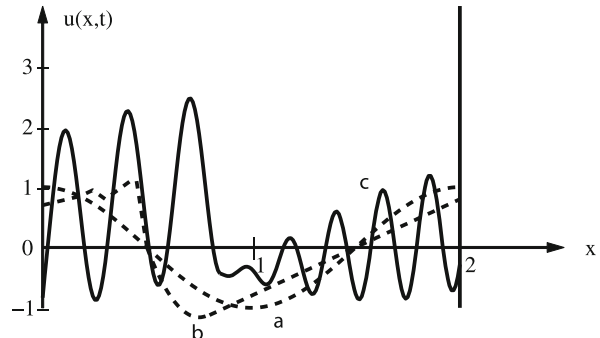
and the periodic boundary conditions with period 2, so that $u(x, t) = u(x + 2, t)$ for all t . Their numerical study with $\sqrt{\delta} = 0.022$ produced remarkably new interesting results, which are shown in Fig. 7.

They observed that, initially, the wave steepened in regions where it had a negative slope, a consequence of the dominant effects of nonlinearity over the dispersive term, δu_{xxx} . As the wave steepens, the dispersive effect then becomes significant and balances the nonlinearity. At later times, the solution develops a series of *eight* well-defined waves, each like sech^2 functions with the taller (faster) waves ever catching up and overtaking the shorter (slower) waves. These waves undergo nonlinear interaction according to the KdV equation and then emerge from the interaction without change of form and amplitude, but with only a small change in their phases. So, the most remarkable feature is that these waves retain their identities after the nonlinear interaction. Another surprising fact is that the initial profile reappears, very similarly to the FPU recurrence phenomenon. In view of their preservation of shape and the resemblance to the particle-like character of these waves, Kruskal and Zabusky called these solitary waves, *solitons*, like photon, proton, electron, and other terms for elementary particles in physics.

Historically, the famous 1965 paper of Zabusky and Kruskal (1965) marked the birth of the new

Water Waves and the Korteweg-de Vries Equation,

Fig. 7 Development of solitary waves: **a** initial profile at $t = 0$, **b** profile at $t = \pi^{-1}$, and **c** wave profile at $t = (3.6)\pi^{-1}$ (from (Zabusky and Kruskal 1965))



concept of the *soliton*, a name intended to signify particle-like quantities. Subsequently, Zabusky (1967) confirmed, numerically, the actual physical interaction of two solitons, and Lax (1968) gave a rigorous analytical proof that the identities of two distinct solitons are preserved through the nonlinear interaction governed by the KdV equation. Physically, when two solitons of different amplitudes (and hence, of different speeds) are placed far apart on the real line, the taller (faster) wave to the left of the shorter (slower) wave, the taller one eventually catches up to the shorter one and then overtakes it. When this happens, they undergo a nonlinear interaction according to the KdV equation and emerge from the interaction completely preserved in form and speed with only a phase shift. Thus, these two remarkable features, (i) steady progressive pulse-like solutions and (ii) the preservation of their shapes and speeds, confirmed the particle-like property of the waves and, hence, the definition of the soliton. Subsequently, Gardner et al. (1967, 1974) and Hirota (1971, 1973a, b) constructed analytical solutions of the KdV equation that provide the description of the interaction among N solitons for any positive integral N . After the discovery of the complete integrability of the KdV equation in 1967, the theory of the KdV equation and its relationship to the Euler equations of motion as an approximate model derived from the theory of asymptotic expansions became of major interest. From a physical point of view, the KdV equation is not only a standard nonlinear model for long water waves in a dispersive medium, it also arises as an approximate model in numerous other fields, including ion-acoustic plasma waves, magnetohydrodynamic waves, and anharmonic lattice vibrations. Experimental confirmation of solitons and their interactions has been provided successfully by

Zabusky and Galvin (1971), Hammack and Segur (1974), and Weidman and Maxworthy (1978). Thus, these discoveries have led, in turn, to extensive theoretical, experimental, and computational studies over the last 40 years. Many nonlinear model equations have now been found that possess similar properties, and diverse branches of pure and applied mathematics have been required to explain many of the novel features that have appeared.

The Korteweg–de Vries and Boussinesq Equations

We consider an inviscid liquid of constant mean depth h and constant density ρ without surface tension. We assume that the (x, y) -plane is the undisturbed free surface with the z -axis positive upward. The free surface elevation above the undisturbed mean depth h is given by $z = \eta(x, y, t)$, so that the free surface is at $z = H = h + \eta$ and $z = 0$ is the horizontal rigid bottom (see Fig. 8).

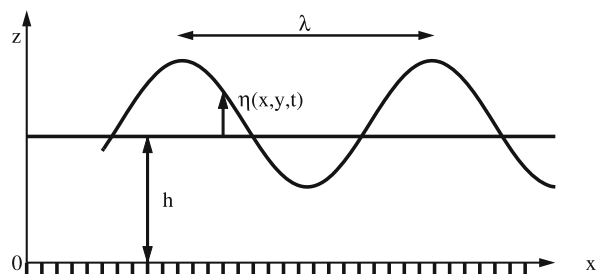
It has already been recognized that the parameters $\varepsilon = (a/h)$ and $\kappa = ak$, where a is the surface wave amplitude and k is the wavenumber, must both be small for the linearized theory of surface waves to be valid. To develop the nonlinear shallow water theory, it is convenient to introduce the following nondimensional flow variables based on a different length scale h (which could be the fluid depth):

$$\begin{aligned} (x^*, y^*) &= \frac{1}{l}(x, y), \quad z^* = \frac{z}{h}, \\ t^* &= \left(\frac{ct}{l}\right), \quad \eta^* = \frac{\eta}{a}, \quad \phi^* = \left(\frac{h}{alc}\right)\phi, \end{aligned} \quad (142)$$

where $c = \sqrt{gh}$ is the typical horizontal velocity (or shallow water wave speed).

Water Waves and the Korteweg-de Vries Equation,

Fig. 8 A shallow water wave model



We next introduce two fundamental parameters to characterize the nonlinear shallow water waves:

$$\varepsilon = \frac{a}{h} \quad \text{and} \quad \delta = \frac{h^2}{l^2}, \quad (143)$$

where ε is called the *amplitude* parameter and $\sqrt{\delta}$ is called the *long wavelength* or *shallowness* parameter.

In terms of the preceding nondimensional flow variables and the parameters, the basic equations for water waves (73)–(76) can be written in the nondimensional form, dropping the asterisks:

$$\begin{aligned} \delta(\phi_{xx} + \phi_{yy}) + \phi_{zz} &= 0, \quad 0 \leq z \\ &\leq 1 + \varepsilon\eta, \end{aligned} \quad (144)$$

$$\begin{aligned} \frac{\partial \phi}{\partial t} + \frac{\varepsilon}{2}(\phi_x^2 + \phi_y^2) + \frac{\varepsilon}{2\delta}\phi_z^2 + \eta \\ = 0 \quad \text{on} \quad z = 1 + \varepsilon\eta, \end{aligned} \quad (145)$$

$$\delta[\eta_t + \varepsilon(\phi_x \eta_x + \phi_y \eta_y)] - \phi_z = 0 \quad \text{on} \quad z = 1 + \varepsilon\eta, \quad (146)$$

$$\phi_z = 0 \quad \text{on} \quad z = 0. \quad (147)$$

It is noted that the parameter $\kappa = ak$ does not enter explicitly in Eqs. (144)–(147), but an equivalent parameter $\gamma = (a/l)$ is associated with ε and δ through $\gamma = (a/h) \cdot (h/l) = \varepsilon\sqrt{\delta}$.

If ε is small, the terms involving ε in (145) and (146) can be neglected to recover the linearized free surface conditions. However, the assumption that δ is small might be interpreted as the characteristic feature of the shallow water theory. So, we expand ϕ in terms of δ without any assumption about ε , and write

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + \dots, \quad (148)$$

and then substitute in (144)–(146). The lowest-order term in (144) is

$$\phi_{0zz} = 0, \quad (149)$$

which, with (147), yields $\phi_{0z} = 0$, for all z , or $\phi_0 = \phi_0(x, y, t)$, which indicates that the horizontal velocity components are independent of the vertical coordinate z in lowest order. Consequently, we use the notation

$$\phi_{0x} = u(x, y, t) \quad \text{and} \quad \phi_{0y} = v(x, y, t). \quad (150ab)$$

The first- and second-order terms in (144) are given by

$$\phi_{0xx} + \phi_{0yy} + \phi_{1zz} = 0, \quad (151)$$

$$\phi_{1xx} + \phi_{1yy} + \phi_{2zz} = 0. \quad (152)$$

Integrating (151) with respect to z and using (150ab) gives

$$\phi_{1z} = -z(u_x + v_y) + C(x, y, t), \quad (153)$$

where the arbitrary function $C(x, y, t)$ becomes zero because of the bottom boundary condition (147). Integrating the resulting Eq. (153), again with respect to z and omitting the arbitrary constant, we obtain

$$\phi_1 = -\frac{z^2}{2}(u_x + v_y), \quad (154)$$

so that $\phi_1 = 0$ at $z = 0$ and u and v are then the horizontal velocity components at the bottom boundary.

We next substitute (154) in (151) and (152), and then integrate with condition (147) to determine the arbitrary function. Consequently,

$$\begin{aligned} \phi_{2z} &= \frac{1}{6}z^3[(\nabla^2 u)_x + (\nabla^2 v)_y], \\ \phi_2 &= \frac{1}{24}z^4[(\nabla^2 u)_x + (\nabla^2 v)_y], \end{aligned} \quad (155ab)$$

where ∇^2 is the two-dimensional Laplacian.

We next consider the free surface boundary conditions retaining all terms up to order δ , ε in

(145), and δ^2 , ε^2 , and $\delta\varepsilon$ in (146). It turns out that conditions (145) and (146) become

$$\phi_{0t} - \frac{\delta}{2}(u_{tx} + v_{ty}) + \eta + \frac{1}{2}\varepsilon(u^2 + v^2) = 0, \quad (156)$$

$$\begin{aligned} & \delta \left[\left\{ \eta_t + \varepsilon(u\eta_x + v\eta_y) \right\} + (1 + \varepsilon\eta)(u_x + v_y) \right] \\ &= \frac{\delta^2}{6} \left[(\nabla^2 u)_x + (\nabla^2 v)_y \right]. \end{aligned} \quad (157)$$

Differentiating (156) first with respect to x and then with respect to y gives two equations:

$$u_t + \varepsilon(uu_x + vv_x) + \eta_x - \frac{1}{2}\delta(u_{txx} + v_{txy}) = 0, \quad (158)$$

$$v_t + \varepsilon(uu_y + vv_y) + \eta_y - \frac{1}{2}\delta(u_{txy} + v_{tyy}) = 0. \quad (159)$$

Simplifying (157) yields

$$\begin{aligned} & \eta_t + [u(1 + \varepsilon\eta)]_x + [v(1 + \varepsilon\eta)]_y \\ &= \frac{\delta}{6} \left[(\nabla^2 u)_x + (\nabla^2 v)_y \right]. \end{aligned} \quad (160)$$

Evidently, Eqs. (158)–(160) represent the non-dimensional *shallow water equations*.

Using the fact that ϕ_0 is irrotational, that is, $u_y = v_x$ and neglecting terms $O(\delta)$ in (158)–(160), we obtain the fundamental shallow water equations

$$u_t + \varepsilon(uu_x + vv_x) + \eta_x = 0, \quad (161)$$

$$v_t + \varepsilon(uv_x + vv_y) + \eta_y = 0, \quad (162)$$

$$\eta_t + [u(1 + \varepsilon\eta)]_x + [v(1 + \varepsilon\eta)]_y = 0. \quad (163)$$

This system of three, coupled, nonlinear equations is closed and admits some interesting and useful solutions for u , v , and η . It is equivalent to the boundary-layer equations in fluid mechanics. Finally, it can be linearized when $\varepsilon = (a/h) \ll 1$ to obtain the following dimensional equations:

$$\begin{aligned} u_t + g\eta_x &= 0, & v_t + g\eta_y \\ &= 0, & \eta_t + h(u_x + v_y) = 0. \end{aligned} \quad (164abc)$$

Eliminating u and v from these equations gives

$$\eta_{tt} = c^2(\eta_{xx} + \eta_{yy}). \quad (165)$$

This is a well-known *two-dimensional wave equation*. It corresponds to the nondispersive shallow water waves that propagate with constant velocity $c = \sqrt{gh}$. This velocity is simply the linearized version of $\sqrt{g(h + \eta)}$. The wave equation has the simple *d'Alembert solution* representing plane progressive waves

$$\begin{aligned} \eta(x, y, t) &= f(k_1x + l_1y - \kappa_1ct) \\ &\quad + g(k_2x + l_2y - \kappa_2ct), \end{aligned} \quad (166)$$

where f and g are arbitrary functions and $\kappa_r^2 = (k_r^2 + l_r^2)$, $r = 1, 2$.

We consider the one-dimensional case retaining both ε and δ order terms in (158)–(160) so that these equations reduce to the *Boussinesq* (1871a, b, 1872, 1877) equations

$$u_t + \varepsilon uu_x + \eta_x - \frac{1}{2}\delta u_{txx} = 0, \quad (167)$$

$$\eta_t + [u(1 + \varepsilon\eta)]_x - \frac{1}{6}\delta u_{xxx} = 0. \quad (168)$$

On the other hand, Eqs. (161)–(163), expressed in dimensional form, read

$$u_t + uu_x + vu_y + gH_x = 0, \quad (169)$$

$$v_t + uv_x + vv_y + gH_y = 0, \quad (170)$$

$$H_t + (uH)_x + (vH)_y = 0, \quad (171)$$

where $H = (h + \eta)$ is the total depth and $H_x = \eta_x$, since the depth h is constant.

In particular, the one-dimensional version of the shallow water equations follows from (169)–(171) and is given by

$$u_t + uu_x + gH_x = 0, \quad (172)$$

$$H_t + (uH)_x = 0. \quad (173)$$

This system of approximate shallow water equations is analogous to the exact governing equations of gas dynamics for the case of a compressible flow involving only one space variable (see Riabouchinsky (1932)).

It is convenient to rewrite these equations in terms of the wave speed $c = \sqrt{gH}$ by using $dc = (g/2c)dH$, so that they become

$$u_t + uu_x + 2cc_x = 0, \quad (174)$$

$$2c_t + cu_x + 2uc_x = 0. \quad (175)$$

The standard method of characteristics can easily be used to solve (174) and (175). Adding and subtracting these equations allows us to rewrite them in the characteristic form

$$\left[\frac{\partial}{\partial t} + (c + u) \frac{\partial}{\partial x} \right] (u + 2c) = 0, \quad (176)$$

$$\left[\frac{\partial}{\partial t} + (c - u) \frac{\partial}{\partial x} \right] (u - 2c) = 0. \quad (177)$$

Equations (176) and (177) show that $u + 2c$ propagates in the positive x -direction with velocity $c + u$, and $u - 2c$ travels in the negative x -direction with velocity $c - u$, that is, both $u + 2c$ and $u - 2c$ propagate in their respective directions with velocity c relative to the water. In other words

$du + 2c = \text{constant}$ on curves C_+ on which

$$\frac{dx}{dt} = u + c$$

$u - 2c = \text{constant}$ on curves C_- on which

$$\frac{dx}{dt} = u - c$$

(178ab)

Where C^+ and C^- are *characteristic curves* of the system of partial differential Eqs. (172) and (173). A disturbance propagates along these characteristic curves at speed c relative to the flow speed. The

quantities $(u \pm 2c)$ are called the *Riemann invariants* of the system, and a simple wave is propagating to the right into water of depth h , that is, $u - 2c = c_0 = \sqrt{gh}$. Then, the solution is given by

$$u = f(\xi), \quad x = \xi + \left(c_0 + \frac{3}{2}u \right) t, \quad (179)$$

where $u(x, t) = f(x)$ at $t = 0$. However, we note that

$$u_x = \left(1 - \frac{3}{2}u_x t \right) f'(\xi),$$

giving

$$u_x = \frac{2f'(\xi)}{2 + 3tf'(\xi)}. \quad (180)$$

Thus, if $f'(\xi)$ is anywhere less than zero, u_x tends to infinity as $t \rightarrow -2/(3f')$. In terms of the free surface elevation, solution (179) implies that the wave profile progressively distorts itself, and, in fact, any forward-facing portion of such a wave continually steepens, or the higher parts of the wave tend to catch up with lower parts in front of them. Thus, all of these waves, carrying an increase of elevation, invariably break. The breaking of water waves on beaches is perhaps the most common and the most striking phenomenon in nature.

An alternative system equivalent to the nonlinear evolution Eqs. (167) and (168) can be derived from the nonlinear shallow water theory, retaining both ε and δ order terms with $\delta < 1$. This system is also known as the *Boussinesq equations*, which, in dimensional variables, are given by

$$\eta_t + [(h + \eta)u]_x = 0, \quad (181)$$

$$u_t + uu_x + g\eta_x = \frac{1}{3}h^2u_{xxt}. \quad (182)$$

They describe the evolution of long water waves that move in both positive and negative x -directions. Eliminating η and neglecting terms smaller than $O(\varepsilon, \delta)$ gives a single *Boussinesq equation* for $u(x, t)$ in the form

$$u_{tt} - c^2 u_{xx} + \frac{1}{2} (u^2)_{xt} = \frac{1}{3} h^2 u_{xxx}. \quad (183)$$

The linearized Boussinesq equation for u and η follows from (181) and (182) as

$$\left[\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} - \frac{1}{3} h^2 \frac{\partial^4}{\partial x^2 \partial t^2} \right] \begin{pmatrix} u \\ \eta \end{pmatrix} = 0. \quad (184)$$

This is in perfect agreement with the infinitesimal wave theory result expanded for small kh . Thus, the third derivative term in (182) may be identified with the frequency dispersion.

Another equivalent version of the Boussinesq equation is given by

$$\eta_{tt} - c^2 \eta_{xx} = \frac{3}{2} \left(\frac{\eta^2}{h} \right)_{xx} + \frac{1}{3} h^2 \eta_{xxx}. \quad (185)$$

There are several features of this equation. It is a nonlinear partial differential equation that incorporates the basic idea of nonlinearity and dispersion. Boussinesq obtained three invariant physical quantities, Q , E , and M , defined by

$$\begin{aligned} Q &= \int_{-\infty}^{\infty} \eta dx, \quad E = \int_{-\infty}^{\infty} \eta^2 dx, \\ M &= \int_{-\infty}^{\infty} \left[\eta_x^2 - 3 \left(\frac{\eta}{h} \right)^3 \right] dx, \end{aligned} \quad (186)$$

provided that $\eta \rightarrow 0$ as $|x| \rightarrow \infty$. Evidently, Q and E represent the volume and the energy of the solitary wave. The third quantity M is called the *moment of instability*, and the variational problem, $\delta M = 0$ with E fixed, leads to the unique solitary-wave solution. Boussinesq also derived the results for the amplitude and volume of a solitary wave of given energy in the form

$$a = \frac{3}{4} \left(\frac{E^{3/2}}{h} \right), \quad Q = 2hE^{1/3}. \quad (187ab)$$

The former result shows that the amplitude of a solitary wave in a channel varies inversely as the channel depth h .

The Boussinesq equation can then be written in the normalized form

$$u_{tt} - u_{xx} - \frac{3}{2} (u^2)_{xx} - u_{xxx} = 0. \quad (188)$$

This particular form is of special interest because it admits inverse scattering formalism. Equation (188) has steady progressive wave solutions in the form

$$u(x, t) = 4k^2 f(X), \quad X = kx - \omega t, \quad (189)$$

where the equation for $f(X)$ can be integrated to obtain

$$f_{XX} = 6A + (4 - 6B)f - 6f^2, \quad (190)$$

where A is a constant of integration and

$$\omega^2 = k^2 + k^4(4 - 6B). \quad (191)$$

For the special case $A = B = 0$, a single solitary-wave solution is given by

$$f(X) = \text{sech}^2(X - X_0), \quad (192)$$

where X_0 is a constant of integration. This result can be used to construct a solution for a series of solitary waves, spaced 2σ apart, in the form

$$f(X) = \sum_{n=-\infty}^{\infty} \text{sech}^2(X - 2n\sigma). \quad (193)$$

This is a 2σ periodic function that satisfies (190) for certain values of A and B .

We next assume that ε and δ are comparable, so that all terms $O(\varepsilon, \delta)$ in (158)–(160) can be retained. For the case of the two-dimensional wave motion ($v = 0$ and $\partial/\partial y = 0$), these equations become

$$u_t + \eta_x + \varepsilon u u_x - \frac{1}{2} \delta u_{txx} = 0, \quad (194)$$

$$\eta_t + [u(1 + \varepsilon\eta)]_x - \frac{1}{6} \delta u_{xxx} = 0. \quad (195)$$

We now seek steady progressive wave solutions traveling to the positive x -direction only, so that $u = u(x - Ut)$ and $\eta = \eta(x - Ut)$. With the

terms of zero order in ε and δ and $U = 1$, we assume a solution of the form

$$u = \eta + \varepsilon P + \delta Q, \quad (196)$$

where P and Q are unknown functions to be determined. Consequently, Eqs. (194) and (195) become

$$(\eta + \varepsilon P + \delta Q)_t + \eta_x + \varepsilon \eta \eta_x - \frac{1}{2} \delta \eta_{xxx} = 0, \quad (197)$$

$$\eta_t + [(1 + \varepsilon \eta)(\eta + \varepsilon P + \delta Q)]_x - \frac{1}{6} \delta \eta_{xxx} = 0. \quad (198)$$

These equations must be consistent so that we stipulate for the zero order

$$\eta_t = -\eta_x, \quad P = -\frac{1}{4}\eta^2, \quad Q = \frac{1}{3}\eta_{xx} = -\frac{1}{3}\eta_{xt}. \quad (199)$$

We use these results to rewrite both (197) and (198) with the assumption that ε and δ are of equal order and small enough for their products and squares to be ignored, so that the ratio $(\varepsilon/\delta) = (a^2/h^3)$ is of the order one. Consequently, we obtain a single equation for $\eta(x, t)$ in the form

$$\eta_t + \left(1 + \frac{3}{2}\varepsilon\eta\right)\eta_x + \frac{1}{6}\delta\eta_{xxx} = 0. \quad (200)$$

This is now universally known as the *Korteweg and de Vries equation* as they discovered it in their 1895 seminal work. We point out that $(\varepsilon/\delta) = a^2/h^3$ is one of the *fundamental* parameters in the theory of nonlinear shallow water waves. Recently, Infeld (1980) considered three-dimensional generalizations of the Boussinesq and Korteweg-de Vries equations.

Solutions of the KdV Equation: Solitons and Cnoidal Waves

To find solutions of the KdV equation, it is convenient to rewrite it in terms of dimensional variables as

$$\eta_t + c\left(1 + \frac{3}{2h}\eta\right)\eta_x + \frac{ch^2}{6}\eta_{xxx} = 0, \quad (201)$$

where $c = \sqrt{gh}$, and the total depth $H = h + \eta$. The first two terms $(\eta_t + c\eta_x)$ describe wave evolution at the shallow water speed c , the third term with coefficient $(3c/2h)$ represents a nonlinear wave steepening, and the last term with coefficient $(ch^2/6)$ describes linear dispersion. Thus, the KdV equation is a balance between time evolution, nonlinearity, and linear dispersion. The dimensional velocity u is obtained from (196) with (199) in the form

$$u = \frac{c}{h}\left(\eta - \frac{1}{4h}\eta^2 + \frac{h}{3}\eta_{xx}\right). \quad (202)$$

We seek a traveling wave solution of (201) in the frame X so that $\eta = \eta(X)$ and $X = x - Ut$ with $\eta \rightarrow 0$, as $|x| \rightarrow \infty$, where U is a constant speed. Substituting this solution in (201) gives

$$(c - U)\eta' + \frac{3c}{2h}\eta\eta' + \frac{ch^2}{6}\eta''' = 0, \quad (203)$$

where $\eta' = d\eta/dX$. Integrating this equation with respect to X yields

$$(c - U)\eta + \frac{3c}{4h}\eta^2 + \frac{ch^2}{6}\eta'' = A, \quad (204)$$

where A is an integrating constant.

We multiply this equation by $2\eta'$ and integrate again to obtain

$$(c - U)\eta^2 + \left(\frac{c}{2h}\right)\eta^3 + \left(\frac{ch^2}{6}\right)\left(\frac{d\eta}{dX}\right)^2 = 2A\eta + B, \quad (205)$$

where B is also a constant of integration.

We now consider a special case when η and its derivatives tend to zero at infinity and $A = B = 0$, so that (205) gives

$$\left(\frac{d\eta}{dX}\right)^2 = \frac{3}{h^3}\eta^2(a - \eta), \quad (206)$$

where

$$a = 2h\left(\frac{U}{c} - 1\right). \quad (207)$$

The right-hand side of (206) vanishes at $\eta = 0$ and $\eta = a$, and the exact solution of (206) represents Russell's solitary wave in the form

$$\begin{aligned} \eta(X) &= a \operatorname{sech}^2(bX), \\ b &= \left(\frac{3a}{4h^3}\right)^{1/2}. \end{aligned} \quad (208ab)$$

Thus, the explicit form of the solution is

$$\eta(x, t) = a \operatorname{sech}^2\left[\left(\frac{3a}{4h^3}\right)^{1/2} (x - Ut)\right], \quad (209)$$

where the velocity of the wave is

$$U = c\left(1 + \frac{a}{2h}\right). \quad (210)$$

This is an *exact* solution of the KdV equation for all (a/h) ; however, the equation is derived with the approximation $(a/h) \ll 1$. The solution (209) is called a *soliton* (or *solitary wave*) describing a single hump of height a above the undisturbed depth h and tending rapidly to zero away from $X = 0$. The solitary wave propagates to the right with velocity $U(>c)$, which is directly proportional to the amplitude a and has width $b^{-1} = (3a/4h^3)^{-1/2}$, that is, b^{-1} is inversely proportional to the square root of the amplitude a . Another significant feature of the soliton solution is that it travels in the medium without change of shape, which is hardly possible without retaining δ -order terms in the governing equation. A solitary wave profile has already been shown in Fig. 6.

In the general case, when both A and B are nonzero, (205) can be rewritten as

$$\begin{aligned} \frac{h^3}{3} \left(\frac{d\eta}{dX}\right)^2 &= -\eta^3 + 2h\left(\frac{U}{c} - 1\right)\eta^2 \\ &+ \frac{2h}{c}(2A\eta + B) = F(\eta), \end{aligned} \quad (211)$$

where $F(\eta)$ is a cubic with simple zeros.

We seek a real bounded solution for $\eta(X)$, which has a minimum value zero and a maximum value a and oscillates between the two values. For bounded solutions, all three zeros η_1, η_2, η_3 must be real. Without loss of generality, we set $\eta_1 = 0$ and $\eta_2 = a$. Hence, the third zero must be negative so that $\eta_3 = -(b - a)$ with $b > a > 0$. With these choices, $F(\eta) = \eta(a - \eta)(\eta - a + b)$ and Eq. (211) assumes the form

$$\frac{h^3}{3} \left(\frac{d\eta}{dX}\right)^2 = \eta(a - \eta)(\eta - a + b), \quad (212)$$

where

$$U = c\left(1 + \frac{2a - b}{2h}\right), \quad (213)$$

which is obtained by comparing the coefficients of η^2 in (211) and (212).

Writing $a - \eta = p^2$, it follows from Eq. (212) that

$$\left(\frac{3}{4h^3}\right)^{1/2} dX = \frac{dp}{[(a - p^2)(b - p^2)]^{1/2}}. \quad (214)$$

Substituting $p = \sqrt{a} \, q$ in (214) gives the standard elliptic integral of the first kind (see Dutta and Debnath (1965) and Helal and Molines (1981))

$$\begin{aligned} \left(\frac{3b}{4h^3}\right)^{1/2} X &= \int_0^q \frac{dq}{[(1 - q^2)(1 - m^2 q^2)]^{1/2}}, \\ m &= \left(\frac{a}{b}\right)^{1/2}, \end{aligned} \quad (215)$$

and then, function q can be expressed in terms of the Jacobian elliptic function, $sn(z, m)$

$$q(X, m) = sn\left[\left(\frac{3b}{4h^3}\right)^{1/2} X, m\right], \quad (216)$$

where m is the modulus of $sn(z, m)$.

Finally,

$$\begin{aligned}\eta(X) &= a \left[1 - \operatorname{sn}^2 \left\{ \left(\frac{3b}{4h^3} \right)^{1/2} X \right\} \right] \\ &= a \operatorname{cn}^2 \left[\left(\frac{3b}{4h^3} \right)^{1/2} X \right],\end{aligned}\quad (217)$$

where $\operatorname{cn}(z, m)$ is also the Jacobian elliptic function with a period $2K(m)$, where $K(m)$ is the complete elliptic integral of the first kind defined by

$$K(m) = \int_0^{\pi/2} (1 - m^2 \sin^2 \theta)^{-1/2} d\theta, \quad (218)$$

and $\operatorname{cn}^2(z) + \operatorname{sn}^2(z) = 1$.

It is important to note that $\operatorname{cn} z$ is periodic, and hence, $\eta(X)$ represents a train of periodic waves in shallow water. Thus, these waves are called *cnoidal waves* with wavelength

$$\lambda = 2 \left(\frac{4h^3}{3b} \right)^{1/2} K(m). \quad (219)$$

The upshot of this analysis is that solution (217) represents a nonlinear wave whose shape and wavelength (or period) all depend on the amplitude of the wave. A typical cnoidal wave is shown in Fig. 9. Sometimes, the cnoidal waves with slowly varying amplitude are observed in rivers. More often, wavetrains behind a weak bore (called an *undular bore*) can be regarded as cnoidal waves. Two limiting cases are of special physical interest: (i) $m \rightarrow 0$ and (ii) $m \rightarrow 1$.

In the first case, $\operatorname{sn} z \rightarrow \sin z$, $\operatorname{cn} z \rightarrow \cos z$ as $m \rightarrow 0$ ($a \rightarrow 0$). This corresponds to small-amplitude waves where the linearized KdV equation is appropriate. So, in this limiting case, the solution (217) becomes

$$\eta(x, t) = \frac{1}{2} a [1 + \cos(kx - \omega t)], \quad k = \left(\frac{3b}{h^3} \right)^{1/2}, \quad (220)$$

where the corresponding dispersion relation is

$$\omega = Uk = ck \left(1 - \frac{1}{6} k^2 h^2 \right). \quad (221)$$

This corresponds to the first two terms of the series expansion of $(gk \tanh kh)^{1/2}$. Thus, these results are in perfect agreement with the linearized theory.

In the second limiting case, $m \rightarrow 1$ ($a \rightarrow b$), $\operatorname{cn} z \rightarrow \operatorname{sech} z$. Thus, the cnoidal wave solution tends to the classical KdV solitary-wave solution where the wavelength λ , given by (219), tends to infinity because $K(a) = \infty$ and $K(0) = \pi/2$. The solution identically reduces to (209) with (210).

We next report the numerical computation of the KdV Eq. (201) due to Berezin and Karpman (1966). In terms of new variables defined by

$$x^* = x - ct, t^* = t, \eta^* = \left(\frac{3c}{2h} \right) \eta, \quad (222)$$

omitting the asterisks, Eq. (201) becomes

$$\eta_t + \eta \eta_x + \beta \eta_{xxx} = 0, \quad (223)$$

where $\beta = \left(\frac{1}{6} \right) ch^2$.

We examine the numerical solution of (223) with the initial condition

$$\eta(x, 0) = \eta_0 f\left(\frac{x}{\ell}\right), \quad (224)$$

where η_0 is constant and $f(\xi)$ is a nondimensional function characterizing the initial wave profile. It is convenient to introduce the dimensionless variables

$$\xi = \frac{x}{\ell}, \quad \tau = \frac{\eta_0 t}{\ell}, \quad u = \frac{\eta}{\eta_0} \quad (225)$$

so that Eqs. (223) and (224) reduce to

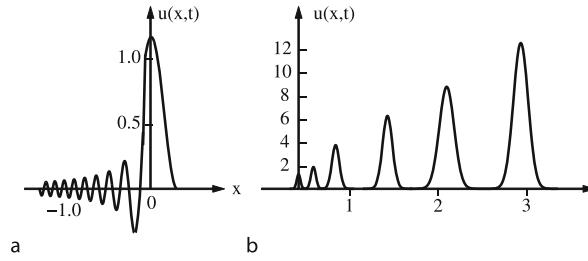
$$u_\tau + uu_\xi + \sigma^{-2} u_{\xi\xi\xi} = 0, \quad (226)$$

$$u(\xi, 0) = f(\xi), \quad (227)$$

where the dimensionless parameter σ is defined by $\sigma = \ell(\eta_0/\beta)^{1/2}$.



Water Waves and the Korteweg-de Vries Equation, Fig. 9 A cnoidal wave



Water Waves and the Korteweg-de Vries Equation, Fig. 10 The solutions of the KdV equation $u(x, t)$ for large values of t with the values of the similarity parameter σ : **a** $\sigma = 1.9$, **b** $\sigma = 16.5$ (from (Berezin and Karpman 1966))

Berezin and Karpman (1966) obtained the numerical solution of (226) with the Gaussian initial pulse of the form $u(\xi, 0) = f(\xi) = \exp(-\xi^2)$ and values of the parameter $\sigma = 1.9$ and $\sigma = 16.5$. Their numerical solutions are shown in Fig. 10.

As shown in Fig. 10 for case (a), the perturbation splits into a soliton and a wavepacket. In case (b), there are six solitons. It is readily seen that the peaks of the solitons lie nearly on a straight line. This is due to the fact that the velocity of the soliton is proportional to its amplitude, so that the distances traversed by the solitons would also be proportional to their amplitudes.

Zabusky's (1967) numerical investigation of the interaction of two solitons reveals that the taller soliton, initially behind, catches up to the shorter one, they undergo a nonlinear interaction and, then, emerge from the interaction without any change in shape and amplitude. The end result is that the taller soliton reappears in front and the shorter one behind. This is essentially strong computational evidence of the stability of solitons.

Using the transformation

$$x^* = \varepsilon \beta(x - ct), \quad t^* = \varepsilon^3 t, \quad \eta^* = (\alpha \varepsilon^2)^{-1} \eta$$

with $\alpha \beta (= 3c/2h) = 6$ and $\beta^3 (= ch^2/6) = 1$, we write the KdV Eq. (201) in the normalized form, dropping the asterisks,

$$\eta_t + 6\eta\eta_x + \eta_{xxx} = 0. \quad (228)$$

We next seek a steady progressive wave solution of (228) in the form

$$\eta = 2k^2 f(X), \quad X = kx - \omega t. \quad (229)$$

Then, the equation for $f(X)$ can be integrated once to obtain

$$f''(X) = 6A + (4 - 6B)f - 6f^2, \quad (230)$$

where A is a constant of integration and the frequency ω is given by

$$\omega = k^3(4 - 6B). \quad (231)$$

A single-soliton solution corresponds to the special case $A = B = 0$ and is given by

$$f(X) = \text{sech}^2(X - X_0), \quad (232)$$

where X_0 is a constant.

Thus, for a series of solitons spaced 2σ apart, we write

$$f(X) = \sum_{n=-\infty}^{\infty} \text{sech}^2(X - 2n\sigma). \quad (233)$$

This is a 2σ periodic function that satisfies (230) for some A and B .

The general elliptic function solution of (228) can be obtained from the integral of (230), which can be written as

$$f_X^2 = -4C + 12Af + (4 - 6B)f^2 - 4f^3, \quad (234)$$

where C is a constant of integration. Various asymptotic and numerical results lead to the relations (see Whitham (1984))

$$C = -\frac{1}{2} \frac{dA(\sigma)}{d\sigma} = \frac{1}{4} \frac{d^2B(\sigma)}{d\sigma^2}, \quad (235a)$$

and the cubic in (234) can be factorized as

$$\begin{aligned}
 & -C + 3Af + \left(1 - \frac{3}{2}B\right)f^2 - f^3 \\
 & = (f_1 - f)(f - f_2)(f - f_3), \quad (236)
 \end{aligned}$$

where $f_r(\sigma)$ ($r = 1, 2, 3$) are determined from $A(\sigma)$, $B(\sigma)$, and $C(\sigma)$. If we set $f_1 > f_2 > f_3$ and, then, the periodic solution oscillates between f_1 at $X = 0$ and f_2 at $X = \sigma$, one particular form of the solution is given by

$$f(X) = f_2 + (f_1 - f_2)cn^2\left(\sqrt{(f_1 - f_3)X}\right), \quad (237)$$

where the modulus m of $cn(z, m)$ is given by

$$m^2 = \left(\frac{f_1 - f_2}{f_1 - f_3}\right). \quad (238)$$

Thus, it follows from (233) and (237) that the following identity holds:

$$\begin{aligned}
 & f_2 + (f_1 - f_2)cn^2\left(\sqrt{(f_1 - f_3)X}\right) \\
 & = \sum_{n=-\infty}^{\infty} \text{sech}^2(X - 2n\sigma), \quad (239)
 \end{aligned}$$

which can be verified by comparing the periods and poles of the two sides.

Finally, the higher-order modified KdV equation

$$v_t + (p + 1)v^p v_x + v_{xxx} = 0, \quad p > 2, \quad (240)$$

admits single-soliton solutions in the form

$$v(x, t) = a \text{sech}^{2/p}(kx - \omega t). \quad (241)$$

However, in view of the fractional powers of the sech function, it seems, perhaps, unlikely that there will be any simple superposition formula.

Derivation of the KdV Equation from the Euler Equations

This problem was discussed in section “The Korteweg-de Vries and Boussinesq Equations”

by using the Laplace equation for the velocity potential under the assumption that $\delta = O(\varepsilon)$ as $\varepsilon \rightarrow 0$. Here we follow Johnson (1997) to present another derivation of the KdV equation from the Euler equation in $(1 + 1)$ dimensions. This approach can be generalized to derive higher dimensional KdV equations.

We consider the problem of surface gravity waves which propagate in the positive x -direction over stationary water of constant depth. The associated Euler Eq. (59) and the continuity Eq. (60) in $(1 + 1)$ dimensions are given by

$$u_t + \varepsilon(uu_x + wu_z) = -p_x, \quad (242)$$

$$\delta[w_t + \varepsilon(uw_x + ww_z)] = -p_z, \quad (243)$$

$$u_x + w_z = 0. \quad (244)$$

The free surface and bottom boundary conditions are obtained from (57ab) and (62) in the form

$$w = \eta_t + \varepsilon u \eta_x, \quad p = \eta, \quad \text{on } z = 1 + \varepsilon \eta, \quad (245)$$

$$w = 0 \quad \text{on } z = 0. \quad (246)$$

It can easily be shown that, for any $\sqrt{\delta}$ as $\varepsilon \rightarrow 0$, there exists a region in the (x, t) -space where there is a balance between nonlinearity and dispersion which leads to the KdV equation. The region of interest is defined by a scaling of the independent flow variables as

$$x \rightarrow \sqrt{\frac{\delta}{\varepsilon}} x \quad \text{and} \quad t \rightarrow \sqrt{\frac{\delta}{\varepsilon}} t, \quad (247)$$

for any ε and $\sqrt{\delta}$. In order to ensure consistency in the continuity equation, it is necessary to introduce a scaling of w by

$$w \rightarrow \sqrt{\frac{\varepsilon}{\delta}} w. \quad (248)$$

Consequently, the net effect of the scalings is to replace δ by ε in Eqs. (242)–(246) so that they become

$$u_t + \varepsilon(uu_x + wu_z) = -p_x, \quad (249)$$

$$\varepsilon[w_t + \varepsilon(uw_x + ww_z)] = -p_z, \quad (250)$$

$$u_x + w_z = 0, \quad (251)$$

$$w = \eta_t + \varepsilon u \eta_x, \text{ and } p = \eta \quad \text{on } z = 1 + \varepsilon \eta, \quad (252)$$

$$w = 0 \quad \text{on } z = 0. \quad (253)$$

In the limit as $\varepsilon \rightarrow 0$, the first-order approximation of Eqs. (249) and (252) gives

$$\eta = p, \quad 0 \leq z \leq 1, \quad \text{and} \quad u_t + \eta_x = 0. \quad (254)$$

It then follows from (251) that $w = -zu_x$ which satisfies (253). The boundary condition (252) leads to $\eta_t + u_x = 0$ on $z = 1$, which can be combined with (254) to obtain the linear wave equation

$$\eta_{tt} - \eta_{xx} = 0. \quad (255)$$

For waves propagating in the positive x -direction, we introduce the far-field variables

$$\xi = x - t \quad \text{and} \quad \tau = \varepsilon t, \quad (256)$$

so that $\xi = O(1)$ and $\tau = O(1)$ give the far-field region of the problem. This is the region where nonlinearity balances the dispersion to produce the KdV equation.

With the choice of the transformations (256), Eqs. (249)–(253) can be rewritten in the form

$$-u_\xi + \varepsilon(u_\tau + uu_\xi + wu_z) = -p_\xi, \quad (257)$$

$$\varepsilon[-w_\xi + \varepsilon(w_\tau + uw_\xi + ww_z)] = -p_z, \quad (258)$$

$$u_\xi + w_z = 0, \quad (259)$$

$$w = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi), \quad p = \eta, \quad (260)$$

$$\text{on } z = 1 + \varepsilon \eta,$$

$$w = 0 \quad \text{on } z = 0. \quad (261)$$

We seek an asymptotic series expansion of the solutions of the system (257)–(261) in the form

$$\begin{aligned} (\xi, \tau, \varepsilon) &= \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi, \tau), \\ q(\xi, \tau, z; \varepsilon) &= \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi, \tau, z), \end{aligned} \quad (262)$$

where q (and the corresponding q_n) denotes each of the variables u , w , and p .

Consequently, the leading-order problem is given by

$$u_{0\xi} = p_{0\xi}, \quad p_{0z} = 0, \quad u_{0\xi} + w_{0z} = 0, \quad (263)$$

$$p_0 = \eta_0, \quad w + \eta_{0\xi} = 0 \quad \text{on } z = 1, \quad (264)$$

$$w = 0 \quad \text{on } z = 0. \quad (265)$$

These leading-order equations give

$$p_0 = \eta_0, \quad u_0 = \eta_0, \quad w_0 + z\eta_{0\xi} = 0, \quad 0 \leq z \leq 1, \quad (266)$$

with $u_0 = 0$ whenever $\eta_0 = 0$. The boundary condition on w_0 at $z = 1$ is automatically satisfied.

Using the Taylor series expansion of u , w , and p about $z = 1$, the two surface boundary conditions on $z = 1 + \varepsilon \eta$ are rewritten on $z = 1$ and, hence, take the form

$$\begin{aligned} p_0 + \varepsilon \eta_0 p_{0z} + \varepsilon p_1 &= \eta_0 + \varepsilon \eta_1 \\ &+ O(\varepsilon^2) \quad \text{on } z = 1. \end{aligned} \quad (267)$$

$$\begin{aligned} w_0 + \varepsilon \eta_0 w_{0z} + \varepsilon w_1 &= -\eta_{0\xi} - \varepsilon \eta_{1\xi} \\ &+ \varepsilon(\eta_{0\tau} + u_0 \eta_{0\xi}) \\ &+ O(\varepsilon^2) \quad \text{on } z = 1. \end{aligned} \quad (268)$$

These conditions are to be used together with (257), (258), and (261).

The equations in the next order are given by

$$-u_{1\xi} + u_{0\tau} + u_0 u_{0\xi} + w_0 u_{0z} = -p_{1\xi}, \quad p_{1z} = w_{0\xi}, \quad (269)$$

$$u_{1\xi} + w_{1z} = 0, \quad (270)$$

$$\begin{aligned} p_1 + \eta_0 p_0 &= \eta_1, w_1 + \eta_0 + w_{0z} \\ &= -\eta_{1\xi} + \eta_{0\tau} + u_0 \eta_{0\xi} \text{ on } z = 1, \end{aligned} \quad (271)$$

$$w_1 = 0 \quad \text{on } z = 0. \quad (272)$$

Noting that

$$u_{0z} = 0, \quad p_{0z} = 0, \quad \text{and} \quad w_{0z} = -\eta_{0\xi}, \quad (273)$$

we obtain

$$p_1 = \frac{1}{2}(1 - z^2)\eta_{0\xi\xi} + \eta_1, \quad (274)$$

and therefore,

$$\begin{aligned} w_{1z} &= -u_{1\xi} = -p_{1\xi} - u_{0\tau} - u_0 u_{0\xi} \\ &= -\left[\eta_{1\xi} + \frac{1}{2}(1 - z^2)\eta_{0\xi\xi\xi} + \eta_{0\tau} + \eta_0 \eta_{0\xi}\right]. \end{aligned} \quad (275)$$

Finally, we find

$$\begin{aligned} w_1 &= -\left[\eta_{1\xi} + \eta_{0\tau} + \eta_0 + \eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi}\right]z \\ &\quad + \frac{1}{6}z^3\eta_{0\xi\xi\xi}, \end{aligned} \quad (276)$$

which satisfies the bottom boundary condition on $z = 0$.

The free surface boundary condition on $z = 1$ gives

$$\begin{aligned} (w_1)_{z=1} &= -\left(\eta_{1\xi} + \eta_{0\tau} + \eta_0 \eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi}\right) \\ &\quad + \frac{1}{6}\eta_{0\xi\xi\xi} = -\eta_{1\xi} + \eta_{0\tau} + 2\eta_0 \eta_{0\xi}, \end{aligned} \quad (277)$$

which yields the equation for $\eta_0(\xi, \tau)$ in the form

$$\eta_{0\tau} + \frac{3}{2}\eta_0 \eta_{0\xi} + \frac{1}{6}\eta_{0\xi\xi\xi} = 0. \quad (278)$$

This is the *Korteweg-de Vries (KdV) equation*, which describes nonlinear plane gravity waves propagating in the x -direction. The exact solution of the general initial-value problem for the KdV equation can be obtained provided the initial data decay sufficiently rapidly as $|\xi| \rightarrow \infty$. We may raise the question of whether the asymptotic expansion for η (and hence, for the other flow variables) is uniformly valid as $|\xi| \rightarrow \infty$ and as $\tau \rightarrow \infty$. For the case of $\tau \rightarrow \infty$, this question is difficult to answer because the equations for $\eta_n (n \geq 1)$ are not easy to solve. However, all the available numerical evidence suggests that the asymptotic expansion of η is indeed uniformly valid as $\tau \rightarrow \infty$ (at least for solutions which satisfy $\eta \rightarrow 0$ as $|\xi| \rightarrow \infty$). From a physical point of view, if the waves are allowed to propagate indefinitely, then other physical effects cannot be neglected. In the case of real water waves, the most common physical effects include viscous damping. In practice, the viscous damping seems to be sufficiently weak to allow the dispersive and nonlinear effects to dominate before the waves eventually decay completely.

Two-Dimensional and Axisymmetric KdV Equations

It is well known that the KdV equation describes nonlinear plane waves which propagate only in the x -direction. However, there are many physical situations in which waves move on a two-dimensional surface. So it is natural to include both x - and y -directions with the appropriate balance of dispersion and nonlinearity. One of the simplest examples is the classical two-dimensional linear wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad (279)$$

which describes the propagation of long waves.

This equation has a solution in the form

$$u(\mathbf{x}, t) = a \exp [i(\omega t - \mathbf{\kappa} \cdot \mathbf{x})], \quad (280)$$

where a is the wave amplitude, ω is the frequency, $\mathbf{x} = (x, y)$, and the wavenumber vector is $\mathbf{\kappa} = (k, \ell)$.

The dispersion relation is given by

$$\omega^2 = c^2(k^2 + \ell^2). \quad (281)$$

For waves that propagate predominantly in the x -direction, the wavenumber component ℓ becomes small so that the approximate phase velocity is given by

$$c_p = c \left(1 + \frac{1}{2} \frac{\ell^2}{k^2} \right) \quad \text{as } \ell \rightarrow 0. \quad (282)$$

It follows from (282) that the phase velocity suffers from a small correction provided by the wavenumber component ℓ in the y -direction. In order to ensure that this small correction is the same order as the dispersion and nonlinearity, it is necessary to require $\ell = O(\sqrt{\varepsilon})$ or $\ell^2 = O(\varepsilon)$. This requirement can be incorporated in the governing equations by a scaling of the flow variables as

$$y \rightarrow \sqrt{\varepsilon} y \quad \text{and} \quad v \rightarrow \sqrt{\varepsilon} v, \quad (283)$$

and using the same far-field transformations as (256).

Consequently, Eqs. (59)–(62) and the free surface condition (57ab) with the parameters δ replaced by ε reduce to the form

$$\begin{aligned} -u_\xi + \varepsilon(u_\tau + uu_\xi + \varepsilon v u_y + w u_z) \\ = -p_\xi, \end{aligned} \quad (284)$$

$$-v_\xi + \varepsilon(v_\tau + uv_\xi + \varepsilon v v_y + w v_z) = -p_y, \quad (285)$$

$$\begin{aligned} -\varepsilon[-w_\xi + \varepsilon(w_\tau + uw_\xi + \varepsilon v w_y + w w_z)] \\ = -p_z, \end{aligned} \quad (286)$$

$$-u_\xi + \varepsilon v_y + w_z = 0, \quad (287)$$

$$\begin{aligned} w = -\eta_\xi + \varepsilon(\eta_\tau + uu_\xi + \varepsilon v \eta_y) \quad \text{and} \\ p = \eta \quad \text{on } z = 1 + \varepsilon \eta, \end{aligned} \quad (288)$$

$$w = 0 \quad \text{on } z = 0. \quad (289)$$

We seek the same asymptotic series solution (262) valid as $\varepsilon \rightarrow 0$ without any change of the leading order problem except that the flow variables involve y so that

$$\begin{aligned} p_0 = \eta_0, \quad u_0 = \eta_0, w_0 = -z\eta_{0\xi}, \\ 0 \leq z \leq 1, \end{aligned} \quad (290)$$

$$v_{0\xi} = \eta_{0y}. \quad (291)$$

At the next order, the only difference from section “[Solutions of the KdV Equation: Solitons and Cnoidal Waves](#)” is in the continuity equation which becomes

$$w_{1z} = -u_{1\xi} - v_{0y}. \quad (292)$$

This change leads to the following equation:

$$\begin{aligned} w_1 = -\left(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi} + v_{0y}\right)z \\ + \frac{1}{6}z^3\eta_{0\xi\xi\xi}. \end{aligned} \quad (293)$$

Using Eq. (291), the final result is the equation for the leading-order representation of the surface wave in the form

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} + v_{0y} = 0. \quad (294)$$

Consequently, differentiating (294) with respect to ξ and replacing $v_{0\xi}$ by η_{0y} give the evolution equation for $\eta_0(\xi, \tau, y)$ in the form

$$\left(2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi}\right)_\xi + \eta_{0yy} = 0. \quad (295)$$

This is the *two-dimensional* or, more precisely, the *(1 + 2)-dimensional KdV equation*. Obviously, when there is no y -dependence, (295) reduces to the KdV Eq. (278). The two-dimensional KdV equation is also known as the *Kadomtsev–Petviashvili (KP) equation* (Kadomtsev and Petviashvili (1970)). This

is another very special completely integrable equation, and it has an exact analytical solution that represents obliquely crossing nonlinear waves. Physically, any number of waves cross obliquely and interact nonlinearly. In particular, the nonlinear interaction of three waves leads to a *resonance phenomenon*. Such an interaction becomes more pronounced, leading to a strongly nonlinear interaction as the wave configuration is more nearly that of parallel waves. This situation can be interpreted as one in which the waves interact nonlinearly over a large distance so that a strong distortion is produced among these waves. The reader is referred to Johnson's (1997) book for more detailed information on oblique interaction of waves.

We now consider the Euler Eq. (63) and the continuity Eq. (65) in cylindrical polar coordinates (r, θ, z) . It is convenient to use the non-dimensional flow variables, parameters, and scaled variables similar to those defined by (53), (54), and (58) with $r = \ell r^*$ where r^* is a nondimensional variable so that the polar form of Euler Eq. (66), continuity Eq. (65), the free surface boundary conditions (57ab) in nondimensional cylindrical polar form become

$$\begin{aligned} \frac{Du}{Dt} - \frac{\varepsilon v^2}{r} &= -\frac{\partial p}{\partial r}, \quad \frac{Dv}{Dt} + \frac{\varepsilon uv}{r} \\ &= -\frac{1}{r} \frac{\partial p}{\partial \theta}, \quad \delta \frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \end{aligned} \quad (296)$$

$$\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \quad (297)$$

$$w = \eta_t + \varepsilon \left(u\eta_r + \frac{v}{r} \eta_\theta \right) \quad \text{and} \quad (298)$$

$$p = \eta \quad \text{on } z = 1 + \varepsilon\eta,$$

$$w = 0 \quad \text{on } z = 0, \quad (299)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \varepsilon \left(u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z} \right), \quad (300)$$

and ε and δ are defined by (52).

For the case of axisymmetric wave motions ($\frac{\partial}{\partial \theta} = 0$), the governing equations become

$$u_t + \varepsilon(uu_r + ww_z) = -p_r, \quad (301)$$

$$\delta[w_t + \varepsilon(uw_r + ww_z)] = -p_z, \quad (302)$$

$$u_r + \frac{u}{r} + w_z = 0, \quad (303)$$

$$w = \eta_t + \varepsilon u\eta_r \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta, \quad (304)$$

$$w = 0 \quad \text{on } z = 0. \quad (305)$$

In the limit as $\varepsilon \rightarrow 0$, the linearized version of Eqs. (296)–(300) become

$$u_t = -p_r, \quad v_t = -\frac{1}{r} p_\theta, \quad \delta w_t = -p_z, \quad (306)$$

$$\frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \quad (307)$$

$$w = \eta_t \quad \text{and} \quad p = \eta \quad \text{on } z = 1, \quad (308)$$

$$w = 0 \quad \text{on } z = 0, \quad (309)$$

For long waves ($\delta \rightarrow 0$), Eqs. (306)–(309) lead to the classical wave equation

$$\eta_{tt} = \eta_{rr} + \frac{1}{r} \eta_r + \frac{1}{r^2} \eta_{\theta\theta}. \quad (310)$$

For axisymmetric surface waves, the wave Eq. (310) becomes

$$\eta_{tt} = \eta_{rr} + \frac{1}{r} \eta_r. \quad (311)$$

This can be solved by using the Hankel transform (see Debnath and Bhatta (2007)).

It is convenient to introduce the characteristic variable $\xi = r - t$ for outgoing waves and $R = \alpha r$ so that $\alpha \rightarrow 0$ which corresponds to large radius r . In other words, $R = O(1)$, $\alpha \rightarrow 0$ gives $r \rightarrow \infty$. The Eq. (311) reduces to the form

$$2\eta_{\xi R} + \frac{1}{R}\eta_{\xi} + \alpha\left(\eta_{RR} + \frac{1}{R}\eta_R\right) = 0. \quad (312)$$

In the limit as $\alpha \rightarrow 0$, it follows that

$$\sqrt{R}\eta_{\xi} = g(\xi), \quad (313)$$

where $g(\xi)$ is an arbitrary function of ξ .

For outgoing waves, the correct solution takes the form for $\alpha \rightarrow 0$ ($r \rightarrow \infty$),

$$\eta = \frac{1}{\sqrt{R}} \int g(\xi) d\xi = \frac{1}{\sqrt{R}} f(\xi), \quad (314)$$

where $\eta = 0$ when $f = 0$.

This shows that the amplitude of axisymmetric waves decays as the radius $r \rightarrow \infty$ ($R \rightarrow \infty$). This behavior is totally different from the derivation of the KdV equation where the amplitude is uniformly $O(\varepsilon)$. In the present axisymmetric case, the amplitude decreases as the radius r increases so that there is no far-field region where the balance between nonlinearity and dispersion occurs. In other words, the amplitude is so small that nonlinear terms play no role at the leading order. However, there exists a scaling of the flow variables which leads to the *axisymmetric (concentric) KdV equations* shown by Johnson (1997).

We recall the axisymmetric Euler equations of motion and boundary conditions (301)–(305) and introduce scalings in terms of large radial variable R (see Johnson (1997)),

$$\xi = \frac{\varepsilon^2}{\delta}(r - t) \quad \text{and} \quad R = \frac{\varepsilon^6}{\delta^2}r. \quad (315)$$

We next apply the transformations of the flow variables

$$(\eta, p, u, w) = \frac{\varepsilon^3}{\delta} \left(\eta^*, p^*, u^*, \frac{\varepsilon^2}{\delta} w^* \right), \quad (316)$$

where large distance/time is measured by the scale (δ^2/ε^6) so that

$$\left(\frac{\delta^2}{\varepsilon^6} \right)^{-\frac{1}{2}} = \left(\frac{\varepsilon^3}{\delta} \right),$$

which represents the scale of the amplitude of the waves. It is important to point out that the original wave amplitude parameter ε can now be interpreted based on the amplitude of the wave for $r = O(1)$ and $t = O(1)$. Consequently, the governing equations and the boundary conditions (301)–(305) become, dropping the asterisks in the variables,

$$\begin{aligned} -u_{\xi} + \alpha(uu_{\xi} + wu_z + \alpha uu_R) \\ = -(p_{\xi} + \alpha p_R), \end{aligned} \quad (317)$$

$$\alpha[-w_{\xi} + \alpha(uw_{\xi} + ww_z + \alpha uw_R)] = -p_z, \quad (318)$$

$$u_{\xi} + w_z + \alpha\left(u_R + \frac{1}{R}u\right) = 0, \quad (319)$$

$$\begin{aligned} w &= -\eta_{\xi} + \alpha(u\eta_{\xi} + \alpha u\eta_R), \\ p &= \eta \quad \text{on} \quad z = 1 + \alpha\eta, \end{aligned} \quad (320)$$

$$w = 0 \quad \text{on} \quad z = 0, \quad (321)$$

where $\alpha = (\delta^{-1}\varepsilon^4)$ is a new parameter. These equations are similar in structure to those discussed above with parameter ε , which is now replaced by α in (317)–(321) so that the limit as $\alpha \rightarrow 0$ is required. This requirement is satisfied (for example $\varepsilon \rightarrow 0$ with δ fixed), and the scaling introduced by (315) describes the region where the appropriate balance occurs so that the wave amplitude in this region is $O(\alpha)$.

We now seek an asymptotic series solution in the form

$$q(\xi, R, z) = \sum_{n=0}^{\infty} \alpha^n q_n(\xi, R, z), \quad \alpha \rightarrow 0, \quad (322)$$

where q represents each of η , u , w , and p .

In the leading order, we obtain the familiar equations

$$p_0 = \eta_0, \quad u_0 = \eta_0, \quad w_0 = -z\eta_{0\xi}, \quad 0 \leq z \leq 1. \quad (323)$$

It follows from the continuity Eq. (319) that

$$w_{1z} = -u_{1z} - \left(u_{0R} + \frac{1}{R}u_0\right). \quad (324)$$

Without any more algebraic calculation, it turns out that $\eta_0(\xi, R)$ satisfies the nonlinear evolution equation

$$2\eta_{0R} + \frac{1}{R}\eta_0 + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0. \quad (325)$$

This is usually referred to as the *axisymmetric (concentric) KdV equation* which includes a new term $R^{-1}\eta_0$. We may use the large time variable $\tau = (\delta^{-2}\varepsilon^6)t$ so that $R = \tau + \alpha\xi \approx \tau$ (see Johnson (1997) or (1980)).

Johnson (1980) derived a new concentric KdV equation which incorporates weak dependence on the angular coordinate θ . In the derivation of KP Eq. (295), $\sqrt{\varepsilon}$ was chosen as the scaling parameter in the y -direction. Similarly, the appropriate scaling on the angular variable θ may be chosen as $\sqrt{\alpha}$. In the derivation of the concentric KdV Eq. (325), the parameter α plays the role of ε which is used as the small parameter in the asymptotic solution of the KdV equation.

Following the work of Johnson (1980, 1997), we choose the variables ξ and R defined by (315) and the scaled θ variable as

$$\theta = \sqrt{\alpha}\theta^* = (\delta^{-1}\varepsilon^2)\theta^*, \quad (326)$$

which introduces a small angular deviation from purely concentric effects. We also use the scaled velocity component in the θ -direction as

$$v = (\delta^{-3/2}\varepsilon^5)v^*, \quad (327)$$

so that the scalings on $u = \phi_r$ and $v = \frac{1}{r}\phi_\theta$ are found to be consistent.

Using (315)–(316), Eqs. (296)–(300), and following Johnson (1997), the equation in cylindrical polar coordinates can be obtained in the form:

$$\begin{aligned} &\left(2\eta_{0R} + \frac{1}{R}\eta_0 + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi}\right)_\xi \\ &+ \frac{1}{R^2}\eta_{0\theta\theta} = 0. \end{aligned} \quad (328)$$

This is known as *Johnson's (or nearly concentric KdV) equation*, as it was first derived by Johnson (1980). In the absence of the θ -dependence, Eq. (328) reduces to the concentric KdV Eq. (325).

The Nonlinear Schrödinger Equation and Solitary Waves

We describe below that the nonlinear modulation of a quasi-monochromatic wave described by the nonlinear Schrödinger equation. To take into account the nonlinearity and the modulation in the far-field approximation, the wavenumber k and frequency ω in the linear dispersion relation are replaced by $k - i\frac{\partial}{\partial x}$ and $\omega + i\frac{\partial}{\partial t}$, respectively. It is convenient to use the nonlinear dispersion relation in the form

$$D\left(k - i\frac{\partial}{\partial x}, \quad \omega + i\frac{\partial}{\partial t}, \quad |A|^2\right)A = 0. \quad (329)$$

We consider the case of a weak nonlinearity and a slow variation of the amplitude, and hence, the amplitude A is assumed to be a slowly varying function of space and time. We next expand (329) with respect to $|A|^2$, $-i\frac{\partial}{\partial x}$, and $i\frac{\partial}{\partial t}$ to obtain

$$\begin{aligned} &D(k, \omega, 0) - i\left(D_k\frac{\partial}{\partial x} - D_\omega\frac{\partial}{\partial t}\right)A \\ &- \frac{1}{2}\left(D_{kk}\frac{\partial^2}{\partial x^2} - 2D_{k\omega}\frac{\partial^2}{\partial x\partial t} + D_{\omega\omega}\frac{\partial^2}{\partial t^2}\right)A \\ &+ \frac{\partial D}{\partial |A|^2}|A|^2A = 0, \end{aligned} \quad (330)$$

where the first term $D(k, \omega, 0)$ corresponds to the linear dispersion so that $D(k, \omega, 0) = 0$ represents the linear dispersion relation.

Introducing the transformation $x^* = x - Ct$, $t^* = t$, assuming that $A = O(\varepsilon)$, $\frac{\partial}{\partial x^*} = O(\varepsilon)$, and $\frac{\partial}{\partial t^*} = O(\varepsilon^2)$, retaining all terms up to $O(\varepsilon^3)$, and dropping the asterisks, we find that

$$i \frac{\partial A}{\partial t} + p \frac{\partial^2 A}{\partial x^2} + q |A|^2 A = 0, \quad (331)$$

where

$$p = \frac{1}{2} \left(\frac{dC}{dk} \right), \quad q = \frac{1}{D_\omega} \left(\frac{\partial D}{\partial |A|^2} \right), \quad (332)$$

and the following results

$$C = - \frac{D\omega}{Dk}, \frac{dC}{dk} = (D_{kk} + 2CD_{\omega k} + C^2 D_{\omega\omega}) / D_\omega, \quad (333)$$

have been used with D given by (329).

Equation (331) is known as the *cubic nonlinear Schrödinger (NLS) equation*. When the last cubic nonlinear term is neglected, Eq. (331) reduces to the corresponding *linear Schrödinger (NLS) equation* for $A(x, t)$ in the form

$$i \frac{\partial A}{\partial t} + \frac{1}{2} \omega''(k) \frac{\partial^2 A}{\partial x^2} = 0, \quad (334)$$

when $p = \frac{1}{2} C'(k) = \frac{1}{2} \omega''(k)$ is used.

More explicitly, if the nonlinear dispersion relation is given by

$$\omega = \omega(k, a^2), \quad (335)$$

and if we expand ω in a Taylor series about $k = k_0$ and $|a|^2 = 0$, we obtain

$$\omega \approx \omega_0 + (k - k_0) \omega'_0 + \frac{1}{2} (k - k_0)^2 \omega''_0 + \left(\frac{\partial \omega}{\partial |a|^2} \right)_{|a|^2=0} |a|^2. \quad (336)$$

Replacing $(\omega - \omega_0)$ by $i(\frac{\partial}{\partial t})$, $k - k_0$ by $-i(\frac{\partial}{\partial x})$, and assuming that the resulting operators act on the amplitude function $a(x, t)$, it turns out that $a(x, t)$ satisfies the equation

$$i(a_t + \omega'_0 a_x) + \frac{1}{2} \omega''_0 a_{xx} + \gamma |a|^2 a = 0, \quad (337)$$

where

$$\gamma = - \left(\frac{\partial \omega}{\partial |a|^2} \right)_{|a|^2=0} = \text{constant}. \quad (338)$$

Equation (337) is known as the *cubic nonlinear Schrödinger equation*. If we choose a frame of reference moving with the linear group velocity ω'_0 , that is, $\xi = x - \omega'_0 t$ and $\tau = t$, the term involving a_x will drop out from (337), and therefore, the amplitude $a(x, t)$ satisfies the normalized *nonlinear Schrödinger equation*

$$ia_\tau + \frac{1}{2} \omega''_0 a_{\xi\xi} + \gamma |a|^2 a = 0. \quad (339)$$

The corresponding dispersion relation is given by

$$\omega = \frac{1}{2} \omega''_0 k^2 - \gamma a^2. \quad (340)$$

According to the stability criterion established by Whitham (1974), the wave modulation is stable if $\gamma \omega''_0 < 0$ or unstable if $\gamma \omega''_0 > 0$.

To study the solitary wave solution, it is convenient to use the NLS equation in the standard form

$$i\psi_t + \psi_{xx} + \gamma |\psi|^2 \psi = 0, \quad -\infty < x < \infty, \quad t > 0. \quad (341)$$

We then seek waves of permanent form by assuming the solution

$$\psi = f(X) e^{i(mX - nt)}, \quad X = x - Ut, \quad (342)$$

for some functions f and constant wave speed U to be determined, and where m, n are constants.

Substitution of (342) in (341) gives

$$f'' + i(2m - U)f' + (n - m^2)f + \gamma |f|^2 f = 0. \quad (343)$$

We eliminate f' by setting $2m - U = 0$, and then write $n = m^2 - \alpha$, so that f can be assumed real. Thus, Eq. (343) becomes

$$f'' - \alpha f + \gamma f^3 = 0. \quad (344)$$

Multiplying this equation by $2f'$ and integrating, we find that

$$f'^2 = A + \alpha f^2 - \frac{\gamma}{2} f^4 = F(f), \quad (345)$$

where $F(f) = (\alpha_1 - \alpha_2 f^2)(\beta_1 - \beta_2 f^2)$, so that $\alpha = -(\alpha_1 \beta_2 + \alpha_2 \beta_1)$, $A = \alpha_1 \beta_1$, $\gamma = -2(\alpha_2 \beta_2)$, and the α 's and β 's are assumed to be real and distinct.

Evidently, it follows from (345) that

$$X = \int_0^f \frac{df}{\sqrt{(\alpha_1 - \alpha_2 f^2)(\beta_1 - \beta_2 f^2)}}. \quad (346)$$

Setting $(\alpha_2/\alpha_1)^{1/2} f = u$ in this integral (346), we deduce the following elliptic integral of the first kind (see Dutta and Debnath (1965) and Helal and Molines (1981)):

$$\sigma X = \int_0^u \frac{du}{\sqrt{(1-u^2)(1-\kappa^2 u^2)}}, \quad (347)$$

where $\sigma = (\alpha_2 \beta_1)^{1/2}$ and $\kappa = (\alpha_1 \beta_2)/(\beta_1 \alpha_2)$.

Thus, the final solution can be expressed in terms of the Jacobian elliptic function

$$u = sn(\sigma X, \kappa). \quad (348a)$$

Thus, the solution for $f(X)$ is given by

$$f(X) = \left(\frac{\alpha_1}{\alpha_2} \right)^{1/2} sn(\sigma X, \kappa). \quad (348b)$$

In particular, when $A = 0$, $\alpha > 0$, and $\gamma > 0$, we obtain a solitary wave solution. In this case, (345) can be rewritten

$$\sqrt{\alpha} X = \int_0^f \left\{ f'^2 \left(1 - \frac{\gamma}{2\alpha} f^2 \right) \right\}^{-1/2} df. \quad (349)$$

Substitution of $(\gamma/2\alpha)^{1/2} f = \text{sech } \theta$ in this integral gives the exact solution

$$f(X) = \left(\frac{2\alpha}{\gamma} \right)^{1/2} \text{sech}[\sqrt{\alpha}(x - Ut)]. \quad (350)$$

This represents a *solitary wave* solution that propagates without change of shape with constant velocity U . Unlike the solitary wave solution of the KdV equation, the amplitude and the velocity of the wave are independent parameters. It is noted that the solitary wave exists only for the unstable case ($\gamma > 0$). This means that small modulations of the unstable wavetrain lead to a series of solitary waves.

The well-known nonlinear dispersion relation (116) for deep water waves is

$$\omega = \sqrt{gk} \left(1 + \frac{1}{2} a^2 k^2 \right). \quad (351)$$

Therefore,

$$\begin{aligned} \omega'_0 &= \frac{\omega_0}{2k_0}, \quad \omega''_0 = \frac{\omega_0}{4k_0^2}, \quad \text{and} \\ \gamma &= -\frac{1}{2} \omega_0 k_0^2, \end{aligned} \quad (352)$$

and the NLS equation for deep water waves is obtained from (337) in the form

$$i \left(a_t + \frac{\omega_0}{2k_0} a_x \right) - \frac{\omega_0}{8k_0^2} a_{xx} - \frac{1}{2} \omega_0 k_0^2 |a|^2 a = 0. \quad (353)$$

The normalized form of this equation in a frame of reference moving with the linear group velocity ω'_0 is

$$ia_t - \left(\frac{\omega_0}{8k_0^2} \right) a_{xx} - \frac{1}{2} \omega_0 k_0^2 |a|^2 a = 0. \quad (354)$$

Since $\gamma \omega''_0 = (\omega_0^2/8) > 0$, this equation confirms the instability of deep water waves. This is one of the most remarkable recent results in the theory of water waves.

We next discuss the uniform solution and the solitary wave solution of the NLS Eq. (354). We look for solutions in the form

$$a(x, t) = A(X) \exp(i\gamma^2 t), \quad X = x - \omega'_0 t, \quad (355)$$

and substitute this in Eq. (354) to obtain the following equation:

$$A_{XX} = \frac{8k_0^2}{\omega_0} \left(\gamma^2 A + \frac{1}{2} \omega_0 k_0^2 A^3 \right). \quad (356)$$

We multiply this equation by $2A_X$ and, then, integrate to find

$$\begin{aligned} A_X^2 &= - \left(A_0^4 m'^2 + \frac{8}{\omega_0} \gamma^2 k_0^2 A^2 + 2k_0^4 A^4 \right) \\ &= (A_0^2 - A^2)(A^2 - m'^2 A_0^3), \end{aligned} \quad (357)$$

where $A_0^4 m'^2$ is an integrating constant, $2k_0^4 = 1$, $m'^2 = 1 - m^2$, and $A_0^2 = 4\gamma^2 / \omega_0 k_0^2 (m^2 - 2)$, which relates A_0 , γ , and m .

Finally, we rewrite Eq. (357) in the form

$$A_0^2 dX = \frac{dA}{\left[\left(1 - \frac{A^2}{A_0^2} \right) \left(\frac{A^2}{A_0^2} - m'^2 \right) \right]^{1/2}}, \quad (358a)$$

or, equivalently,

$$\begin{aligned} A_0(X - X_0) &= \int \frac{ds}{[(1 - s^2)(s^2 - m'^2)]^{1/2}}, \\ &= (A/A_0). \end{aligned} \quad (358b)$$

This can readily be expressed in terms of the Jacobi dn function (see Dutta and Debnath (1965) and Helal and Molines (1981)):

$$A = A_0 dn[A_0(X - X_0), m], \quad (359)$$

where m is the modulus of the dn function.

In the limit, $m \rightarrow 0$, $dnz \rightarrow 1$, and $\gamma^2 \rightarrow -\frac{1}{2} \omega_0 k_0^2 A_0^2$. Hence, the solution becomes

$$a(x, t) = A(t) = A_0 \exp \left(-\frac{1}{2} i \omega_0 k_0^2 A_0^2 t \right). \quad (360)$$

On the other hand, when $m \rightarrow 1$, $dnz \rightarrow \text{sech } z$, and $\gamma^2 \rightarrow -\frac{1}{4} \omega_0 k_0^2 A_0^2$. Therefore, the solitary wave solution is

$$\begin{aligned} a(x, t) &= A_0 \exp \left(-\frac{i}{4} \omega_0 k_0^2 A_0^2 t \right) \\ &\quad \text{sech} [A_0(x - \omega'_0 t - X_0)]. \end{aligned} \quad (361)$$

An analysis of this section reveals several remarkable features of the nonlinear Schrödinger equation. Like the KdV equation, the nonlinear Schrödinger equation is the lowest-order approximation for weakly and strongly nonlinear dispersive waves system. This equation can also be used to investigate instability phenomena in many other physical systems. Like the various forms of the KdV equation, the NLS equation arises in many physical problems, including nonlinear water waves and ocean waves, waves in plasma, propagation of heat pulses in a solid, self-trapping phenomena in nonlinear optics, nonlinear waves in a fluid-filled viscoelastic tube, and various nonlinear instability phenomena in fluids and plasmas.

Whitham's Equations of Nonlinear Dispersive Waves

To describe a slowly varying nonlinear and non-uniform oscillatory wavetrain in a dispersive medium, we assume the existence of a one-dimensional solution in the form (see Whitham (1974) or Debnath (2005)), so that

$$\phi(x, t) = a(x, t) \exp \{ i\theta(x, t) \} + c.c., \quad (362)$$

where $c.c.$ stands for the complex conjugate and $a(x, t)$ is the complex amplitude (see Debnath (2005)). The phase function $\theta(x, t)$ is given by

$$\theta(x, t) = xk(x, t) - t\omega(x, t), \quad (363)$$

and k , ω , and a are slowly varying functions of space variable x and time t .

Because of the slow variations of k and ω , it is reasonable to assume that these quantities still satisfy the linear Dispersion relation of the form

$$\omega = W(k). \quad (364)$$

Differentiating (363) with respect to x and t , respectively, we obtain

$$\theta_x = k + \{x - tW'(k)\}k_x, \quad (365)$$

$$\theta_t = -W(k) + \{x - tW'(k)\}k_t. \quad (366)$$

In the neighborhood of stationary points defined by $W'(k) = (x/t) > 0$, these equations become

$$\theta_x = k(x, t) \quad \text{and} \quad \theta_t = -\omega(x, t). \quad (367ab)$$

These results can be used as a definition of *local wavenumber* and *local frequency* of a slowly varying nonlinear wavetrain.

In view of (367ab), relation (364) gives a nonlinear partial differential equation for the phase $\theta(x, t)$ in the form

$$\frac{\partial \theta}{\partial t} + W\left(\frac{\partial \theta}{\partial x}\right) = 0. \quad (368)$$

The solution of this equation determines the geometry of the wave pattern.

We eliminate θ from (367ab) to obtain the equation

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \quad (369)$$

This is known as the *Whitham equation* for the conservation of waves, where k represents the *density* of waves and ω is the *flux* of waves.

Using the dispersion relation (364), Eq. (369) gives

$$\frac{\partial k}{\partial t} + C(k) \frac{\partial k}{\partial x} = 0, \quad (370)$$

where $C(k) = W'(k)$ is the group velocity. This represents the simplest nonlinear wave (hyperbolic) equation for the propagation of wavenumber k with group velocity $C(k)$.

Equations (370) and (364) reveal that ω also satisfies the first-order, nonlinear wave (hyperbolic) equation

$$\frac{\partial \omega}{\partial t} + W'(k) \frac{\partial \omega}{\partial x} = 0. \quad (371)$$

It follows from Eqs. (370) and (371) that both k and ω remain constant on the characteristic curves defined by

$$\frac{dx}{dt} = W'(k) = C(k) \quad (372)$$

in the (x, t) -plane. Since k or ω is constant on each characteristic, the characteristics are straight lines with slope $C(k)$.

Finally, it follows from the preceding analysis that any constant value of the phase θ propagates according to $\theta(x, t) = \text{constant}$, and hence,

$$\theta_t + \left(\frac{dx}{dt}\right) \theta_x = 0, \quad (373)$$

which gives, by (367ab),

$$\frac{dx}{dt} = -\frac{\theta_t}{\theta_x} = \frac{\omega}{k} = c(k). \quad (374)$$

Thus, the phase of the waves propagates with the phase speed $c(k)$. On the other hand, Eq. (370) ensures that the wavenumber k propagates with group velocity $C(k) = (d\omega/dk) = W'(k)$.

We next follow Whitham (1974) or Debnath (2005) to investigate how wave energy propagates in a dispersive medium. We consider the following integral involving the square of the wave amplitude (energy) between any two points $x = x_1$ and $x = x_2$ ($0 < x_1 < x_2$):

$$Q(t) = \int_{x_1}^{x_2} |A|^2 dx = \int_{x_1}^{x_2} AA^* dx \quad (375)$$

$$= 2\pi \int_{x_1}^{x_2} \frac{F(k)F^*(k)}{t |W''(k)|} dx, \quad (376)$$

which, due to a change of variable $x = tW'(k)$,

$$= 2\pi \int_{k_1}^{k_2} F(k)F^*(k)dk, \quad (377)$$

where $x_r = tW'(k_r)$, $r = 1, 2$.

When k_r is kept fixed as t varies, $Q(t)$ remains constant so that

$$\begin{aligned} 0 &= \frac{dQ}{dt} = \frac{d}{dt} \int_{x_1}^{x_2} |A|^2 dx \\ &= \int_{x_1}^{x_2} \frac{\partial}{\partial t} |A|^2 dx + |A|^2 W'(k_2) - |A|^2 W'(k_1). \end{aligned} \quad (378)$$

In the limit, as $x_2 - x_1 \rightarrow 0$, this result reduces to the first-order, partial differential equation

$$\frac{\partial}{\partial t} |A|^2 + \frac{\partial}{\partial x} [W'(k)|A|^2] = 0. \quad (379)$$

This represents the equation for the conservation of wave energy where $|A|^2$ and $|A|^2 W'(k)$ are the energy density and energy flux, respectively. It also follows that the energy propagates with group velocity $W'(k)$. It has been shown that the wavenumber k also propagates with the group velocity. Thus, the group velocity plays a double role.

The preceding analysis reveals another important fact that (364), (369), and (379) constitute a closed set of equations for the three functions k , ω , and A . Indeed, these are the fundamental equations for nonlinear dispersive waves and are known as *Whitham's equations*.

In his pioneering work on nonlinear dispersive waves, Whitham (1974) formulated a more general energy equation based on the amount of energy $Q(t)$ between two points x_1 and x_2

$$Q(t) = \int_{x_1}^{x_2} g(k)A^2 dx, \quad (380)$$

where $g(k)$ is an arbitrary proportionality factor associated with the square of the amplitude and energy.

In a new coordinate system moving with the group velocity, that is, along the rays $x = C(k)t$, Eq. (380) reduces to the form

$$Q(t) = 2\pi \int_{x_1}^{x_2} g(k)F(k)F^*(k)dk, \quad (381)$$

where $\omega = \Omega(k)$ and $\Omega''(k) > 0$ and k_1 and k_2 are defined by $x_1 = C(k_1)t$ and $x_2 = C(k_2)t$, respectively.

Using the principle of conservation of energy, that is, stating that the energy between the points x_1 and x_2 traveling with the group velocities $C(k_1)$ and $C(k_2)$ remains invariant, it turns out from (380) that

$$\begin{aligned} \frac{dQ}{dt} &= \int_{x_1}^{x_2} \frac{\partial}{\partial t} \{g(k)A^2\} dx + g(k_2)C(k_2)A^2(x_2, t) \\ &\quad - g(k_1)C(k_1)A^2(x_1, t) = 0, \end{aligned} \quad (382)$$

which is, in the limit as $x_2 - x_1 \rightarrow 0$,

$$\frac{\partial}{\partial t} \{g(k)A^2\} + \frac{\partial}{\partial x} \{g(k)C(k)A^2\} = 0. \quad (383)$$

This may be treated as the more general energy equation. Quantities $g(k)A^2$ and $g(k)C(k)A^2$ represent the *energy density* and the *energy flux*, so that they are proportional to $|A|^2$ and $C(k)|A|^2$, respectively. The flux of energy propagates with the group velocity $C(k)$. Hence, the group velocity has a double role: it is the propagation velocity for the wavenumber k and for the energy $g(k)|A|^2$.

Thus, (369) and (383) are known as *Whitham's conservation equations* for nonlinear dispersive waves. The former represents the conservation of wave-number k and the latter is the conservation of energy (or more generally, the conservation of wave action). Whitham also derived the conservation equations more rigorously from a general and extremely powerful approach that is now known as *Whitham's averaged variational principle*.

For slowly varying dispersive wavetrains, the solution maintains the elementary form $u = \Phi(\theta, a)$, but ω , $\mathbf{k}$, and a are no longer constants, so that θ is not a linear function of x_i and t . The local wavenumber and local frequency are defined by

$$k_i = \frac{\partial \theta}{\partial x_i}, \quad \omega = -\frac{\partial \theta}{\partial t}. \quad (384ab)$$

The *Whitham averaged Lagrangian* over the phase of the integral of the Lagrangian L is defined by

$$\mathcal{L}(\omega, \mathbf{k}, a, \mathbf{x}, t) = \frac{1}{2\pi} \int_0^{2\pi} L \, d\theta \quad (385)$$

and is calculated by assuming the uniform periodic solution $u = \Phi(\theta, a)$ in L . Whitham first formulated the *averaged variational principle* in the form

$$\delta \iint \mathcal{L} \, d\mathbf{x} dt = 0, \quad (386)$$

to derive the equations for ω , $\mathbf{k}$, and a .

It is noted that the dependence of $\mathcal{L}$ on $\mathbf{x}$ and t reflects possible nonuniformity of the medium supporting the wave motion. In a uniform medium, $\mathcal{L}$ is independent of $\mathbf{x}$ and t , so that the Whitham function $\mathcal{L} = \mathcal{L}(\omega, \mathbf{k}, a)$. However, in a uniform medium, some additional variables also appear only through their derivatives. They represent potentials whose derivatives are important physical quantities.

The Euler equations resulting from the independent variations of δa and $\delta \theta$ in (386) with $\mathcal{L} = \mathcal{L}(\omega, \mathbf{k}, a)$ are

$$\delta a : \mathcal{L}_a(\omega, \mathbf{k}, a) = 0, \quad (387)$$

$$\delta \theta : \frac{\partial}{\partial t} \mathcal{L}_\omega - \frac{\partial}{\partial x_i} \mathcal{L}_{k_i} = 0. \quad (388)$$

The θ -eliminant of (384ab) gives the consistency equations

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0, \quad \frac{\partial k_i}{\partial x_j} - \frac{\partial k_j}{\partial x_i} = 0. \quad (389ab)$$

Thus, (387, 388, 389ab) represent the *Whitham equations* for describing the slowly varying wavetrain in a nonuniform medium and constitute a closed set from which the triad ω , $\mathbf{k}$, and a can be determined.

In linear problems, the Lagrangian L , in general, is a quadratic in u and its derivatives. Hence, if $\Phi(\theta) = a \cos \theta$ is substituted in (385), $\mathcal{L}$ must always take the form

$$\mathcal{L}(\omega, \mathbf{k}, a) = D(\omega, \mathbf{k}) a^2, \quad (390)$$

so that the dispersion relation ($\mathcal{L}_a = 0$) must take the form

$$D(\omega, \mathbf{k}) = 0. \quad (391)$$

We note that the stationary value of $\mathcal{L}$ is, in fact, zero for linear problems. In the simple case, L equals the difference between kinetic and potential energy. This proves the well-known principle of equipartition of energy, stating that average potential and kinetic energies are equal.

Whitham's Instability Analysis of Water Waves

Section “[The Nonlinear Schrödinger Equation and Solitary Waves](#)” deals with Whitham's new remarkable variational approach to the theory of slowly varying, nonlinear, dispersive waves. Based upon Whitham's ideas and, especially, Whitham's fundamental dispersion Eq. (388), Lighthill (1965, 1967) developed an elegant and remarkably simple result determining whether very gradual – not necessarily small – variations in the properties of a wavetrain are governed by hyperbolic or elliptic partial differential equations. A general account of Lighthill's work with special reference to the instability of wavetrains on deep water was described by Debnath (1994). This section is devoted to the Whitham instability theory with applications to water waves.

According to Whitham's nonlinear dispersive wave theory $\mathcal{L}_a = 0$ gives a dispersion relation that depends on wave amplitude a has the form

$$\omega = \omega(k, a), \quad (392)$$

where equations for k and a are no longer uncoupled and constitute a system of partial differential equations. The first important question is

whether these equations are hyperbolic or elliptic. This can be answered by a standard and simple method combined with Whitham's conservation Eqs. (389ab) and (383). For moderately small amplitudes, we use the Stokes expansion of ω in terms of k and a^2 in the form

$$\omega(k) = \omega_0(k) + \omega_2(k)a^2 + \dots \quad (393)$$

We substitute this result in (389ab) and (383), replace $\omega'(k)$ by its linear value $\omega'_0(k)$, and retain the terms of order a^2 to obtain the equations for k and a^2 in the form

$$\frac{\partial k}{\partial t} + [\omega'(k) + \omega'_2(k)a^2] \frac{\partial k}{\partial x} + \omega_2(k) \frac{\partial a^2}{\partial x} = 0, \quad (394)$$

$$\frac{\partial a^2}{\partial t} + \omega'_0(k) \frac{\partial a^2}{\partial x} + \omega''_0(k)a^2 \frac{\partial k}{\partial x} = 0. \quad (395)$$

Neglecting the term $O(a^2)$, these equations can be rewritten as

$$\frac{\partial k}{\partial t} + \omega'_0 \frac{\partial k}{\partial x} + \omega_2 \frac{\partial a^2}{\partial x} = 0, \quad (396)$$

$$\frac{\partial a^2}{\partial t} + \omega'_0(k) \frac{\partial a^2}{\partial x} + \omega''_0 a^2 \frac{\partial k}{\partial x} = 0. \quad (397)$$

These describe the modulations of a linear dispersive wavetrain and represent a coupled system due to the nonlinear dispersion relation (393) exhibiting the dependence of ω on both k and a . In matrix form, these equations read

$$\begin{pmatrix} \omega'_0 & \omega_2 \\ [1mm]\omega''_0 a^2 & \omega'_0 \end{pmatrix} \begin{pmatrix} \frac{\partial k}{\partial x} \\ [1mm]\frac{\partial a^2}{\partial x} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ [1mm]0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial k}{\partial t} \\ [1mm]\frac{\partial a^2}{\partial t} \end{pmatrix} = 0. \quad (398)$$

Hence, the eigenvalues λ are the roots of the determinant equation

$$|a_{ij} - \lambda b_{ij}| = \begin{vmatrix} \omega'_0 - \lambda & \omega_2 \\ \omega''_0 a^2 & \omega'_0 - \lambda \end{vmatrix} = 0, \quad (399)$$

where a_{ij} and b_{ij} are the coefficient matrices of (398). This determinant equation gives the characteristic velocities

$$\begin{aligned} \lambda &= \left(\frac{dx}{dt} \right) = C(k) \\ &= C_0(k) \pm a [\omega_2(k)\omega''_0(k)]^{1/2} + O(a^2), \end{aligned} \quad (400ab)$$

where $C_0(k) = \omega'_0(k)$ is the linear group velocity, and, in general, $\omega''_0(k) \neq 0$ for dispersive waves. The equations are hyperbolic or elliptic depending on whether $\omega_2(k)\omega''_0(k) > 0$ or < 0 .

In the hyperbolic case, the characteristics are real, and the double characteristic velocity splits into two separate velocities and provides a generalization of the group velocity of nonlinear dispersive waves. In fact, the characteristic velocities (400ab) are used to define the *nonlinear group velocities*. The splitting of the double characteristic velocity into two different velocities is one of the most significant results of the Whitham theory. This means that any initial disturbance of finite extent will eventually split into two separate disturbances. This prediction is radically different from that of the linearized theory, where an initial disturbance may suffer from a distortion due to dependence of the linear group velocity $C_0(k) = \omega'_0(k)$ on the wavenumber k , but would *never split* into two. Another significant consequence of nonlinearity in the hyperbolic case is that compressive modulations will suffer from gradual distortion and steepening in the typical hyperbolic manner discussed earlier. This leads to the multiple-valued solutions and hence, eventually, breaking of waves.

In the elliptic case ($\omega_2, \omega''_0 < 0$), the characteristics are imaginary. This leads to *ill-posed problems* in the theory of nonlinear wave propagation. Any small sinusoidal disturbances in a and k may be represented by solutions of the form $\exp[ia\{x - C(k)t\}]$, where $C(k)$ is given by (400ab) for the unperturbed values of a and k . Thus, when

$C(k)$ is complex, these disturbances will grow exponentially in time, and hence, the periodic wavetrains become definitely *unstable*.

An application of this analysis to the Stokes waves on deep water reveals that the associated dispersion equation is elliptic in this case. For waves on deep water, the dispersion relation is

$$\omega = \sqrt{gk} \left(1 + \frac{1}{2} a^2 k^2 \right) + O(a^4). \quad (401)$$

This result is compared with the Stokes expansion (393) to give $\omega_0(k) = \sqrt{gk}$ and $\omega_2(k) = \frac{1}{2} k^2 \sqrt{gk}$. Hence, $\omega_0'' \omega_2 < 0$, the velocities (400ab) are complex, and the Stokes waves on deep water are definitely *unstable*. The instability of deep water waves came as a real surprise to researchers in the field in view of the very long and controversial history of the subject. The question of instability went unrecognized for a long period of time, even though special efforts have been made to prove the existence of a permanent shape for periodic water waves for all amplitudes less than the critical value at which the waves assume a sharp-crested form. However, Lighthill's (1967) theoretical investigation into the elliptic case and Benjamin and Feir's (1967) theoretical and experimental findings have provided conclusive evidence of the instability of Stokes waves on deep water.

For more details of nonlinear dispersive wave phenomena, the reader is also referred to Debnath (1994).

In his pioneering work on nonlinear water waves, Whitham (1974) observed that the neglect of dispersion in the nonlinear shallow water equations leads to the development of multivalued solutions with a vertical slope, and hence, eventually breaking occurs. It seems clear that the third derivative dispersive term in the KdV equation produces the periodic and solitary waves which are not found in the shallow water theory. However, the KdV equation cannot describe the observed symmetrical peaking of the crests with a finite angle. On the other hand, the Stokes waves include full effects of dispersion, but they are limited to small amplitude, and

describe neither the solitary waves nor the peaking phenomenon.

Although both breaking and peaking are without doubt involved in the governing equations of the exact potential theory, Whitham (1974) developed a mathematical equation that can include all these phenomena. It has been shown earlier that the breaking of shallow water waves is governed by the nonlinear equation

$$\eta_t + c_0 \eta_x + d \eta \eta_x = 0, \quad d = 3c_0(2h_0)^{-1}. \quad (402)$$

On the other hand, the linear equation corresponding to a general linear dispersion relation

$$\frac{\omega}{k} = c(k) \quad (403)$$

is given by the integrodifferential equation in the form

$$\eta_t + \int_{-\infty}^{\infty} K(x-s) \eta_s(s, t) ds = 0, \quad (404)$$

where the kernel K is given by the inverse Fourier transform of $c(k)$:

$$K(x) = \mathcal{F}^{-1}\{c(k)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} c(k) dk. \quad (405)$$

Whitham combined the above ideas to formulate a new class of *nonlinear nonlocal* equations

$$\eta_t + d \eta \eta_x + \int_{-\infty}^{\infty} K(x-s) \eta_s(s, t) ds = 0. \quad (406)$$

This is well known as the *Whitham equation*, which can, indeed, describe symmetric waves that propagate without change of shape and peak at a critical height, as well as asymmetric waves that invariably break.

Once a wave breaks, it usually continues to travel in the form of a bore as observed in tidal waves. The weak bores have a smooth but oscillatory structure, whereas the strong bores have a structure similar to turbulence with no coherent

oscillations. Since the region where waves break is a zone of high energy dissipation, it is natural to include a second derivative dissipative term in the KdV equation to obtain

$$\eta_t + c_0 \eta_x + d\eta \eta_x + \mu \eta_{xxx} - \nu \eta_{xx} = 0, \quad (407)$$

where $\mu = \frac{1}{6} c_0 h_0^2$. This is known as the *KdV–Burgers equation*, which also arises in nonlinear acoustics for fluids with gas bubbles (see (Karpman 1975)).

Whitham's (1974) equations for the slow modulation of the wave amplitude a and the wavenumber k in the case of two-dimensional deep water waves are given by

$$\frac{\partial}{\partial t} \left(\frac{a^2}{\omega_0} \right) + \frac{\partial}{\partial x} \left(C \frac{a^2}{\omega_0} \right) = 0, \quad (408)$$

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0, \quad (409)$$

where $\omega_0 = \sqrt{gk}$ is the first-order approximation for the wave frequency $\omega(k)$ and $C = (g/2\omega_0)$ is the group velocity.

Chu and Mei (1970, 1971) observed that certain terms of the dispersive type, neglected in Whitham's equations to the same order of approximation, must be included to extend the validity of these equations. Whitham's theory is based on the direct use of Stokes' dispersion relation for a uniform wave train,

$$\omega = \omega_0 \left(1 + \frac{1}{2} \varepsilon^2 a^2 k^2 \right), \quad (410)$$

whereas Chu and Mei added terms of higher derivatives and dispersive type to the expression for ω , so that

$$\omega = \omega_0 \left[1 + \varepsilon^2 \left(\frac{1}{2} a^2 k^2 + \left\{ \left(\frac{a}{\omega_0} \right)_{tt} \div 2\omega_0 a \right\} \right) \right]. \quad (411)$$

They used the expression (411) to transform (408) and (409) in a frame of reference moving with the group velocity $C(k)$ and obtained the following nondimensional equations:

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x} (a^2 \phi_x) = 0, \quad (412)$$

$$-2 \frac{\partial^2 \phi}{\partial x \partial t} + \frac{\partial}{\partial x} \left[-\phi_x^2 + \frac{a^2}{4} + \frac{a_{xx}}{16a} \right] = 0, \quad (413)$$

where we have used Chu and Mei's result $W = -2\phi_x$ and ϕ is a small phase variation. Integrating (413) with respect to x and setting the constant of integration to be zero gives

$$\phi_t + \frac{1}{2} \phi_x^2 - \frac{1}{8} a^2 - \frac{a_{xx}}{32a} = 0. \quad (414)$$

A transformation $\Psi = a \exp(4i\phi)$ is used to simplify (412) and (414), which reduces to the *nonlinear Schrödinger equation*

$$i\Psi_t + \frac{1}{8} \Psi_{xx} + \frac{1}{2} \Psi |\Psi|^2 = 0. \quad (415)$$

This equation has also been derived and exploited by several authors, including Benney and Roskes (1969), Hasimoto and Ono (1972), and Davey and Stewartson (1974) to examine the nonlinear evolution of Stokes' waves on water.

An alternative way to study nonlinear evolution of two- and three-dimensional wavepackets is to use the method of multiple scales in which the small parameter ε is explicitly built into the expansion procedure. The small parameter ε characterizes the wave steepness. This method has been employed by several authors in various fields and has also been used by Hasimoto and Ono (1972) and Davey and Stewartson (1974).

The Benjamin–Feir Instability of the Stokes Water Waves

One of the simplest solutions of the nonlinear Schrödinger Eq. (354) is given by (360), that is,

$$A(t) = A_0 \exp \left(-\frac{1}{2} i \omega_0 k_0^2 A_0^2 t \right), \quad (416)$$

where A_0 is a constant. This essentially represents the fundamental component of the Stokes wave.

We consider a perturbation of (416) and express it in the form

$$a(x, t) = A(t)[1 + B(x, t)], \quad (417)$$

where $B(x, t)$ is the perturbation function. Substituting this result in (354) gives

$$\begin{aligned} i(1+B)A_t + iAB_t - \left(\frac{\omega_0}{8k_0^2}\right)AB_{xx} \\ = \frac{1}{2}\omega_0 k_0^2 A_0^2 \left[(1+B) + BB^*(1+B) \right. \\ \left. + (B+B^*)B + (B+B^*) \right] A, \end{aligned} \quad (418)$$

where $B^*(x, t)$ is the complex conjugate of the perturbed function $B(x, t)$. Neglecting squares of B , Eq. (418) reduces to

$$iB_t - \left(\frac{\omega_0}{8k_0^2}\right)B_{xx} = \frac{1}{2}\omega_0 k_0^2 A_0^2 (B + B^*). \quad (419)$$

We look for a solution for perturbed quantity $B(x, t)$ in the form

$$B(x, t) = B_1 \exp(\Omega t + i\ell x) + B_2 \exp(\Omega^* t - i\ell x), \quad (420)$$

where B_1 and B_2 are complex constants, ℓ is a real wavenumber, and Ω is a growth rate (possibly a complex quantity) to be determined. Substituting the solution for B in (419) yields a pair of coupled equations:

$$\left(i\Omega + \frac{\omega_0 \ell^2}{8k_0^2}\right)B_1 - \frac{1}{2}\omega_0 k_0^2 A_0^2 (B_1 + B_2^*) = 0, \quad (421)$$

$$\left(i\Omega^* + \frac{\omega_0 \ell^2}{8k_0^2}\right)B_2 - \frac{1}{2}\omega_0 k_0^2 A_0^2 (B_1^* + B_2) = 0. \quad (422)$$

We take the complex conjugate of (422) to transform it into the form

$$\left(-i\Omega + \frac{\omega_0 \ell^2}{8k_0^2}\right)B_2^* - \frac{1}{2}\omega_0 k_0^2 A_0^2 (B_1 + B_2^*) = 0. \quad (423)$$

The pair of linear homogeneous Eqs. (421) and (423) for B_1 and B_2^* admits a nontrivial eigenvalue for Ω provided

$$\begin{vmatrix} i\Omega + \frac{\omega_0 \ell^2}{8k_0^2} - \frac{1}{2}\omega_0 k_0^2 A_0^2 & -\frac{1}{2}\omega_0 k_0^2 A_0^2 \\ [1ex] -\frac{1}{2}\omega_0 k_0^2 A_0^2 & i\Omega + \frac{\omega_0 \ell^2}{8k_0^2} - \frac{1}{2}\omega_0 k_0^2 A_0^2 \end{vmatrix} = 0, \quad (424)$$

which is equivalent to

$$\Omega^2 = \frac{1}{2} \left(\frac{\omega_0 \ell}{2k_0}\right)^2 \left(k_0^2 A_0^2 - \frac{\ell^2}{8k_0^2}\right). \quad (425)$$

The growth rate Ω is purely imaginary or real (and positive) depending on whether $\ell^2 > 8k_0^4 A_0^2$ or $\ell^2 < 8k_0^4 A_0^2$. The former case represents a wave solution for B , and the latter corresponds to the *Benjamin–Feir* (or *modulational*) *instability* with a criterion in terms of the nondimensional wavenumber $\tilde{\ell} = (\ell/k_0)$ as

$$\tilde{\ell}^2 < 8k_0^2 A_0^2. \quad (426)$$

Thus, the range of instability is given by

$$0 < \tilde{\ell} < \tilde{\ell}_c = 2\sqrt{2}k_0 A_0. \quad (427)$$

Since Ω is a function of $\tilde{\ell}$, the maximum instability occurs at $\tilde{\ell} = \tilde{\ell}_{\max} = 2k_0 A_0$, with a maximum growth rate given by

$$(\operatorname{Re} \Omega)_{\max} = \frac{1}{2}\omega_0 k_0^2 A_0^2. \quad (428)$$

To establish the connection with the Benjamin–Feir instability (Benjamin and Feir 1967),

we have to find the velocity potential for the fundamental wave mode multiplied by $\exp(kz)$. It turns out that the term proportional to B_1 is the upper sideband, whereas that proportional to B_2 is the lower sideband. The main conclusion of this analysis is that Stokes water waves are definitely *unstable*.

Future Directions

Although this entry has been devoted to water waves and the Korteweg and de Vries equations, there are some challenging problems dealing with solitary waves envelopes in the wake of a ship in oceans, and the KdV equation with variable coefficients in an ocean of variable depth. Despite some recent progress, there is no complete theory for ships in waves that takes into account of the effects of the finite ship volume, and possible interactions between Kelvin wake and soliton envelopes. In order to respond to experimental results with a rough bottom, theories based on an empirical formula for the bottom stress have been developed, but they are not yet completely satisfactory when compared to experiments.

Of the systems that conserve energy, some are completely integrable in the sense of solitary wave theory, but many including those most important for applications are not. In some cases, useful analytical techniques are currently available, and others are waiting to be discovered. So there is a need for research in the area of nonlinear lattices. When we deal with real world problems, the conservation of energy does not hold because of dissipative effects and forcing terms. Consequently, the KdV equation and other relevant evolution equations need modification by including terms that describe such effects. So, there are challenging problems dealing with these modified evolution equations, their methods of solutions and the qualitative and quantitative understanding of the solutions. On the other hand, prospects for research in the transverse coupling between solitary waves and parallel systems are indeed bright.

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Soliton Solutions for Some Nonlinear Water Wave Dynamical Models

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Article Outline

Introduction
The Description of the AEM
Soliton Extraction
Conclusion
Bibliography

Introduction

Nonlinear partial differential equations (NPDEs) are involved nonlinear complex physical phenomena, which play a vital role in physical sciences. Taking into account the merits and demerits of analytic methods, it is observed that there is no single outstanding preferable method which can be applied to any kind of nonlinear problems to obtain exact solutions. Consequently, it is apprehended that all of these methods are problem dependent, viz., some approaches work well with certain problems but not the others. Therefore, it is rather substantial to relate some established techniques to find the analytic solution for nonlinear models. Many articles have been devoted to find the numerical and exact solutions for a verity of nonlinear models (see Seadawy 2014a; Korteweg

and de Vries 1895; Kochanov et al. 2013; Seadawy 2014b; Peng 2003; Peng 2004; Seadawy and El-Rashidy 2013; Huang 2006; Seadawy 2017a; Helal and Seadawy 2009; Zhao et al. 2006; Seadawy and El-Rashidy 2014; Khater et al. 2000; Khater et al. 2006).

In the last some decades, both mathematician and physicist have made much important effort in this area and demonstrated various useful techniques, such as inverse scattering scheme (Ablowitz and Clarkson 1991), Bäcklund transform method (Hirota 1971), homogeneous balance scheme (Wang 1996), Painlevé expansion (Weiss et al. 1985), sine-cosine method, Darboux transformation, Lie group analysis, mapping method and extended mapping method (Zheng and Fang 2006; Wen 2010), Exp-function method (Zhang et al. 2009), rational expansion method (Xin and Deng-Shan 2009), Adomian decomposition method (Yavuz et al. 2011; Dogan and El-Sayed Salah 2003), homotopy perturbation method (Biazar et al. 2009), homotopy analysis method (Dinarvand et al. 2008), differential transform method, reduced differential method, tanh-coth method (Kara and Khalique 2005; Khalique and Adem 2011; Kichenassamy and Olver 1992), simplified Hirota's method, extended homoclinic test approach (Darvishi and Najafi 2011), transformed rational function method (Khalique and Adem 2011), Hirota's bilinear method (Mehdipoor and Neirameh 2015), and many more. In the literature, a variety of suitable methods are adopted to investigate the behavior of the nonlinear equations. The study about solutions, structures, interaction, and further properties of soliton abstracts gives much more attention, and various meaningful results are successfully obtained (as in Seadawy 2012; Seadawy 2015; Seadawy 2016; Seadawy 2017b; Arshad et al. 2017; Geng and Ma 2007; Seadawy and Manafian 2018; Zhang et al. 2007; Khan et al. 2013; Misirli and Gurefe 2010; Wena et al. 2009; Yao and Li 2002; Liu and Liu 2002; Noor et al. 2011; Benjamin et al. 1972; Abdel Rady et al. 2010).

In this entry, we further extend the ansatz equation in more general form in the modified extended direct algebraic method to construct traveling wave solutions of modified Benjamin-Bona-Mahony (m-BBM), coupled Drinfel'd-Sokolov-Wilson (c-DSW), and $(3 + 1)$ -dimensional extended Jimbo-Miwa (e-JM) equations. As a result, some new and more general exact traveling wave solutions are obtained. Mathematica 10.4 is used to carry out simulations and to visualize the behavior of nonlinear waves. The governing equations are as follows:

In (Noor et al. 2011) the Exp-function method, the modified simple equation method is used (Khan et al. 2013), and the homogeneous balance method in Abdel Rady et al. (2010)) is applied to investigate the behavior of Benjamin-Bona-Mahony equation. The m-BBM model equation is given as

$$u_t + \varsigma_1 u_x + \varsigma_2 u^2 u_x + \varsigma_3 u_{xxt} = 0 \quad (1)$$

The exact solution for the nonlinear coupled Drinfel'd-Sokolov-Wilson system is obtained by various authors (as in Misirli and Gurefe 2010); the Exp-function method has been carried out, while the bifurcation method is adopted in Wena et al. (2009)), whereas Yao and Li (2002) and Liu and Liu (2002)) employed the direct algebraic method and the modified simple equation method by Khan et al. (2013)). The governing c-DSW system reads

$$\left. \begin{aligned} u_t + \varsigma_1 v v_x &= 0, \\ v_t + \varsigma_2 v_{xxx} + \varsigma_3 u v_x + \varsigma_4 u_x v &= 0 \end{aligned} \right\} \quad (2)$$

The second equation in the renowned KP hierarchy of integrable systems is Jimbo-Miwa equation, illustrates the most crucial $(3 + 1)$ -dimensional waves equation in contemporary physics (Mehdipoor and Neirameh 2015; Seadawy 2012; Seadawy 2015; Seadawy 2016), but does not permit any of the conventional integrability tests. The two types of $(3 + 1)$ - D e-JM equations are read as

$$u_{xxx} + \varsigma_1 u_y u_{xx} + \varsigma_2 u_x u_{xy} + \varsigma_3 u_{yt} + \varsigma_4 (u_{xz} + u_{yz} + u_{zz}) = 0 \quad (3)$$

$$u_{xxx} + \varsigma_1 u_y u_{xx} + \varsigma_2 u_x u_{xy} + \varsigma_3 (u_{xt} + u_{yt} + u_{zt}) + \varsigma_4 u_{xz} = 0 \quad (4)$$

where the coefficients ς_i for $i = 1, 2, 3, 4$ are real constants.

This entry has been devised as follows: in section “[The Description of the AEM](#),” the auxiliary equation method is introduced, while in section “[Soliton Extraction](#),” the solutions of the NPDEs have been presented. In last section “[Conclusion](#),” the conclusions have been drawn.

The Description of the AEM

We will briefly present the AEM in the following steps:

Step 1. Let us have a general form of nonlinear PDE

$$F(u, u_t, u_x, u_y, u_{xx}, u_{yy}, \dots) = 0. \quad (5)$$

where F is a polynomial function with respect to the indicated variables.

Step 2. The following wave variable is presented to solve (5)

$$u(x, t) = F(\xi), \quad (6)$$

The transformations (6) convert the PDE (5) to an ODE

$$O_i(F, F_\xi, F_{\xi\xi}, F_{\xi\xi\xi}, \dots), \quad (7)$$

where $F = F(\xi)$ is unknown function.

Step 3. The main idea of the auxiliary equation method based on expanding the traveling wave solution $F(\xi)$ of Eq. (7) as a finite series

$$F(\xi) = \sum_{i=0}^n e^{it} \psi^i(\xi), \quad (8)$$

ψ satisfies

$$\frac{d\psi}{d\xi} = \zeta_0 + \zeta_1 \psi(\xi) + \zeta_2 \psi^2(\xi) + \zeta_3 \psi^3(\xi) + \zeta_4 \psi^4(\xi), \quad (9)$$

$$\xi = x \pm \omega t \quad (10)$$

where $c_i (i = 0, 1, 2, 3, 4)$ are constants.

Step 4. Applying the homogenous balance to (5), the parameters n in (8) can be obtained.

Step 5. Substituting (8), (9), and (10) in (5) and collecting the coefficients of $\psi^j \psi^{(k)}$ and then solving the system for ω and ζ_i .

Step 6. Substituting $\omega \zeta_i$ and $\psi(\xi)$ obtained in step 5 into (8), to obtain the solutions for (1), (2), (3), and (4).

Soliton Extraction

m-BBM Equation

Consider the transformation

$$u(x, t) = u(\xi), \quad \xi = x + \omega t, \quad (11)$$

using (11) into (1),

$$(\omega + \zeta_1)u' + \zeta_2 u^2 u' + \omega \zeta_3 u''' = 0 \quad (12)$$

integrating

$$(\omega + \zeta_1)u + \frac{1}{3} \zeta_2 u^3 + \omega \zeta_3 u'' = 0 \quad (13)$$

Considering the homogeneous balance between u^3 and u'' gives $n = 1$. Suppose the solution of (13) is of the form

$$u = 1 + e^t \psi(\xi) \quad (14)$$

Substituting from equations (8), (9), and (14) into (13) and collecting the coefficients of $\psi^j \psi^{(k)}$,

Case I

$$\psi_1(\xi) = e^{-t} (\eta_2 \tanh(\eta_1 \xi) - 1) \quad (15)$$

where

$$\eta_1 = \frac{\sqrt{\zeta_1 + \omega}}{\sqrt{2} \sqrt{\zeta_3} \sqrt{\omega}}, \quad (16)$$

$$\eta_2 = -\frac{i\sqrt{3} \sqrt{\zeta_1 + \omega}}{\sqrt{\zeta_2}} \quad (17)$$

The parameters ζ_i become

$$\zeta_0 = -\frac{ie^{-t}(3\zeta_1 + \zeta_2 + 3\omega)}{\sqrt{6} \sqrt{\zeta_2} \sqrt{\zeta_3} \sqrt{\omega}},$$

$$\zeta_1 = -\frac{i\sqrt{\frac{2}{3}} \sqrt{\zeta_2}}{\sqrt{\zeta_3} \sqrt{\omega}},$$

$$\zeta_2 = -\frac{i\sqrt{\zeta_2} e^t}{\sqrt{6} \sqrt{\zeta_3} \sqrt{\omega}},$$

$$\zeta_3 = 0,$$

$$\zeta_4 = 0$$

so the solution of (1) will be

$$\begin{aligned} u_1(x, t) &= \eta_1 \eta_2 \operatorname{sech}^2(\eta_1 \xi) \\ &\times (2\eta_1^2 \zeta_3 \omega (2 - 3 \operatorname{sech}^2(\eta_1 \xi)) \\ &+ \eta_2^2 \zeta_2 \tanh^2(\eta_1 \xi) + \zeta_1 + \omega) \end{aligned} \quad (18)$$

Case II

$$\psi_1(\xi) = e^{-t} (\eta_2 \tanh(\eta_1 \xi) - 1) \quad (19)$$

where

$$\eta_1 = \frac{\sqrt{\zeta_1 + \omega}}{\sqrt{2} \sqrt{\zeta_3} \sqrt{\omega}}, \quad (20)$$

$$\eta_2 = \frac{i\sqrt{3} \sqrt{\zeta_1 + \omega}}{\sqrt{\zeta_2}} \quad (21)$$

The parameters ζ_i become

$$\begin{aligned}\zeta_0 &= \frac{ie^{-t}(3\varsigma_1 + \varsigma_2 + 3\omega)}{\sqrt{6}\sqrt{\varsigma_2}\sqrt{\varsigma_3}\sqrt{\omega}}, \\ \zeta_1 &= \frac{i\sqrt{\frac{2}{3}}\sqrt{\varsigma_2}}{\sqrt{\varsigma_3}\sqrt{\omega}}, \\ \zeta_2 &= \frac{i\sqrt{\varsigma_2}e^t}{\sqrt{6}\sqrt{\varsigma_3}\sqrt{\omega}}, \\ \zeta_3 &= 0, \\ \zeta_4 &= 0\end{aligned}$$

Case I

$$\begin{aligned}\psi_1(\xi) &= e^{-t} \left(\frac{\sqrt{6}\omega \tanh\left(\frac{\xi\sqrt{\varsigma_2}(\varsigma_3 + 2\varsigma_4)\omega}{\sqrt{2}\varsigma_2\sqrt{\varsigma_3 + 2\varsigma_4}}\right)}{\sqrt{\varsigma_1}\sqrt{\varsigma_3 + 2\varsigma_4}} - 1 \right) \\ (29)\end{aligned}$$

The parameters ζ_i become

$$\begin{aligned}\zeta_0 &= \frac{e^{-t}(6\omega^2 - \varsigma_1(\varsigma_3 + 2\varsigma_4))}{2\sqrt{3}\sqrt{\varsigma_1}\sqrt{\varsigma_2}(\varsigma_3 + 2\varsigma_4)\omega}, \\ \zeta_1 &= -\frac{\sqrt{\varsigma_1}(\varsigma_3 + 2\varsigma_4)}{\sqrt{3}\varsigma_2\varsigma_3\omega + 6\varsigma_2\varsigma_4\omega}, \\ \zeta_2 &= -\frac{\sqrt{\varsigma_1}e^t\sqrt{\varsigma_2}(\varsigma_3 + 2\varsigma_4)\omega}{2\sqrt{3}\varsigma_2\omega}, \\ \zeta_3 &= 0, \\ \zeta_4 &= 0\end{aligned}$$

so the solution of (1) will be

$$\begin{aligned}u_2(x, t) &= \eta_1\eta_2\text{sech}^2(\eta_1\xi) \\ &\quad \times (2\eta_1^2\varsigma_3\omega(2 - 3\text{sech}^2(\eta_1\xi)) \\ &\quad + \eta_2^2\varsigma_2 \tanh^2(\eta_1\xi) + \varsigma_1 + \omega)\end{aligned} \quad (22)$$

c-DSW Equation

Consider the transformation

$$\begin{aligned}u(x, t) &= u(\xi), \quad v(x, t) = v(\xi), \quad \xi \\ &= x + \omega t,\end{aligned} \quad (23)$$

using (23) into (2),

$$\omega u' + \varsigma_1 v' = 0 \quad (24a)$$

$$\varsigma_4 v u' + \omega v' + \varsigma_3 u v' + \varsigma_2 v''' = 0 \quad (24b)$$

integrating (24a)

$$u = -\frac{\varsigma_1 v^2}{2\omega} \quad (25)$$

substituting in (24b)

$$2\omega^2 v' + 2\omega\varsigma_2 v''' - \varsigma_1(\varsigma_3 + 2\varsigma_4)v^2 v' = 0 \quad (26)$$

integrating (26)

$$2\omega^2 v + 2\omega\varsigma_2 v'' - \frac{1}{3}\varsigma_1(\varsigma_3 + 2\varsigma_4)v^3 = 0 \quad (27)$$

Considering the homogeneous balance between v^3 and v'' gives $n = 1$ (Fig. 1). Suppose the solution of (27) is of the form

$$v = 1 + e^t \psi(\xi) \quad (28)$$

Substituting from equations (8), (9), and (28) into (27) and collecting the coefficients of $\psi^j \psi^{(k)}$,

so the solution of (2) will be

$$\begin{aligned}v_1(x, t) &= \eta_1\eta_2\text{sech}^2(\eta_1\xi) \\ &\quad \times (4\eta_1^2\varsigma_2\omega(\cosh(2\eta_1\xi) - 2)\text{sech}^2(\eta_1\xi) \\ &\quad - \eta_2^2\varsigma_1(\varsigma_3 + 2\varsigma_4)\tanh^2(\eta_1\xi) + 2\omega^2)\end{aligned} \quad (30)$$

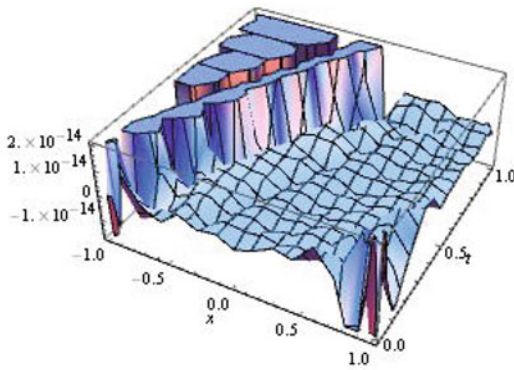
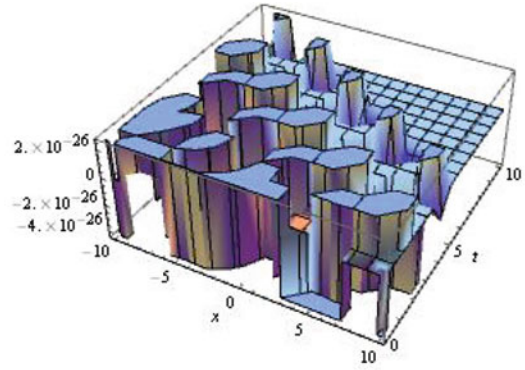
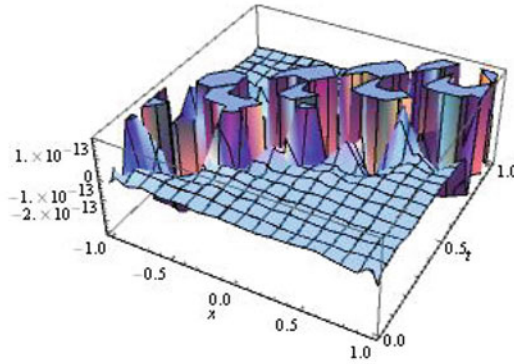
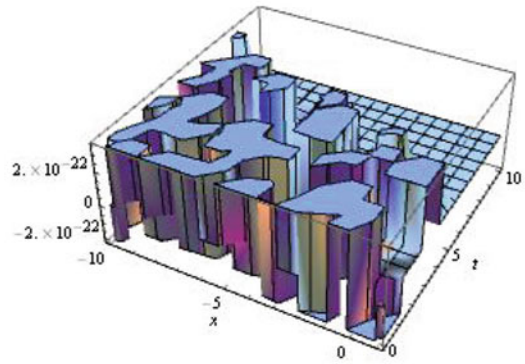
where

$$\eta_1 = \frac{\sqrt{\frac{\varsigma_3}{2} + \varsigma_4\omega}}{\sqrt{\varsigma_2}(\varsigma_3 + 2\varsigma_4)\omega}, \quad (31)$$

$$\eta_2 = \frac{\sqrt{6}\omega}{\sqrt{\varsigma_1}\sqrt{\varsigma_3 + 2\varsigma_4}} \quad (32)$$

Case II

$$\begin{aligned}\psi_2(\xi) &= e^{-t} \left(-\frac{\sqrt{6}\omega \tanh\left(\frac{\xi\sqrt{\varsigma_2}(\varsigma_3 + 2\varsigma_4)\omega}{\sqrt{2}\varsigma_2\sqrt{\varsigma_3 + 2\varsigma_4}}\right)}{\sqrt{\varsigma_1}\sqrt{\varsigma_3 + 2\varsigma_4}} - 1 \right) \\ (33)\end{aligned}$$

(a) $u_1(x, t)$: $c_1=1, c_2=1, c_3=-1, \omega=1$ (b) $u_1(x, t)$: $c_1=1, c_2=1, c_3=1, \omega=-3$ (c) $u_2(x, t)$: $c_1=1, c_2=1, c_3=-1, \omega=1$ (d) $u_2(x, t)$: $c_1=1, c_2=1, c_3=1, \omega=-3$ **Soliton Solutions for Some Nonlinear Water Wave Dynamical Models, Fig. 1** m-BBM equation

The parameters ζ_i become

$$\begin{aligned}\zeta_0 &= \frac{e^{-t}(\zeta_1(\zeta_3 + 2\zeta_4) - 6\omega^2)}{2\sqrt{3}\sqrt{\zeta_1}\sqrt{\zeta_2(\zeta_3 + 2\zeta_4)\omega}}, \\ \zeta_1 &= \frac{\sqrt{\zeta_1}(\zeta_3 + 2\zeta_4)}{\sqrt{3}\zeta_2\zeta_3\omega + 6\zeta_2\zeta_4\omega}, \\ \zeta_2 &= \frac{\sqrt{\zeta_1}e^t\sqrt{\zeta_2(\zeta_3 + 2\zeta_4)\omega}}{2\sqrt{3}\zeta_2\omega}, \\ \zeta_3 &= 0, \\ \zeta_4 &= 0\end{aligned}$$

so the solution of (2) will be (Fig. 2)

$$\begin{aligned}v_2(x, t) &= \eta_1\eta_2\text{sech}^2(\eta_1\xi) \\ &\times (4\eta_1^2\zeta_2\omega(\cosh(2\eta_1\xi) - 2)\text{sech}^2(\eta_1\xi) \\ &- \eta_2^2\zeta_1(\zeta_3 + 2\zeta_4)\tanh^2(\eta_1\xi) + 2\omega^2) \\ &\quad (34)\end{aligned}$$

where

$$\eta_1 = \frac{\sqrt{\frac{\zeta_3}{2} + \zeta_4\omega}}{\sqrt{\zeta_2(\zeta_3 + 2\zeta_4)\omega}}, \quad (35)$$

$$\eta_2 = -\frac{\sqrt{6}\omega}{\sqrt{\zeta_1}\sqrt{\zeta_3 + 2\zeta_4}} \quad (36)$$

(3 + 1) - D e-JM Equation

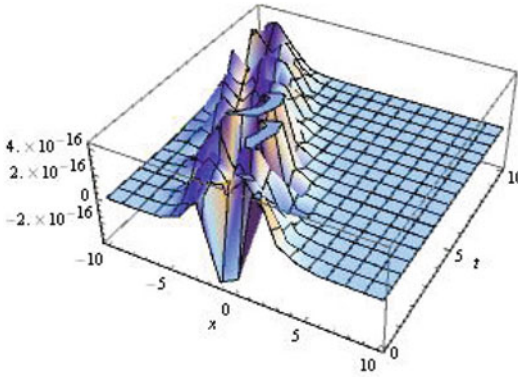
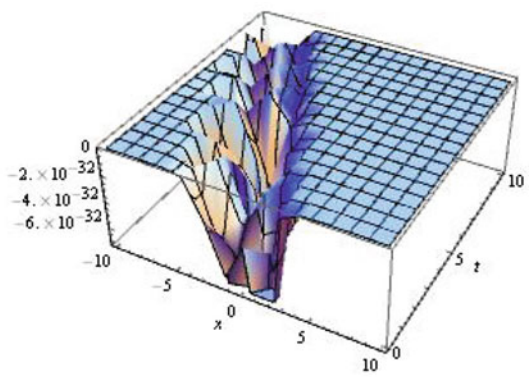
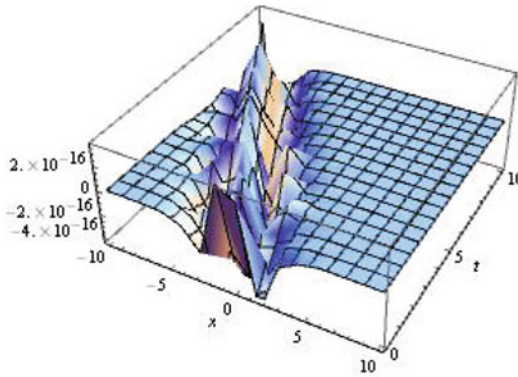
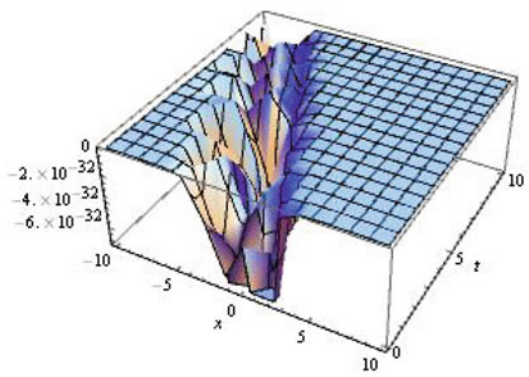
Type I. Consider the transformation

$$u(x, y, z, t) = u(\xi), \quad \xi = x + y + z - \omega t, \quad (37)$$

using (37) into (3),

$$(3\zeta_4 - \omega\zeta_3)u'' + (\zeta_1 + \zeta_2)u'u'' + u'''' = 0 \quad (38)$$

integrating

(a) $v_1(x, t)$: $\varsigma_1=1, \varsigma_2=1, \varsigma_3=1, \omega=1$ (b) $u_1(x, t)$: $\varsigma_1=1, \varsigma_2=1, \varsigma_3=1, \omega=1$ (c) $v_2(x, t)$: $\varsigma_1=1, \varsigma_2=1, \varsigma_3=1, \omega=1$ (d) $u_2(x, t)$: $\varsigma_1=1, \varsigma_2=1, \varsigma_3=1, \omega=1$

Soliton Solutions for Some Nonlinear Water Wave Dynamical Models, Fig. 2 DSW equation

$$(3\varsigma_4 - \omega\varsigma_3)u' + \frac{1}{2}(\varsigma_1 + \varsigma_2)(u')^2 + u''' = 0 \quad (39)$$

Considering the homogeneous balance between $(u')^2$ and u''' gives $n = 2$. Suppose the solution of (39) is of the form

$$u = 1 + e^t \psi(\xi) + e^{2t} \psi^2(\xi) \quad (40)$$

Substituting from equations (8), (9), and (40) into (39) and collecting the coefficients of $\psi^j \psi^{(k)}$,

Case I

$$\psi_1(\xi) = -\frac{e^{-t}}{2} - \frac{\sqrt{12}\eta_2 e^{-\xi\eta_1}}{\sqrt{\eta_2 e^{t-2\xi\eta_1} (e^{\xi\eta_1} - (\varsigma_1 + \varsigma_2)^3)}} \quad (41)$$

where

$$\eta_1 = \frac{\sqrt{(\varsigma_1 + \varsigma_2)^4 (\omega\varsigma_3 - 3\varsigma_4)}}{(\varsigma_1 + \varsigma_2)^2}, \quad (42)$$

$$\eta_2 = \sqrt{(\varsigma_1 + \varsigma_2)^4 (\omega\varsigma_3 - 3\varsigma_4)}, \quad (43)$$

The parameters C_i become

$$C_0 = \frac{e^{-t} \left(-(\varsigma_1 + \varsigma_2)^3 - 48 \sqrt{(\varsigma_1 + \varsigma_2)^4 (\omega\varsigma_3 - 3\varsigma_4)} \right)}{192(\varsigma_1 + \varsigma_2)^2}$$

$$C_1 = -\frac{(\varsigma_1 + \varsigma_2)^3 + 16 \sqrt{(\varsigma_1 + \varsigma_2)^4 (\omega\varsigma_3 - 3\varsigma_4)}}{32(\varsigma_1 + \varsigma_2)^2},$$

$$C_2 = -\frac{e^t (\varsigma_1 + \varsigma_2)}{16},$$

$$C_3 = -\frac{e^{2t} (\varsigma_1 + \varsigma_2)}{24},$$

$$C_4 = 0$$

so the solution of (3) will be

$$u_1(x, y, z, t) = 12\eta_1^2 e^{\xi\eta_1} \left(\frac{\eta_1^2 \eta_2 \left(e^{3\xi\eta_1} + 11e^{2\xi\eta_1}(\varsigma_1 + \varsigma_2)^3 + 11e^{\xi\eta_1}(\varsigma_1 + \varsigma_2)^6 + (\varsigma_1 + \varsigma_2)^9 \right)}{\left(e^{\xi\eta_1} - (\varsigma_1 + \varsigma_2)^3 \right)^5} \right. \\ \left. + \frac{12\eta_1 e^{\xi\eta_1}(\varsigma_1 + \varsigma_2)^5 \left(e^{\xi\eta_1} + (\varsigma_1 + \varsigma_2)^3 \right) (\omega\varsigma_3 - 3\varsigma_4)}{\left(-e^{\xi\eta_1} + (\varsigma_1 + \varsigma_2)^3 \right)^5} - \frac{\eta_2 \left(e^{\xi\eta_1} + (\varsigma_1 + \varsigma_2)^3 \right) (-\omega\varsigma_3 + 3\varsigma_4)}{\left(-e^{\xi\eta_1} + (\varsigma_1 + \varsigma_2)^3 \right)^3} \right) \quad (44)$$

Case II

$$\psi_2(\xi) = \frac{e^{-t} - e^{-t+\xi\eta_1} \left((\varsigma_1 + \varsigma_2)^3 + \sqrt{48\eta_2 \left(-e^{-\xi\eta_1} + (\varsigma_1 + \varsigma_2)^3 \right)} \right)}{2 \left(-1 + e^{\xi\eta_1} (\varsigma_1 + \varsigma_2)^3 \right)} \quad (45)$$

where

$$\eta_1 = \frac{\sqrt{(\varsigma_1 + \varsigma_2)^4 (\omega\varsigma_3 - 3\varsigma_4)}}{(\varsigma_1 + \varsigma_2)^2}, \quad (46)$$

$$\eta_2 = \sqrt{(\varsigma_1 + \varsigma_2)^4 (\omega\varsigma_3 - 3\varsigma_4)}, \quad (47)$$

The parameters C_i become

$$C_0 = \frac{e^{-t} \left(-(\varsigma_1 + \varsigma_2)^3 + 48\sqrt{(\varsigma_1 + \varsigma_2)^4 (\omega\varsigma_3 - 3\varsigma_4)} \right)}{192(\varsigma_1 + \varsigma_2)^2}$$

$$C_1 = -\frac{(\varsigma_1 + \varsigma_2)^3 - 16\sqrt{(\varsigma_1 + \varsigma_2)^4 (\omega\varsigma_3 - 3\varsigma_4)}}{32(\varsigma_1 + \varsigma_2)^2},$$

$$C_2 = -\frac{e^t(\varsigma_1 + \varsigma_2)}{16},$$

$$C_3 = -\frac{e^{2t}(\varsigma_1 + \varsigma_2)}{24},$$

$$C_4 = 0$$

so the solution of (3) will be (Fig. 3)

$$u_2(x, y, z, t) = \frac{16e^{2\xi\eta_1}(\varsigma_1 + \varsigma_2)^3(\omega\varsigma_3 - 3\varsigma_4) \left(1 + e^{\xi\eta_1}(\varsigma_1 + \varsigma_2)^3 \right)}{-1 + e^{\xi\eta_1}(\varsigma_1 + \varsigma_2)^3} \phi_2 \quad (48)$$

where

$$\phi_2 = \frac{3(\varsigma_1 + \varsigma_2)(\eta_2 + 16(\varsigma_1 + \varsigma_2)(\omega\varsigma_3 - 3\varsigma_4)) - 3\eta_1 \left(16\eta_2 + (\varsigma_1 + \varsigma_2)^3 \right)}{\quad} \quad (49)$$

Type II. Consider the transformation

$$u(x, y, z, t) = u(\xi), \quad \xi = x + y + z - \omega t, \quad (50)$$

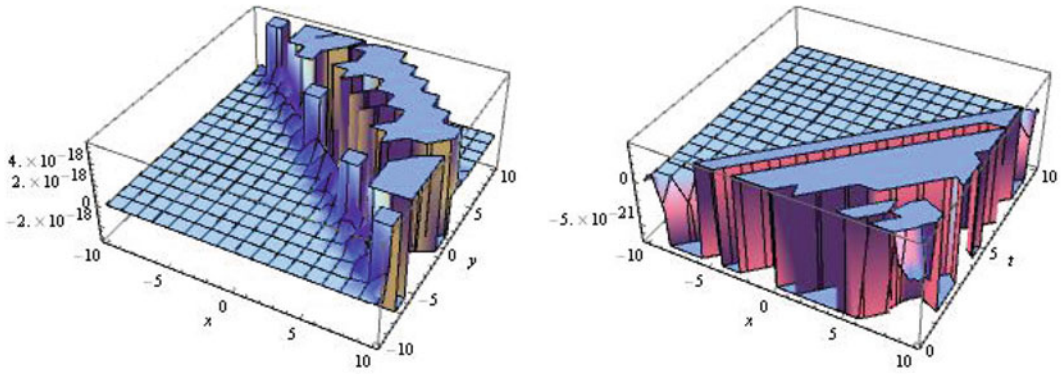
using (50) into (4),

$$(\varsigma_4 - 3\omega\varsigma_3)u'' + (\varsigma_1 + \varsigma_2)u'u'' + u'''' = 0 \quad (51)$$

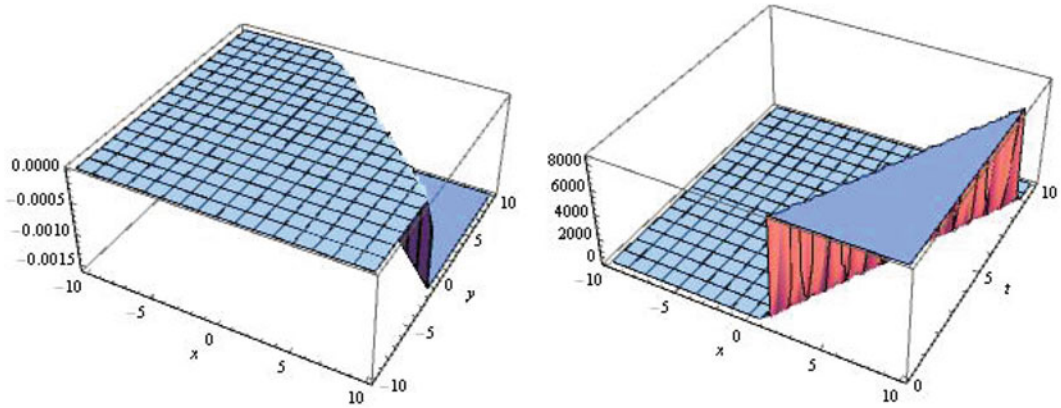
integrating

$$(\varsigma_4 - 3\omega\varsigma_3)u' + \frac{1}{2}(\varsigma_1 + \varsigma_2)(u')^2 + u''' = 0 \quad (52)$$

Considering the homogeneous balance between $(u')^2$ and u''' gives $n = 2$. Suppose the solution of (52) is of the form



(a) $u_1(x, y, z, t) : \varsigma_1=1, \varsigma_2=1, \varsigma_3=1, \varsigma_4=-1, z=1$, (b) $u_1(x, y, z, t) : \varsigma_1=3, \varsigma_2=3, \varsigma_3=2, \varsigma_4=-3, \omega=1, \omega=2, t=3$
 $y=1, z=1$



(c) $u_2(x, y, z, t) : \varsigma_1=1, \varsigma_2=1, \varsigma_3=1, \varsigma_4=-1, z=1$, (d) $u_2(x, y, z, t) : \varsigma_1=3, \varsigma_2=3, \varsigma_3=2, \varsigma_4=-3, \omega=1, \omega=2, t=3$
 $y=1, z=1$

Soliton Solutions for Some Nonlinear Water Wave Dynamical Models, Fig. 3 Type-I (3 + 1) - D e-JM equation

$$u = 1 + e^t \psi(\xi) + e^{2t} \psi^2(\xi) \quad (53) \quad \text{where}$$

Substituting from equations (8), (9), and (53) into (52) and collecting the coefficients of ψ^k ,

$$\eta_1 = \frac{\sqrt{(\varsigma_1 + \varsigma_2)^4 (3\omega\varsigma_3 - \varsigma_4)}}{(\varsigma_1 + \varsigma_2)^2}, \quad (55)$$

Case I

$$\psi_1(\xi) = -\frac{e^{-t}}{2} - \frac{\sqrt{12\eta_2 e^{-2t} (e^{\xi\eta_1} - (\varsigma_1 + \varsigma_2)^3)}}{e^{\xi\eta_1} - (\varsigma_1 + \varsigma_2)^3} \quad (54) \quad \eta_2 = \sqrt{(\varsigma_1 + \varsigma_2)^4 (3\omega\varsigma_3 - \varsigma_4)}, \quad (56)$$

The parameters C_i become

$$C_0 = \frac{e^{-t} \left(-(\zeta_1 + \zeta_2)^3 - 48\sqrt{(\zeta_1 + \zeta_2)^4(3\omega\zeta_3 - \zeta_4)} \right)}{192(\zeta_1 + \zeta_2)^2} \quad \eta_2 = \sqrt{(\zeta_1 + \zeta_2)^4(3\omega\zeta_3 - \zeta_4)}, \quad (61)$$

$$C_1 = -\frac{(\zeta_1 + \zeta_2)^3 + 16\sqrt{(\zeta_1 + \zeta_2)^4(3\omega\zeta_3 - \zeta_4)}}{32(\zeta_1 + \zeta_2)^2},$$

$$C_2 = -\frac{e^t(\zeta_1 + \zeta_2)}{16},$$

$$C_3 = -\frac{e^{2t}(\zeta_1 + \zeta_2)}{24},$$

$$C_4 = 0$$

so the solution of (4) will be

$$u_1(x, y, z, t) = \frac{12e^{\xi\eta_1} \left(e^{\xi\eta_1} - (\zeta_1 + \zeta_2)^3 \right) \left((\zeta_1 + \zeta_2)^4(3\omega\zeta_3 - \zeta_4) \right)^{\frac{3}{2}}}{(\zeta_1 + \zeta_2)^8 \left(e^{\xi\eta_1} - (\zeta_1 + \zeta_2)^3 \right)^5} \phi_1$$

(57) so the solution of (4) will be (Fig. 4)

where

$$\begin{aligned} \phi_1 = & \left(-12e^{\xi\eta_1}(\zeta_1 + \zeta_2)^7(3\omega\zeta_3 - \zeta_4) \right. \\ & + (\zeta_1 + \zeta_2)^4(3\omega\zeta_3 - \zeta_4) \\ & \times \left(e^{2\xi\eta_1} + 10e^{\xi\eta_1}(\zeta_1 + \zeta_2)^3 + (\zeta_1 + \zeta_2)^6 \right) \\ & \left. + (\zeta_1 + \zeta_2)^4(-3\omega\zeta_3 + \zeta_4) \left(-e^{\xi\eta_1}(\zeta_1 + \zeta_2)^3 \right)^2 \right) \end{aligned}$$

(58)

Case II

$$\begin{aligned} \psi_2(\xi) = & \frac{e^{-t}}{2} \\ & + \frac{\sqrt{12\eta_2 e^{-2t+2\xi\eta_1} \left(-e^{\xi\eta_1} + (\zeta_1 + \zeta_2)^3 \right)}}{-1 + e^{\xi\eta_1}(\zeta_1 + \zeta_2)^3} \end{aligned} \quad (59)$$

where

$$\eta_1 = \frac{\sqrt{(\zeta_1 + \zeta_2)^4(3\omega\zeta_3 - \zeta_4)}}{(\zeta_1 + \zeta_2)^2}, \quad (60)$$

The parameters C_i become

$$C_0 = \frac{e^{-t} \left(-(\zeta_1 + \zeta_2)^3 + 48\sqrt{(\zeta_1 + \zeta_2)^4(3\omega\zeta_3 - \zeta_4)} \right)}{192(\zeta_1 + \zeta_2)^2}$$

$$C_1 = -\frac{(\zeta_1 + \zeta_2)^3 - 16\sqrt{(\zeta_1 + \zeta_2)^4(3\omega\zeta_3 - \zeta_4)}}{32(\zeta_1 + \zeta_2)^2},$$

$$C_2 = -\frac{e^t(\zeta_1 + \zeta_2)}{16},$$

$$C_3 = -\frac{e^{2t}(\zeta_1 + \zeta_2)}{24},$$

$$C_4 = 0$$

$u_2(x, y, z, t)$

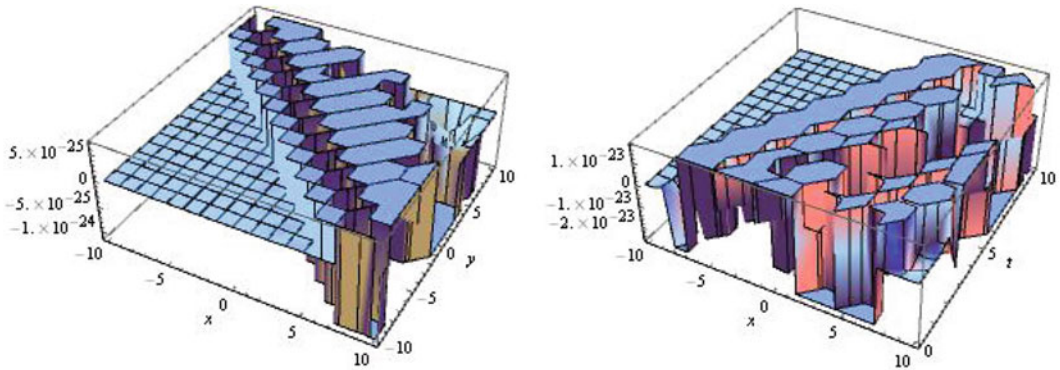
$$= \frac{12e^{\xi\eta_1} \left(1 + e^{\xi\eta_1}(\zeta_1 + \zeta_2)^3 \right) \left((\zeta_1 + \zeta_2)^4(3\omega\zeta_3 - \zeta_4) \right)^{\frac{3}{2}}}{(\zeta_1 + \zeta_2)^8 \left(-1 + e^{\xi\eta_1}(\zeta_1 + \zeta_2)^3 \right)^5} \phi_2 \quad (62)$$

where

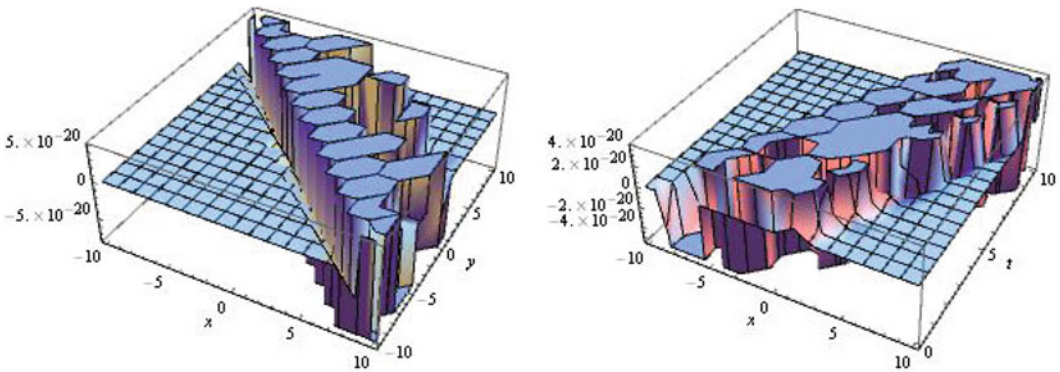
$$\begin{aligned} \phi_2 = & \left(-12e^{\xi\eta_1}(\zeta_1 + \zeta_2)^7(3\omega\zeta_3 - \zeta_4) \right. \\ & + (\zeta_1 + \zeta_2)^4(3\omega\zeta_3 - \zeta_4) \\ & \times \left(1 + 10e^{\xi\eta_1}(\zeta_1 + \zeta_2)^3 + e^{2\xi\eta_1}(\zeta_1 + \zeta_2)^6 \right) \\ & \left. + (\zeta_1 + \zeta_2)^4(-3\omega\zeta_3 + \zeta_4) \left(-1 + e^{\xi\eta_1}(\zeta_1 + \zeta_2)^3 \right)^2 \right) \end{aligned} \quad (63)$$

Conclusion

The aim of the study is to find some new traveling wave solutions for modified Benjamin-Bona-Mahony, coupled Drinfel'd-Sokolov-Wilson, and (3 + 1)-dimensional extended Jimbo-Miwa equations. It is observed that the auxiliary equation



(a) $u_1(x, y, z, t) : \varsigma_1=1, \varsigma_2=1, \varsigma_3=1, \varsigma_4=-1, z=1$, (b) $u_1(x, y, z, t) : \varsigma_1=3, \varsigma_2=3, \varsigma_3=2, \varsigma_4=-3, \omega=1, \omega=2, t=3$
 $y=1, z=1$



(c) $u_2(x, y, z, t) : \varsigma_1=1, \varsigma_2=1, \varsigma_3=1, \varsigma_4=-1, z=1$, (d) $u_2(x, y, z, t) : \varsigma_1=3, \varsigma_2=3, \varsigma_3=2, \varsigma_4=-3, \omega=1, \omega=2, t=3$
 $y=1, z=1$

Soliton Solutions for Some Nonlinear Water Wave Dynamical Models, Fig. 4 Type-II (3 + 1) - D e-JM equation

method is one of the most powerful tools to find a variety of analytical solutions for more complex problems. These results are very auspicious for further investigation and stances on a strong basis for the solution of NPDEs.

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Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models

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Article Outline

Introduction
Applications
Discussion of the Results
Conclusion
Bibliography

Introduction

In different areas of applied science, the investigation of exact travelling wave solutions has played a vital role for describing the character wave problems. For plentiful demonstration of dilemmas in mathematical physics, different nonlinear wave systems have been discussed such as the phenomena flow of heat, plasma physics, and optical fibers. Peripheral of solitary wave is growing daily, so essential to find for exact traveling wave solutions to NLEEs. Exact solutions assist us understand the necessary back ground of problems.

The exact solutions of nonlinear partial differential equations have been investigated by many authors [1–36]. The research of traveling wave solutions of some nonlinear evolution equations derived from such fields played an imperative role in the analysis of some phenomena such as the Bernoulli's sub-ODE method (Wang et al. 2007),

Exp($-\Phi(\xi)$)-expansion method (Islam et al. 2015), extended simple equation method (Ali et al. 2017; Lu et al. 2017a, 2018a), modified extended direct algebraic method (Arshad et al. 2017; Seadawy et al. 2017), the extended sinh-cosh and sin-cos methods (Wang et al. 2005), the Jacobi elliptic function method (Ali 2011), the Exp-function method (He and Wu 2006; Akbar and Ali 2012; Naher et al. 2012), the ansatz method (Triki et al. 2012; Sassaman and Biswas 2009; Chowdhury and Biswas 2012; Song et al. 2013), the perturbation method (Biswas et al. 2012), modified extended mapping method (Abdullah et al. 2017), the extended tanh-function method (Zhou et al. 2017; Lu et al. 2017b; Mirzazadeh 2016), modified Kudryashov and hyperbolic function methods (Seadawy et al. 2018), and many more (Helal and Seadawy 2009, 2011, 2012; Khater et al. 2006; Seadawy 2011, 2015, 2016, 2017).

Modified Liouville and the symmetric regularized long wave equations have been useful for the description of weakly nonlinear ion acoustic and space charge waves in mathematical physics. Similarly the tremendous benefit of fourth-order nonlinear Ablowitz-Kaup-Newell-Segur water wave equation is that it can be concentrated to few eminent nonlinear evolution equations such as the KdV, the mKdV, and the sine-Gordon equations which have enormous applications in applied mathematics. Now recently Lu. D (Lu et al. 2018a) investigated exact and solitary wave solutions on modified Liouville and the symmetric regularized long wave equation by applying the modified simple equation and exp($-\Psi(\xi)$) expansion methods. Furthermore, Ali. A (Ali et al. 2018) lately investigated the soliton solutions of AKNS equation by employing simple equation and modified simple equation methods. Our focus in this current entry is to demonstrate the so-called modified F-expansion method for construction of soliton solutions above illustrated three important models, having valuable advantages for handling the nonlinear wave problems in nonlinear science.

The entry is organized as follows: section “Applications,” applied modified F-expansion

method to our selective three nonlinear wave models; results discussion in section “[Discussion of the Results](#)”; and finally section “[Conclusion](#).”

Applications

Application of Modified Liouville Equation

The generalized form of Modified Liouville equation in Lu et al. (2018a)

$$d_1^2 v_{xx} - v_{tt} + d_2 e^{\beta v} = 0, \quad (1)$$

where d_1 , β , d_2 are arbitrary constants. Let wave transformation;

$$v(x, t) = V(\xi), \quad \xi = kx + \omega t, \quad (2)$$

$$V = e^{\beta v}$$

Replacement of Eq. (2) in Eq. (1) gives

$$\left(\frac{k^2 d_1^2}{\beta} - \frac{\omega^2}{\beta} \right) V'' V - \left(\frac{k^2 d_1^2}{\beta} - \frac{\omega^2}{\beta} \right) V'^2 + d_2 V^3 = 0. \quad (3)$$

Suppose Eq. (3) has solution (Aasaraai 2015)

$$V = a_0 + a_1 F(\xi) + a_2 F^2(\xi) + b_1 F^{-1}(\xi) + b_2 F^{-2}(\xi) \quad (4)$$

Here $F(\xi)$ convinces the following Riccati equation (Helal and Seadawy 2012)

$$F'(\xi) = H_1 + H_2 F(\xi) + H_3 F^2(\xi) \quad (5)$$

Submission of Eq. (4), Eq. (5) in Eq. (3), gives a large equations system with $a_0, a_1, b_1, a_2, b_2, H_1, H_2, H_3, d_1, d_2, k, \omega$. With help of Mathematica, have subsequent solutions categories.

The Soliton-Like Solutions of Eq. (1)

1. When $H_1 = 0$, $H_2 = 1$, $H_3 = -1$, then we have,

$$a_0 = 0, \quad a_2 = -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, \quad (6)$$

$$a_1 = \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, \quad b_1 = 0, \quad b_2 = 0.$$

Substitute Eq. (6) in Eq. (4),

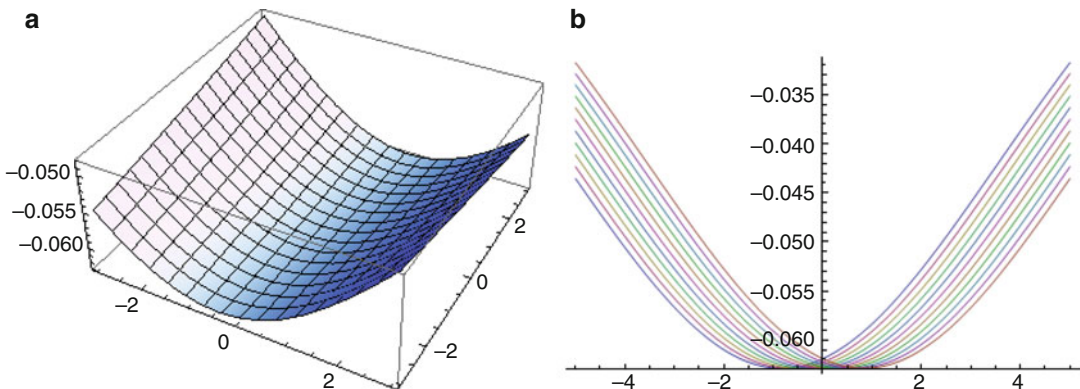
$$V_1(\xi) = \frac{(d_1^2 k^2 - \omega^2)}{d_2 \beta} \left(1 + \tanh\left(\frac{1}{2}\xi\right) \right) \left(1 - \frac{1}{2} \left(1 + \tanh\left(\frac{1}{2}\xi\right) \right) \right) \quad (7)$$

2. When $H_1 = 0$, $H_2 = -1$, $H_3 = 1$, we have,

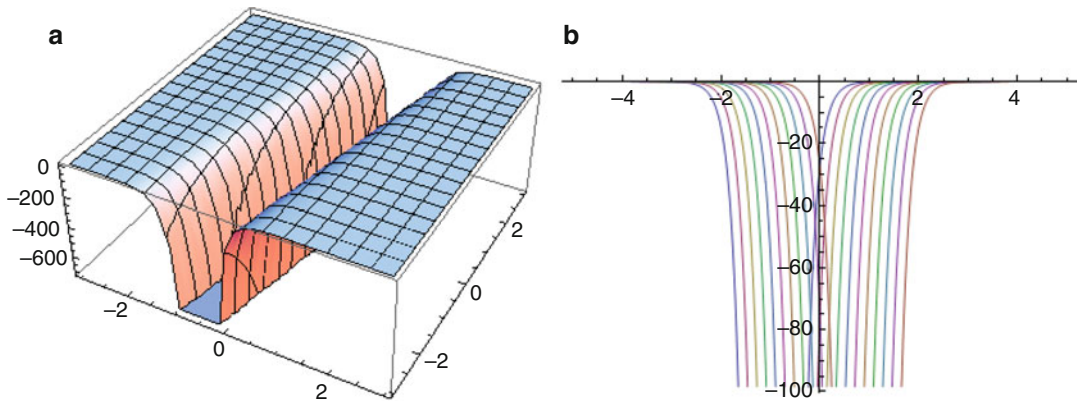
$$a_0 = 0, \quad a_2 = -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, \quad (8)$$

$$a_1 = \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, \quad b_1 = 0, \quad b_2 = 0.$$

Replace (8) in (4) (Figs. 1 and 2),



Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models, Fig. 1 The 3D and 2D for solution (7) with; $d_1 = 2.9$, $d_2 = 4$, $k = 0.3$, $\beta = -1.5$, $\omega = -0.05$



Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models, Fig. 2 Dark Solitary waves (9) are plotted with; $d_1 = -1.9$, $d_2 = -0.05$, $k = 0.3$, $\beta = -1.5$, $\omega = 1$

$$V_2(\xi) = \frac{(d_1^2 k^2 - \omega^2)}{d_2 \beta} \left(1 - \coth\left(\frac{1}{2}\xi\right) \right) \left(1 - \frac{1}{2} \left(1 - \coth\left(\frac{1}{2}\xi\right) \right) \right) \quad (9)$$

3. When $H_1 = \frac{1}{2}$, $H_2 = 0$, $H_3 = -\frac{1}{2}$, we have,

Family-I

$$a_0 = \frac{d_1^2 k^2 - \omega^2}{2d_2 \beta}, \quad a_1 = 0, \quad (10)$$

$$a_2 = 0, \quad b_1 = 0, \quad b_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2 \beta}$$

Put Eq. (10) in Eq. (4),

$$V_{31}(\xi) = \frac{d_1^2 k^2 - \omega^2}{2d_2 \beta} + \frac{(\omega^2 - d_1^2 k^2)}{2d_2 \beta (\coth(\xi) \pm \operatorname{csch}(\xi))^2} \quad (11)$$

Family-II

$$a_0 = \frac{d_1^2 k^2 - \omega^2}{2d_2 \beta}, \quad a_1 = 0, \quad (12)$$

$$a_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2 \beta}, \quad b_1 = 0, \quad b_2 = 0$$

Substitute Eq. (12) in (4),

$$V_{32}(\xi) = \frac{d_1^2 k^2 - \omega^2}{2d_2 \beta} + \frac{(\omega^2 - d_1^2 k^2) (\coth(\xi) \pm \operatorname{csch}(\xi))^2}{2d_2 \beta} \quad (13)$$

Family-III

$$a_0 = \frac{d_1^2 k^2 - \omega^2}{d_2 \beta}, \quad a_1 = 0, \quad (14)$$

$$a_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2 \beta}, \quad b_1 = 0,$$

$$b_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2 \beta}$$

Substituting Eq. (14) into Eq. (4), we obtained,

$$V_{33}(\xi) = \frac{d_1^2 k^2 - \omega^2}{d_2 \beta} + \frac{(\omega^2 - d_1^2 k^2)}{2d_2 \beta} \left((\coth(\xi) \pm \operatorname{csch}(\xi))^2 + \frac{1}{(\coth(\xi) \pm \operatorname{csch}(\xi))^2} \right) \quad (15)$$

4. When $H_1 = 1$, $H_2 = 0$, $H_3 = -1$,

Family-I

$$a_0 = \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, \quad a_1 = a_2 = b_1 = 0, \quad (16)$$

$$b_2 = -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}$$

Substituting (16) in (4),

$$V_{41}(\xi) = \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta} - \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta (\tanh(\xi))^2} \quad (17)$$

Family-II

$$\begin{aligned} a_0 &= \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, \\ a_2 &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, \\ b_1 &= b_2 = a_1 = 0, \end{aligned} \quad (18)$$

Put (18) in (4),

$$\begin{aligned} V_{42}(\xi) &= \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta} \\ &\quad - \frac{2(d_1^2 k^2 - \omega^2)(\tanh(\xi))^2}{d_2 \beta} \end{aligned} \quad (19)$$

Family-III

$$\begin{aligned} a_0 &= \frac{4(d_1^2 k^2 - \omega^2)}{d_2 \beta}, \quad a_1 = 0, \\ a_2 &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, \quad b_1 = 0, \\ b_2 &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta} \end{aligned} \quad (20)$$

Putting Eq. (20) in Eq. (4),

$$\begin{aligned} V_{43}(\xi) &= \frac{4(d_1^2 k^2 - \omega^2)}{d_2 \beta} - \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta} \\ &\quad \left((\tanh(\xi))^2 + \frac{1}{(\tanh(\xi))^2} \right) \end{aligned} \quad (21)$$

The Trigonometric Function Solutions of Eq. (1)

1. When $H_1 = \frac{1}{2}$, $H_2 = 0$, $H_3 = \frac{1}{2}$,

Family-I

$$\begin{aligned} a_0 &= \frac{\omega^2 - d_1^2 k^2}{2d_2 \beta}, \quad a_1 = 0, \\ a_2 &= \frac{\omega^2 - d_1^2 k^2}{2d_2 \beta}, \quad b_1 = 0, \quad b_2 = 0 \end{aligned} \quad (22)$$

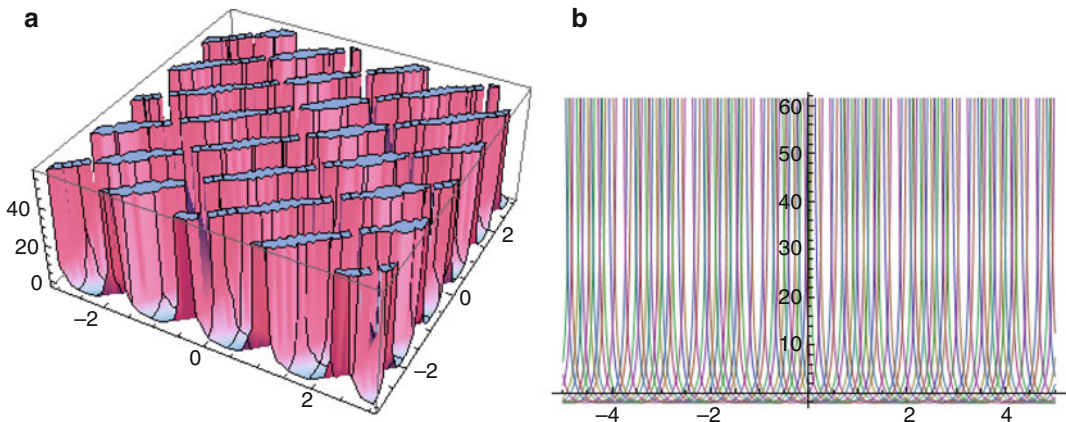
Substitute Eq. (22) in Eq. (4) (Fig. 3),

$$\begin{aligned} V_{51}(\xi) &= \frac{\omega^2 - d_1^2 k^2}{2d_2 \beta} \\ &\quad + \frac{(\omega^2 - d_1^2 k^2)}{2d_2 \beta} (\sec(\xi) + \tan(\xi))^2 \end{aligned} \quad (23)$$

Family-II

$$\begin{aligned} a_0 &= \frac{\omega^2 - d_1^2 k^2}{2d_2 \beta}, \quad a_1 = 0, \quad a_2 = 0, \\ b_1 &= 0, \quad b_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2 \beta} \end{aligned} \quad (24)$$

Eq. (24) into Eq. (4) gives,



Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models, Fig. 3 Periodic Solitary waves (23) are drawn with: $d_1 = 0.7$, $d_2 = 1$, $k = 4.5$, $\beta = -4.5$, $\omega = -5.3$

$$\begin{aligned}
 V_{52}(\xi) &= \frac{\omega^2 - d_1^2 k^2}{2d_2\beta} + \frac{(\omega^2 - d_1^2 k^2)}{2d_2\beta} \left(\frac{1}{\sec(\xi) + \tan(\xi)} \right)^2 \\
 a_0 &= \frac{\omega^2 - d_1^2 k^2}{d_2\beta}, \quad a_1 = 0, \quad a_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2\beta}, \\
 b_1 &= 0, \quad b_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2\beta}
 \end{aligned} \tag{26}$$

Family-III

Substituting (26) in (4),

$$\begin{aligned}
 V_{53}(\xi) &= \frac{\omega^2 - d_1^2 k^2}{d_2\beta} + \frac{(\omega^2 - d_1^2 k^2)}{2d_2\beta} \\
 &\quad \left((\sec(\xi))^2 + (\tan(\xi))^2 + 2(\sec(\xi))(\tan(\xi)) + \frac{1}{(\sec(\xi) + \tan(\xi))^2} \right)
 \end{aligned} \tag{27}$$

2. When $H_1 = -\frac{1}{2}$, $H_2 = 0$, $H_3 = -\frac{1}{2}$,

$$a_0 = \frac{\omega^2 - d_1^2 k^2}{2d_2\beta}, \quad a_1 = 0, \quad a_2 = 0,$$

$$b_1 = 0, \quad b_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2\beta} \tag{30}$$

Family-I

$$a_0 = \frac{\omega^2 - d_1^2 k^2}{2d_2\beta}, \quad a_1 = 0, \tag{28}$$

$$a_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2\beta}, \quad b_1 = 0, \quad b_2 = 0$$

Substituting (30) in (4),

$$V_{62}(\xi) = \frac{\omega^2 - d_1^2 k^2}{2d_2\beta} - \frac{(\omega^2 - d_1^2 k^2)}{2d_2\beta(\sec(\xi) - \tan(\xi))^2} \tag{31}$$

Putting (28) in (4),

$$\begin{aligned}
 V_{61}(\xi) &= \frac{\omega^2 - d_1^2 k^2}{2d_2\beta} \\
 &\quad + \frac{(\omega^2 - d_1^2 k^2)}{2d_2\beta} (\sec(\xi) - \tan(\xi))^2
 \end{aligned} \tag{29}$$

Family-III

$$a_0 = \frac{\omega^2 - d_1^2 k^2}{d_2\beta}, \quad a_1 = 0,$$

$$a_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2\beta}, \quad b_1 = 0, \quad b_2 = \frac{\omega^2 - d_1^2 k^2}{2d_2\beta} \tag{32}$$

Family-II

Replace of Eq. (33) in Eq. (4),

$$\begin{aligned}
 V_{63}(\xi) &= \frac{\omega^2 - d_1^2 k^2}{d_2\beta} - \frac{(\omega^2 - d_1^2 k^2)}{2d_2\beta} \\
 &\quad \left((\sec(\xi))^2 + (\tan(\xi))^2 - 2(\sec(\xi))(\tan(\xi)) - \frac{1}{(\sec(\xi) - \tan(\xi))^2} \right)
 \end{aligned} \tag{33}$$

3. When $H_1 = 1(-1)$, $H_2 = 0$, $H_3 = 1(-1)$,

Family-I

$$\begin{aligned} a_0 &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, & a_1 &= b_1 = 0, \\ a_2 &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, & b_2 &= 0 \end{aligned} \quad (34)$$

Put Eq. (34) in Eq. (4),

$$\begin{aligned} V_{71}(\xi) &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta} \\ &\quad - \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta} (\tan(\xi)(\cot(\xi)))^2 \end{aligned} \quad (35)$$

Family-II

$$\begin{aligned} a_0 &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, & a_1 &= 0, & a_2 &= 0, \\ b_1 &= 0, & b_2 &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta} \end{aligned} \quad (36)$$

Substitute Eq. (36) in Eq. (4),

$$\begin{aligned} V_{72}(\xi) &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta} \\ &\quad - \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta (\tan(\xi)(\cot(\xi)))^2} \end{aligned} \quad (37)$$

Family-III

$$\begin{aligned} a_0 &= -\frac{4(d_1^2 k^2 - \omega^2)}{d_2 \beta}, & a_1 &= 0, \\ a_2 &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta}, & b_1 &= 0, \\ b_2 &= -\frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta} \end{aligned} \quad (38)$$

Put Eq. (38) in Eq. (4),

$$\begin{aligned} V_{73}(\xi) &= -\frac{4(d_1^2 k^2 - \omega^2)}{d_2 \beta} - \frac{2(d_1^2 k^2 - \omega^2)}{d_2 \beta} \\ &\quad \left((\tan(\xi)(\cot(\xi)))^2 + \frac{1}{(\tan(\xi)(\cot(\xi)))^2} \right) \end{aligned} \quad (39)$$

The Rational Function Solutions of Eq. (1)

1. When $H_1 = 0$, $H_2 = 0$, $H_3 \neq 0$,

$$\begin{aligned} a_0 &= a_1 = 0, \\ a_2 &= -\frac{2(d_1^2 H_3^2 k^2 - H_3^2 \omega^2)}{d_2 \beta}, \\ b_1 &= b_2 = 0, \end{aligned} \quad (40)$$

Substitute Eq. (40) in Eq. (4),

$$V_8(\xi) = -\frac{2(d_1^2 H_3^2 k^2 - H_3^2 \omega^2)}{d_2 \beta} \left(\frac{1}{H_3 \xi + \varepsilon} \right)^2 \quad (41)$$

2. When $H_2 = 0$, $H_3 = 0$,

$$\begin{aligned} a_0 &= 0, & a_1 &= 0, & a_2 &= 0, & b_1 &= 0, \\ b_2 &= -\frac{2(d_1^2 H_1^2 k^2 - H_1^2 \omega^2)}{d_2 \beta} \end{aligned} \quad (42)$$

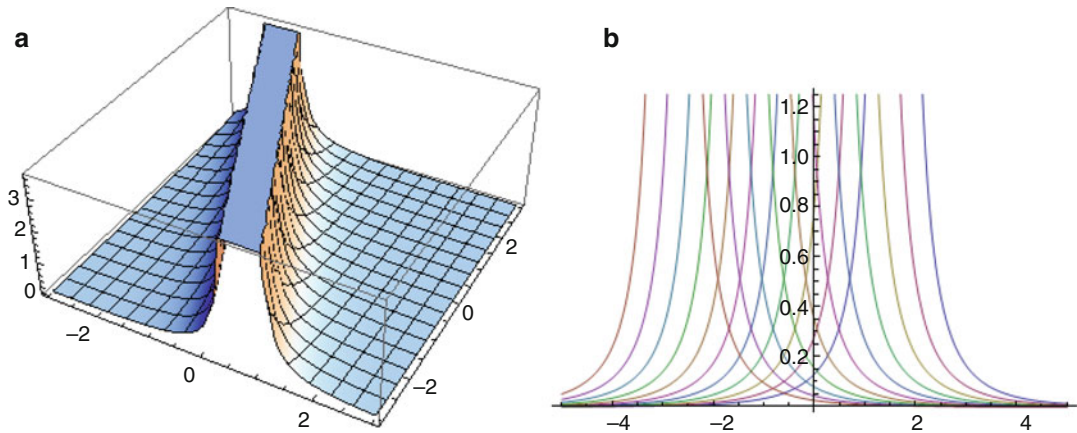
Replace Eq. (42) in Eq. (4),

$$V_9(\xi) = -\frac{2(d_1^2 H_1^2 k^2 - H_1^2 \omega^2)}{d_2 \beta \xi^2} \quad (43)$$

3. When $H_1 \neq 0$, $H_2 \neq 0$, $H_3 = 0$,

$$\begin{aligned} a_0 &= 0, & a_1 &= 0, & a_2 &= 0, \\ b_1 &= -\frac{2(d_1^2 H_1 H_2 k^2 - H_1 H_2 \omega^2)}{d_2 \beta}, \\ b_2 &= -\frac{2(d_1^2 H_1^2 k^2 - H_1^2 \omega^2)}{d_2 \beta} \end{aligned} \quad (44)$$

Putting Eq. (44) in (4) (Fig. 4),



Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models, Fig. 4 The 3D and 2D solitary waves (45) are drawn with: $H_1 = 0.5$, $H_2 = 0.5$, $d_1 = 0$, $d_2 = -0.3$, $k = 2.3$, $\beta = 3$, $\omega = 1$

$$V_{10}(\xi) = -\frac{2(d_1^2 H_1 H_2 k^2 - H_1 H_2 \omega^2)}{d_2 \beta} \left(\frac{H_2}{\exp(H_2 \xi) - H_1} \right) - \frac{2(d_1^2 H_1^2 k^2 - H_1^2 \omega^2)}{d_2 \beta} \left(\frac{H_2}{\exp(H_2 \xi) - H_1} \right)^2 \quad (45)$$

Application of the Symmetric Regularized Long Wave Equation

Consider the SRLW equation (Lu et al. 2018a),

$$v_{tt} - v_{xx} + \left(\frac{v^2}{2} \right)_{xt} - v_{xxt} = 0 \quad (46)$$

Let the transformation,

$$v(x, t) = V(\xi), \quad \xi = x + kt \quad (47)$$

Substitute of Eq. (47) in (46),

$$(k^2 - 1)V'' - k \left(\frac{V^2}{2} \right)'' - k^2 V''' = 0 \quad (48)$$

Taking two times integration of Eq. (48) with disregard constant of integral.

$$(k^2 - 1)V - k \left(\frac{V^2}{2} \right) - k^2 V'' = 0 \quad (49)$$

Let formal solution of Eq. (49) is like Eq. (4). Revolution of (4), (5) in Eq. (49), gives varieties of solutions

The Soliton-Like Solutions of Eq. (46)

1. For $H_1 = 0$, $H_2 = 1$, $H_3 = -1$, then we have,

$$a_0 = \pm\sqrt{2}, \quad a_2 = a_1 = \pm 6\sqrt{2}, \quad b_1 = 0, \quad k = \pm \frac{1}{\sqrt{2}}, \quad b_2 = 0 \quad (50)$$

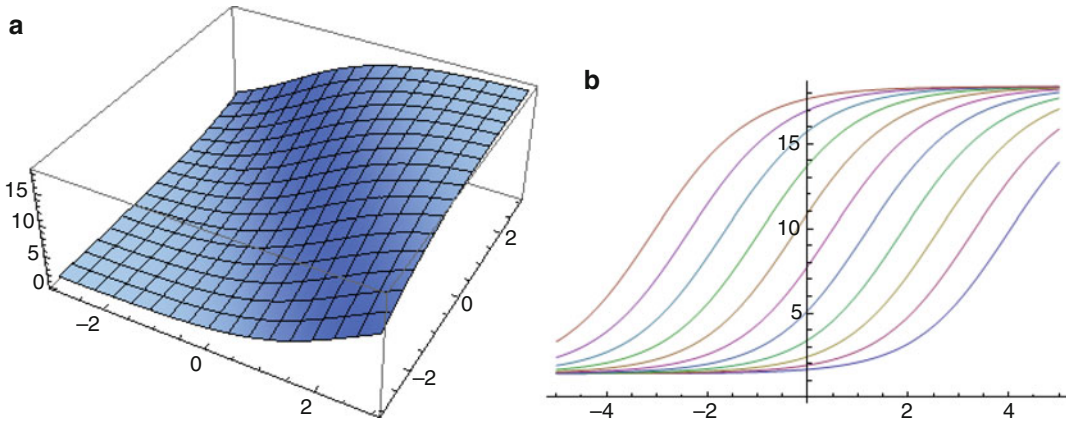
Substitute Eq. (50) in (4),

$$V_{11}(\xi) = \pm\sqrt{2} \pm 3\sqrt{2} \left(1 + \tanh\left(\frac{1}{2}\xi\right) \right) \pm \frac{3\sqrt{2}}{2} \left(1 + \tanh\left(\frac{1}{2}\xi\right) \right)^2 \quad (51)$$

2. For $H_1 = 0$, $H_2 = -1$, $H_3 = 1$, then we have (Fig. 5),

$$a_0 = \pm\sqrt{2}, \quad a_2 = \pm 6\sqrt{2}, \quad a_1 = \pm 6\sqrt{2}, \quad b_1 = 0, \quad k = \pm \frac{1}{\sqrt{2}}, \quad b_2 = 0 \quad (52)$$

Replace (52) in (4),



Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models, Fig. 5 The solitary wave of solution (51) is plotted at (a) and (b), respectively

$$V_{12} = \pm\sqrt{2} \pm 3\sqrt{2} \left(1 - \coth\left(\frac{1}{2}\xi\right)\right) \pm \frac{6\sqrt{2}}{4} \left(1 - \coth\left(\frac{1}{2}\xi\right)\right)^2 \quad (53)$$

3. For $H_1 = \frac{1}{2}$, $H_2 = 0$, $H_3 = -\frac{1}{2}$,

Family-I

$$a_0 = \pm \frac{1}{\sqrt{2}}, \quad k = \pm \frac{1}{\sqrt{2}}, \quad a_1 = a_2 = b_1 = 0, \quad b_2 = \pm \frac{3}{\sqrt{2}} \quad (54)$$

substitute Eq. (54) in Eq. (4),

$$V_{131} = \pm \frac{1}{\sqrt{2}} \pm \frac{3}{\sqrt{2}} \left(\frac{1}{(\coth(\xi))^2 + (\csch(\xi))^2 + 2(\coth(\xi)) \pm (\csch(\xi))} \right) \quad (55)$$

Family-II

$$a_0 = \pm \frac{1}{\sqrt{2}}, \quad k = \pm \frac{1}{\sqrt{2}}, \quad a_1 = b_2 = 0, \quad b_1 = 0, \quad a_2 = \pm \frac{3}{\sqrt{2}} \quad (56)$$

Putting Eq. (56) in Eq. (4),

$$V_{132}(\xi) = \pm \frac{1}{\sqrt{2}} \pm \frac{3}{\sqrt{2}} (\coth(\xi) \pm \csch(\xi))^2 \quad (57)$$

Family-III

$$a_0 = \pm \frac{2}{\sqrt{5}}, \quad k = \pm \frac{1}{\sqrt{5}}, \quad a_1 = 0, \quad a_2 = \pm \frac{3}{\sqrt{5}}, \quad b_1 = 0, \quad b_2 = \pm \frac{3}{\sqrt{5}} \quad (58)$$

Putting Eq. (58) into Eq. (4),

$$V_{133} = \pm \frac{2}{\sqrt{5}} \pm \frac{3}{\sqrt{5}} \left((\coth(\xi) \pm \csch(\xi))^2 + \frac{1}{(\coth(\xi) \pm \csch(\xi))^2} \right) \quad (59)$$

4. For $H_1 = 1$, $H_2 = 0$, $H_3 = -1$,

Family-I

$$a_0 = \pm \frac{4}{\sqrt{5}}, \quad k = \pm \frac{1}{\sqrt{5}}, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = \pm \frac{12}{\sqrt{5}} \quad (60)$$

Replacing Eq. (60) into Eq. (4),

$$V_{141} = \pm \frac{4}{\sqrt{5}} \pm \frac{12}{\sqrt{5}} \left(\frac{1}{\tanh(\xi)} \right)^2. \quad (61)$$

Family-II

$$\begin{aligned} a_0 &= \pm \frac{4}{\sqrt{5}}, & k &= \pm \frac{1}{\sqrt{5}}, & a_1 &= 0, \\ b_2 &= 0, & b_1 &= 0, & a_2 &= \pm \frac{12}{\sqrt{5}} \end{aligned} \quad (62)$$

After replace Eq. (62) in Eq. (4),

$$V_{142}(\xi) = \pm \frac{4}{\sqrt{5}} \pm \frac{12}{\sqrt{5}} (\tanh(\xi))^2, \quad (63)$$

Family-III

$$\begin{aligned} a_0 &= \pm \frac{8}{\sqrt{17}}, & k &= \pm \frac{1}{\sqrt{17}}, & a_1 &= 0, \\ a_2 &= \pm \frac{12}{\sqrt{17}}, & b_1 &= 0, & b_2 &= \pm \frac{12}{\sqrt{17}} \end{aligned} \quad (64)$$

Substitute Eq. (64) in Eq. (4),

$$V_{143}(\xi) = \pm \frac{8}{\sqrt{17}} \pm \frac{12}{\sqrt{17}} \left((\tanh(\xi))^2 + \frac{1}{(\tanh(\xi))^2} \right). \quad (65)$$

The Trigonometric Function Solutions of Eq. (46)

1. For $H_1 = \frac{1}{2}$, $H_2 = 0$, $H_3 = \frac{1}{2}$,

Family-I

$$\begin{aligned} a_0 &= \pm \frac{3}{\sqrt{2}}, & k &= \pm \frac{1}{\sqrt{2}}, & a_1 &= 0, \\ b_2 &= 0, & b_1 &= 0, & a_2 &= \pm \frac{3}{\sqrt{2}} \end{aligned} \quad (66)$$

Put Eq. (66) into Eq. (4) (Fig. 6),

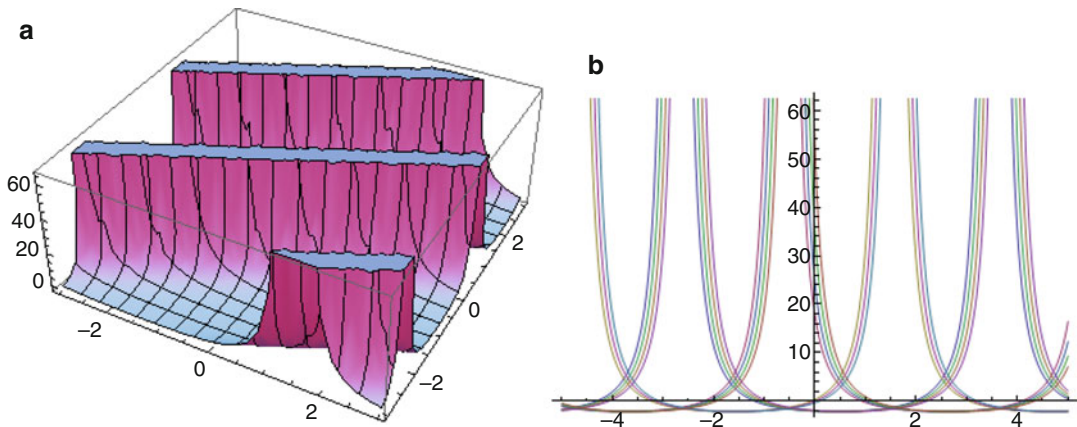
$$V_{151}(\xi) = \pm \frac{3}{\sqrt{2}} \pm \frac{3}{\sqrt{2}} (\sec(\xi) + \tan(\xi))^2. \quad (67)$$

Family-II

$$\begin{aligned} a_0 &= \pm \frac{3}{\sqrt{2}}, & k &= \pm \frac{1}{\sqrt{2}}, & a_1 &= 0, & a_2 &= 0, \\ b_1 &= 0, & b_2 &= \pm \frac{3}{\sqrt{2}} \end{aligned} \quad (68)$$

Alternate Eq. (68) into Eq. (4)

$$V_{152}(\xi) = \pm \frac{3}{\sqrt{2}} \pm \frac{3}{\sqrt{2}} \left(\frac{1}{(\sec(\xi))^2 + (\tan(\xi))^2 + 2(\sec(\xi))(\tan(\xi))} \right). \quad (69)$$



Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models, Fig. 6 Kink solitary waves (67) at (a) and (b)

Family-III

$$\begin{aligned} a_0 &= \pm \frac{6}{\sqrt{5}}, & k &= \frac{1}{\sqrt{5}}, & a_1 &= 0, \\ a_2 &= -\pm \frac{3}{\sqrt{5}}, & b_1 &= 0, & b_2 &= \pm \frac{3}{\sqrt{5}} \end{aligned} \quad (70)$$

Substituting (70) into (4)

$$\begin{aligned} V_{153}(\xi) &= \pm \frac{6}{\sqrt{5}} \pm \frac{3}{\sqrt{5}} \\ &\left((\sec(\xi) + \tan(\xi))^2 + \frac{1}{(\sec(\xi) + \tan(\xi))^2} \right). \end{aligned} \quad (71)$$

2. For $H_1 = -\frac{1}{2}$, $H_2 = 0$, $H_3 = -\frac{1}{2}$,

Family-I

$$\begin{aligned} a_0 &= \pm \frac{3}{\sqrt{2}}, & k &= \frac{1}{\sqrt{2}}, & a_1 &= 0, \\ b_2 &= 0, & b_1 &= 0, & a_2 &= \pm \frac{3}{\sqrt{2}} \end{aligned} \quad (72)$$

Substituting Eq. (72) into Eq. (4)

$$V_{161}(\xi) = \pm \frac{3}{\sqrt{2}} \pm \frac{3}{\sqrt{2}} (\sec(\xi) - \tan(\xi))^2. \quad (73)$$

Family-II

$$\begin{aligned} a_0 &= \pm \frac{3}{\sqrt{2}}, & k &= \frac{1}{\sqrt{2}}, & a_1 &= 0, & a_2 &= 0, \\ b_1 &= 0, & b_2 &= \pm \frac{3}{\sqrt{2}} \end{aligned} \quad (74)$$

Substitution of (74) with Eq. (4) gives,

$$\begin{aligned} V_{162}(\xi) &= \pm \frac{3}{\sqrt{2}} \pm \frac{3}{\sqrt{2}} \\ &\left(\frac{1}{(\sec(\xi))^2 + (\tan(\xi))^2 - 2(\sec(\xi))(\tan(\xi))^2} \right). \end{aligned} \quad (75)$$

Family-III

$$\begin{aligned} a_0 &= \pm \frac{6}{\sqrt{5}}, & k &= \frac{1}{\sqrt{5}}, & a_1 &= 0, \\ a_2 &= \pm \frac{3}{\sqrt{5}}, & b_1 &= 0, & b_2 &= \pm \frac{3}{\sqrt{5}} \end{aligned} \quad (76)$$

Substituting Eq. (76) to Eq. (4)

$$\begin{aligned} V_{163}(\xi) &= \pm \frac{6}{\sqrt{5}} \pm \frac{6}{\sqrt{5}} \\ &\left((\sec(\xi) - \tan(\xi))^2 + \frac{1}{(\sec(\xi) - \tan(\xi))^2} \right). \end{aligned} \quad (77)$$

3. For $H_1 = 1(-1)$, $H_2 = 0$, $H_3 = 1(-1)$,

Family-I

$$\begin{aligned} a_0 &= \pm \frac{12}{\sqrt{5}}, & k &= \frac{1}{\sqrt{5}}, & a_1 &= 0, & b_2 &= 0, \\ b_1 &= 0, & a_2 &= \pm \frac{12}{\sqrt{5}} \end{aligned} \quad (78)$$

Replace Eq. (78) in Eq. (4)

$$V_{171}(\xi) = \pm \frac{12}{\sqrt{5}} \pm \frac{12}{\sqrt{5}} ((\tan(\xi)(\cot(\xi)))^2. \quad (79)$$

Family-II

$$\begin{aligned} a_0 &= \pm \frac{12}{\sqrt{5}}, & k &= \frac{1}{\sqrt{5}}, & a_1 &= b_2 = 0, \\ a_2 &= 0, & b_2 &= \pm \frac{12}{\sqrt{5}} \end{aligned} \quad (80)$$

Put Eq. (80) in Eq. (4)

$$V_{172}(\xi) = \pm \frac{12}{\sqrt{5}} \pm \frac{12}{\sqrt{5}} \left(\frac{1}{\tan(\xi)(\cot(\xi))} \right)^2. \quad (81)$$

Family-III

$$\begin{aligned} a_0 &= \pm \frac{24}{\sqrt{17}}, \quad k \pm \frac{1}{\sqrt{17}}, \quad a_1 = 0, \\ a_2 &= \pm \frac{12}{\sqrt{17}}, \quad b_1 = 0, \quad b_2 = \pm \frac{12}{\sqrt{17}} \end{aligned} \quad (82)$$

Surrogate Eq. (82) within Eq. (4),

$$\begin{aligned} V_{173}(\xi) &= \pm \frac{24}{\sqrt{17}} \pm \frac{12}{\sqrt{17}} \\ &\left((\tan(\xi)(\cot(\xi))^2 + \frac{1}{(\tan(\xi)(\cot(\xi))^2}) \right). \end{aligned} \quad (83)$$

The Rational Function Solution of Eq. (46)

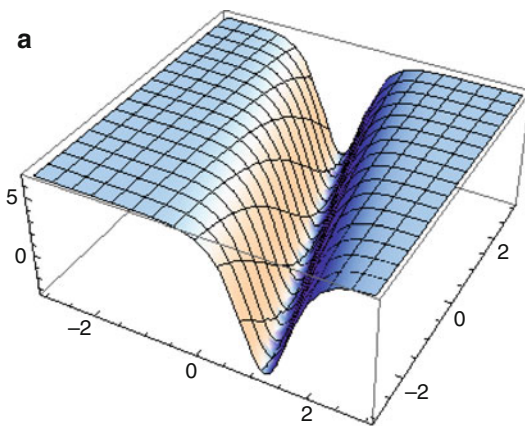
For $H_1 = 0$, $H_2 = 0$, $H_3 \neq 0$,

$$\begin{aligned} a_0 &= 0, \quad k = 1, \quad a_1 = 0, \quad a_2 = -12H_3^2, \\ b_1 &= 0, \quad b_2 = 0 \end{aligned} \quad (84)$$

Substitute Eq. (84) inside Eq. (4)

$$V_{18}(\xi) = 12P_3^2 \left(\frac{1}{H_3\xi + \varepsilon} \right)^2 \quad (85)$$

For $H_2 = 0$, $H_3 = 0$,



$$\begin{aligned} a_0 &= 0, \quad k = 1.0, \quad a_1 = 0, \quad a_2 = 0, \\ b_1 &= 0, \quad b_2 = -12H_1^2 \end{aligned} \quad (86)$$

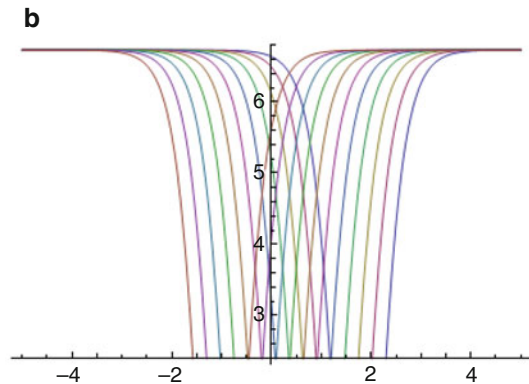
Transfer Eq. (86) in Eq. (4), provides

$$V_{19}(\xi) = -\frac{12}{\xi^2} \quad (87)$$

For $H_1 \neq 0$, $H_2 \neq 0$, $H_3 = 0$,

$$\begin{aligned} a_0 &= \pm \frac{2H_2^2}{\sqrt{H_2^2 + 1}}, \quad k = \pm \frac{1}{\sqrt{H_2^2 + 1}}, \\ a_2 &= 0, \quad b_1 \pm \frac{12H_1H_2}{\sqrt{H_2^2 + 1}}, \\ b_2 &= \pm \frac{12H_1^2}{\sqrt{H_2^2 + 1}}, \quad a_1 = 0, \end{aligned} \quad (88)$$

Substituting Eq. (88) into Eq. (4) (Fig. 7),



Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models, Fig. 7 The graphical description of (89) at (a), (b)

$$V_{18} = \pm \frac{2H_2^2}{\sqrt{H_2^2 + 1}} \pm \frac{12H_1H_2}{\sqrt{H_2^2 + 1}} \left(\frac{H_2}{(\exp(H_2\xi) - H_1)} \right) \pm \frac{12H_1^2}{\sqrt{H_2^2 + 1}} \left(\frac{H_2}{(\exp(H_2\xi) - H_1)} \right)^2 \quad (89)$$

Application of the Fourth-Order Nonlinear Ablowitz-Kaup-Newell-Segur Water Wave Equation

Consider the general form of AKNS (Ali et al. 2018):

$$4u_{xt} + u_{xxx}t + 8u_xu_{xy} + 4u_{xx}v_y - \delta u_{xx} = 0, \quad (90)$$

Consider the traveling waves transformation,

$$u(x, y, t) = V(\xi), \quad \xi = x + y + \mu t, \quad (91)$$

Using Eq. (91) into Eq. (90), we have

$$(4\mu - \delta)V' + 6V'^2 + \mu V''' = 0 \quad (92)$$

Balancing the order of V'^2 and V''' , Eq. (92) has solution (Aasaraai 2015)

$$V(\xi) = a_0 + a_1F(\xi) + b_1F^{-1}(\xi) \quad (93)$$

From Eq. (93), Eq. (5) and Eq. (92), we acquired several equations involving these, a_0 ,

a_1 , b_1 , H_1 , H_2 , H_3 , δ , and μ . We have the following solutions

The Soliton-Like Solutions of Eq. (90)

When $H_1 = 0$, $H_2 = 1$, $H_3 = -1$,

$$\mu = \frac{\delta}{5}, \quad a_1 = \frac{\delta}{5}, \quad b_1 = 0, \quad (94)$$

From Eq. (94) and (93), we have

$$V_{19}(\xi) = a_0 + \frac{\delta}{10} \left(1 + \tanh\left(\frac{1}{2}\xi\right) \right) \quad (95)$$

When $H_1 = 0$, $H_2 = -1$, $H_3 = 1$,

$$\mu = \frac{\delta}{5}, \quad a_1 = -\frac{\gamma}{5}, \quad b_1 = 0, \quad (96)$$

Substituting Eq. (96) into Eq. (93), we have

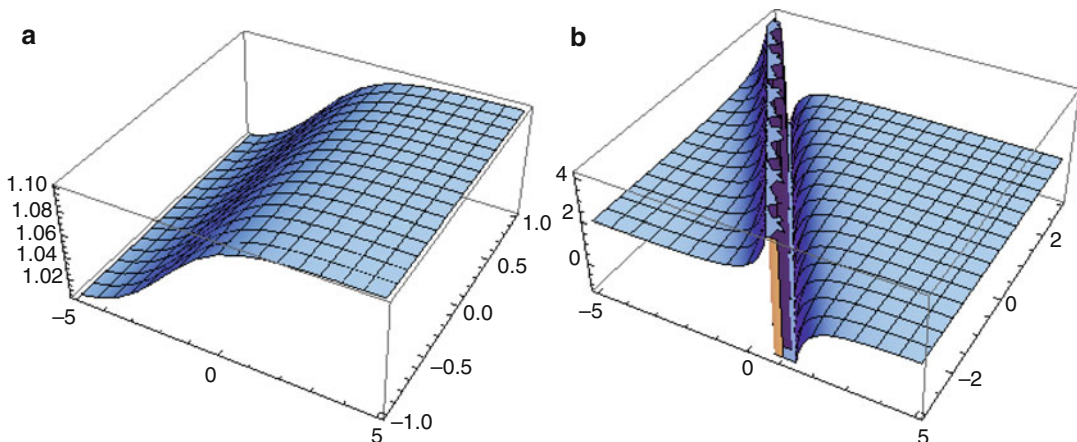
$$V_{20}(\xi) = a_0 - \frac{\delta}{10} \left(1 - \coth\left(\frac{1}{2}\xi\right) \right) \quad (97)$$

When $H_1 = \frac{1}{2}$, $H_2 = 0$, $H_3 = -\frac{1}{2}$,

Family-I

$$\mu = \frac{\gamma}{3}, \quad a_1 = 0, \quad b_1 = -\frac{\delta}{6}, \quad (98)$$

Substituting Eq. (98) into Eq. (93), we obtained (Fig. 8),



Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models, Fig. 8 The movement of (95) at (a) and (97) at (b) under these values of parameters $a_0 = 1$, $\delta = 0.5$ and $a_0 = 1$, $\delta = 5$ respectively

$$V_{211}(\xi) = a_0 - \frac{\delta}{6} \left(\frac{1}{\coth(\xi) \pm \operatorname{csch}(\xi)} \right) \quad (99)$$

Family-II

$$\mu = \frac{\gamma}{3}, \quad b_1 = 0, \quad a_1 = \frac{\delta}{6} \quad (100)$$

Substituting Eq. (100) into Eq. (93), which gives,

$$V_{212}(\xi) = a_0 + \frac{\delta}{6} (\coth(\xi) \pm \operatorname{csch}(\xi)) \quad (101)$$

When $H_1 = 1$, $H_2 = 0$, $H_3 = -1$,

Family-I

$$\mu = \frac{\delta}{8}, \quad a_1 = 0, \quad b_1 = \frac{\delta}{8}, \quad (102)$$

Substituting Eq. (106) into Eq. (93), We achieved

$$V_{221}(\xi) = a_0 + \frac{\delta}{8} \left(\frac{1}{\tanh(\xi)} \right) \quad (103)$$

Family-II

$$\mu = \frac{\delta}{8}, \quad b_1 = 0, \quad a_1 = \frac{\delta}{8}, \quad (104)$$

Substituting Eq. (104) into Eq. (93), this action gives,

$$V_{222}(\xi) = a_0 + \frac{\delta}{8} \tanh(\xi) \quad (105)$$

Family-III

$$\mu = \frac{\delta}{20}, \quad b_1 = \frac{\delta}{20}, \quad a_1 = \frac{\delta}{20}, \quad (106)$$

Substituting Eq. (108) into Eq. (93)

$$V_{223}(\xi) = a_0 + \frac{\gamma}{20} \left(\tanh(\xi) + \frac{1}{\tanh(\xi)} \right) \quad (107)$$

The Trigonometric Function Solutions of Eq. (90)

When $A_1 = \frac{1}{2}$, $A_2 = 0$, $A_3 = \frac{1}{2}$,

Family-I

$$\mu = \frac{\delta}{3}, \quad a_1 = -\frac{\delta}{6}, \quad b_1 = 0, \quad (108)$$

Substituting Eq. (108) into Eq. (93),

$$V_{231}(\xi) = a_0 - \frac{\delta}{6} (\sec(\xi) + \tan(\xi)) \quad (109)$$

Family-II

$$\mu = \frac{\delta}{3}, \quad b_1 = \frac{\delta}{6}, \quad a_1 = 0, \quad (110)$$

Substituting Eq. (110) into Eq. (93),

$$V_{232}(\xi) = a_0 + \frac{\delta}{6} \left(\frac{1}{\sec(\xi) + \tan(\xi)} \right) \quad (111)$$

When $H_1 = -\frac{1}{2}$, $H_2 = 0$, $H_3 = -\frac{1}{2}$,

Family-I

$$\mu = \frac{\delta}{3}, \quad a_1 = \frac{\delta}{6}, \quad b_1 = 0, \quad (112)$$

Substituting Eq. (112) into Eq. (93),

$$V_{241}(\xi) = a_0 + \frac{\gamma}{6} (\sec(\xi) - \tan(\xi)) \quad (113)$$

Family-II

$$\mu = \frac{\delta}{3}, \quad b_1 = -\frac{\delta}{6}, \quad a_1 = 0, \quad (114)$$

Substituting Eq. (114) into Eq. (93),

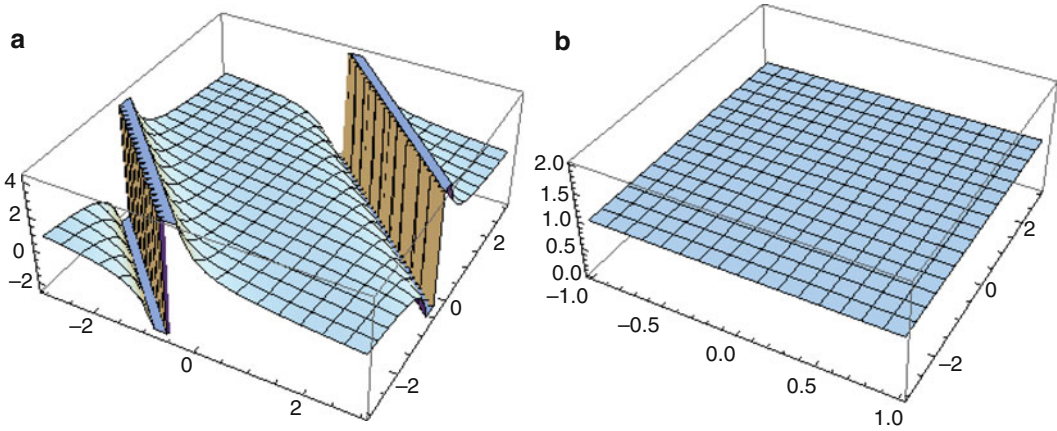
$$V_{242}(\xi) = a_0 - \frac{\gamma}{6} \left(\frac{1}{\sec(\xi) - \tan(\xi)} \right) \quad (115)$$

$$H_1 = -1, \quad H_2 = 0, \quad H_3 = 1(-1),$$

$$\mu = -\frac{\delta}{12}, \quad a_1 = -\frac{\delta}{12}, \quad b_1 = \frac{\delta}{12}, \quad (116)$$

Substituting Eq. (116) into Eq. (93) (Fig. 9),

$$V_{25}(\xi) = a_0 - \frac{\delta}{12} \left(\tan(\xi)(\cot(\xi)) - \frac{1}{\tan(\xi)(\cot(\xi))} \right) \quad (117)$$



Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models, Fig. 9 The 3D movement of the solitary waves solution (111) on (a) and for (117) on (b), $a_0 = 1$, $\delta = 5$

The Rational Function Solutions of Eq. (6)

When $H_1 = 0$, $H_2 = 0$, $H_3 \neq 0$,

Family-I

$$b_1 = \frac{4\mu - \delta}{6H_3}, \quad a_1 = 0, \quad (118)$$

Substituting Eq. (118) into Eq. (93),

$$V_{261}(\xi) = a_0 - \frac{4\mu - \delta}{6H_3} (H_3\xi + \varepsilon) \quad (119)$$

Family-II

$$\mu = \frac{\delta}{4}, \quad a_1 = -\frac{H_3\delta}{4}, \quad b_1 = 0, \quad (120)$$

Substituting Eq. (120) into Eq. (93),

$$V_{262}(\xi) = a_0 + \frac{H_3\delta}{4} \left(\frac{1}{(H_3\xi + \varepsilon)} \right) \quad (121)$$

When $H_2 = 0$, $H_3 = 0$, then we have,

Family-I

$$a_1 = \frac{-4\mu + \delta}{6H_1}, \quad b_1 = 0, \quad (122)$$

Substituting Eq. (122) into Eq. (93),

$$V_{271}(\xi) = a_0 - \frac{4\mu + \delta}{6} (\xi) \quad (123)$$

Family-II

$$\mu = \frac{\delta}{4}, \quad b_1 = \frac{H_1\delta}{4}, \quad a_1 = 0, \quad (124)$$

Substituting Eq. (124) into Eq. (93),

$$V_{272}(\xi) = a_0 + \frac{H_1\delta}{4} \left(\frac{1}{(\xi)} \right) \quad (125)$$

When $H_1 \neq 0$, $H_2 \neq 0$, $H_3 = 0$,

$$\mu = \frac{\delta}{4 + A_2^2}, \quad b_1 = \frac{H_1\delta}{4 + H_2^2}, \quad (126)$$

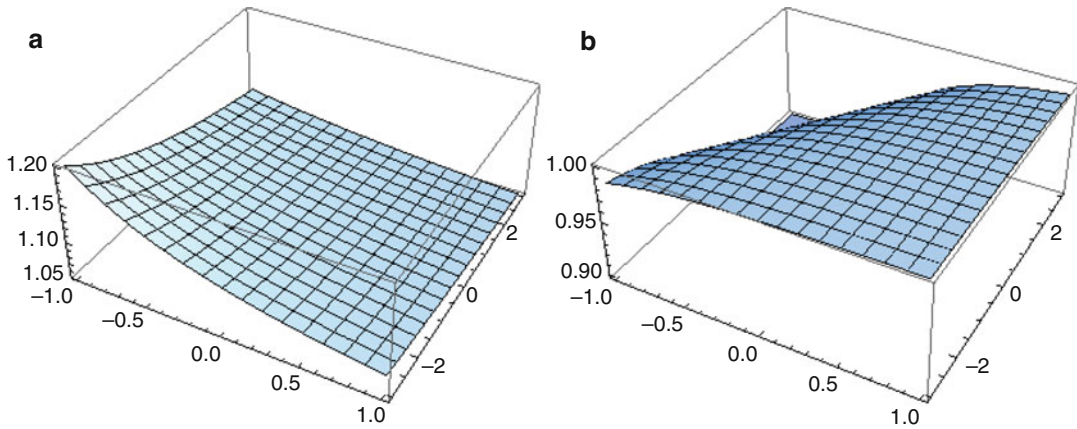
$$a_1 = 0,$$

Substituting Eq. (126) into Eq. (93) (Fig. 10),

$$V_{28}(\xi) = a_0 + \frac{H_1\delta}{4 + H_2^2} \left(\frac{H_2}{(\exp(A_2\xi) - H_1)} \right) \quad (127)$$

Discussion of the Results

After successfully employing the modified F-expansion technique on our three selective models, i.e., Modified Liouville, SRLW, and



Analytical Soliton Solutions for Some Nonlinear Dynamical Water Waves Models, Fig. 10 Solitary waves of (125) at (a) and (127) at (b) with; $\varepsilon = 1$, $H_3 = 1$, $a_0 = 1$, $\delta = 1$ and $H_2 = 2$, $H_1 = 1$, $a_0 = 1$, $\delta = -2$, respectively

AKNS equations, now we have discussed the similarities and dissimilarities of our new constructed results with other results in previous different literature. Few novel constructed results are likely similar with results in different articles with the following points.

- Our solutions (7) and (61) are approximate similar with solutions (24) and (31) in Lu et al. (2018b) respectively.
- Further our solutions of (43) and (45) are nearly same with the solutions (50) and (41) in Lu et al. (2018a), respectively.
- Moreover, our solutions (95) and (97) are likely similar with solutions (38) and (39) in Inca et al. (2017), respectively.
- In the same way, our constructed solutions (101) and (121) are same with the solutions (57) and (42) in Lu et al. (2018b), respectively.
- Fig. 3 of our solution (23) with these values of parameters $d_1 = 0.7$, $d_2 = 1$, $k = 4.5$, $\beta = -4.5$, $\omega = -5.3$ are equal with the solution (51) of Fig. 5 in Lu et al. (2018a) with $\omega = 0.5$, $\omega = 2$, $\omega = 1$, $a = 1$, $b = 1$, $b = 1$, $k = 1$, $k = 0.5$.

The segment, discussion of the results shows that our this new achievement is more efficiency and fruitful to face the nonlinear problems for

solving them in mathematics, physics, and engineering.

Conclusion

Here current work is about the study of modified Liouville, the symmetric regularized long wave, and Fourth order Ablowitz-Kaup-Newell-Segur wave models through the successful implement of the modified F-expansion method, series new soliton solutions in the form of hyperbolic, trigonometric, and rational functions are obtained. The graphical movements some solutions are fruitful to comprehend the physical phenomena of these three nonlinear models. Finally our constructed new results have many potential merits to handling different types of nonlinear wave system.

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Soliton Propagation in Solids: Advances and Applications

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Article Outline

Introduction
Microstructure
Nonlinearity
Bulk and Surface Solitary Waves in Solids
Conclusions
Bibliography

Introduction

The investigation of new materials specially designed for definite purposes, the so-called meta-materials, is at the focus of research in the field of continuum mechanics in recent years. The multitude of applications in the field of technology has been the basis for the appearance of new theoretical models that take into account the complex structure of such materials, as well as new experimental techniques. Nanotechnology has given a new breath to research in this field, due to the associated small length scales which range from 1 nm to a few hundreds of nanometers, thus creating an area where quantum effects may become effective. Nano-inclusions in dielectric materials play a significant role in determining the conductivity and the stiffness of the composites. Generally, experimentation at this scale poses some problems that need further attention.

Maugin (2011) has presented a review on solitons in elastic solids, with extensive literature covering the period from 1938 to 2010. In it, he introduces solitons in solids as solutions to nonlinear equations and systems of sine-Gordon

$$u_{tt} - u_{xx} + m^2 \sin u = 0 \quad (1)$$

and Boussinesq

$$u_{tt} - u_{xx} - \beta u_x u_{xx} - \delta^2 u_{xxxx} = 0 \quad (2)$$

types, where u denotes a field variable, x is a spatial coordinate, t is the time, and m , β , δ are constant parameters.

It is the aim of the present work to give an overall view covering the main aspects of soliton propagation in solids and the advances in this field, as well as the related applications, in the period ranging from 2010 till now. Maugin “historical review” will be a guide and an inspiration throughout the present work. We have preferred to make use mainly of relatively new references belonging to the period from 2010 onward, dealing with the generation, tracking, and use of solitons in solids, in order to convey to the reader a taste of the modern ongoing activities in this wide field of research.

Microstructure

It is pointless to discuss the subject of soliton propagation in solids without referring to microstructure. Microstructure consists of those structural inhomogeneities that may exist at a microscopic level in a continuous medium. Many factors contribute toward microstructure: the distribution of inhomogeneities, their shape and orientation in space, interaction with one another, etc. Microstructure undergoes changes during material processing, for example, by thermal processes or pressure. It is necessary to follow these changes, as microstructure is a determining factor of the different physical properties of a material. It can drastically affect the strength of the material, the appearance of stress gradients, and, consequently, its behavior during plastic deformation

and rupture (Hallberg et al. 2018). Since the phenomenon of plasticity is intimately connected to the microstructure, this latter must be taken in consideration in any modern theory of plasticity.

In continuum mechanics, the media with complex structure, or with microstructure, are characterized by different length scales characterizing their internal structure and by time scales such as relaxation times that provide a rich background for the propagation of solitons. In such systems, the interaction of the different structural constituents of the medium naturally yields nonlinear effects that form the basic ingredients for the propagation of nonlinear waves. Inhomogeneity and anisotropy naturally occur in such materials, due to various factors, for example, granular or layered texture, inclusions, etc., that make such media widely used in technological applications for prescribed purposes. Complexity is strictly understood within such a context. The theoretical basis for investigating such models may be found in the relevant textbooks, while the various applications concern major areas such as control technology, engineering of materials, alloy industry, and others.

There are different approaches to the investigation of microstructure, aiming at defining the different internal length scales and the corresponding hierarchy of microstructures and their connection to the whole structure at a macroscopic scale. Concurrent microstructures (Berezovski et al. 2010) and hierarchical microstructures, i.e., one level containing the other (Casasso et al. 2010), are the most popular models in this respect. Evidently, both ways lead to different systems of partial differential equations and to different descriptions of the medium.

As an example for a two-scale model of elasticity in the one-dimensional case, the free energy function is taken as a quadratic form (Berezovski et al. 2010):

$$F = \frac{\rho c^2}{2} (u_x)^2 + A \phi u_x + \frac{1}{2} B \phi^2 + \frac{1}{2} C (\phi_x)^2, \quad (3)$$

leading to a system of linear field equations

$$\rho u_{tt} = \rho_0 c^2 u_{xx} + A \phi_x, \quad (4)$$

$$I \phi_{tt} = C \phi_{xx} - A u_x - B \phi, \quad (5)$$

where u is the displacement, ϕ is the microdeformation, ρ is the density, c is the wave velocity at macro level, I is an internal inertia measure, and A, B, C are some material parameters.

For a two-level microstructure, a possible form of the strain energy would be (Casasso et al. 2010):

$$W = \frac{1}{2} \alpha (u_x)^2 + \frac{1}{3} \beta (u_x)^3 - A_1 \phi u_x + \frac{1}{2} B \phi^2 + \frac{1}{2} C_1 (\phi_x)^2 - A_2 \phi_x \psi + \frac{1}{2} B_2 \psi^2 + \frac{1}{2} C_2 (\psi_x)^2, \quad (6)$$

yielding a system of nonlinear field equations

$$\rho u_{tt} = \alpha u_{xx} + (\beta u_x^2)_x - A_1 \phi_x, \quad (7)$$

$$I_1 \phi_{tt} = C_1 \phi_{xx} + A_1 u_x - B_1 \phi - A_2 \psi_x, \quad (8)$$

$$I_2 \psi_{tt} = C_2 \psi_{xx} + A_2 \phi_x - B_2 \psi, \quad (9)$$

where ϕ and ψ are the microdeformations for the two levels of microstructure.

As appears from the given examples, the models for elastic solids with microstructure may be investigated using phenomenological approaches. For this, the connection between macro- and micro-fields must be properly formulated within the frame of rigorous thermodynamics, with a view to reach a realistic description of the medium under investigation.

Nonlinearity

Citing Maugin (2011), “The mechanics of deformable solid bodies is always more complicated, if not more difficult, to deal with than that of fluids. The reason for this is multi-fold. But that may explain why nonlinear waves in general, and solitons in particular, entered that field after some

delay. One of the reasons was that – in contrast with fluid mechanics – many specialists of solid mechanics, although dealing with difficult boundary-value problems, do not deal with nonlinearities. It is only with the consideration of physical nonlinearity in crystals and the phenomenon of plasticity (due ultimately to the presence of structural defects such as dislocations), that true nonlinear problems started to appear.”

There exists an ample recent literature on models of continuous media with complex structure, in which the common leading idea is nonlinear modeling. Murnaghan’s model for isotropic elastic materials is known since the 1960s of last century. It involves cubic nonlinearities in strain in the expression for stress and accounts for the propagation of solitons in elastic rods (see Porubov et al. 2009). Maugin (2011) has stressed the importance of nonlinear effects in obtaining localized solutions that would provide experimentalists with an important tool to investigate the microstructure. Propagating waves in this case undergo both dissipation and dispersion. Most of the considered cases, understandably, concern one-dimensional problems, while two-dimensional problems are rare as they present technical difficulties. For concrete choices of the strain energy function and taking into account the different length scales in the microstructure, the used models yield Boussinesq-type nonlinear partial differential equations that describe wave propagation and capture some essential nonlinear effects at the microstructure level. Strain solitons can be studied using the well-known Korteweg-de-Vries equation. Their observation, however, requires the strain amplitude to be sufficiently large; this is achieved through the use of femtosecond laser excitation. Some of the presented solutions are exact, while numerical integration may reveal necessary in more complicated cases (Casasso et al. 2010; Porubov et al. 2009; Pastrone and Engelbrecht 2016; Tamm and Salupere 2012; Sertakov et al. 2014; Engelbrecht et al. 2018). Such models find application in many fields, such as nondestructive testing of materials with microstructure and sound control in nonlinear acoustic metamaterials. The study of nonlinear interactions and dispersion in solids has been boosted by the availability of

high-power laser sources, especially in surface processing techniques and the accompanying change in surface microstructure.

The investigation of nonlinear optical phenomena in photonic crystal fibers and in waveguides represents another area of great importance, as it provides unprecedented opportunities for the study of nonlinear acousto-mechanical interactions and the dispersive properties in these media. This trend started in the 1970s of the last century and witnessed fast development in the field of communications (Chen et al. 2012). Recent advances in the fabrication of microstructured waveguides and fibers to support such different nonlinearities arising in the medium have opened new perspectives in the domain of nonlinear optics and metamaterials in general (Dudley and Taylor 2009).

Bulk and Surface Solitary Waves in Solids

The importance of solitons in solids resides in the fact that they represent localized propagating modes. It is a well-known fact that the stability of solitons originates from a delicate balance between inherent nonlinearity of the medium on one side and dispersion or diffraction provided by the discrete lattice.

Experimentation on the propagation of elastic solitons in rods and plates has indicated that such waves undergo little attenuation, keeping their bell shape and amplitude almost unchanged as they travel in the medium, the wave parameter depending on the material properties and geometry of the waveguide. Problems may arise in the case of polymers due to the variability of material parameters in measurement. When passing through defects, however, strain waves undergo changes in their characteristics, which makes them a valuable tool for nondestructive testing of materials. As example, one of the most important failure phenomena is delamination in composites. It occurs in the form of separation of laminas in the composite under working conditions, due to various causes like manufacturing problems, and generally accompanies failure. This phenomenon may be detected

through the observation of variations occurring in bulk strain solitons, transforming under this defect into a wave train of solitons (Dreiden et al. 2012, 2014; Samsonov et al. 2015, 2017).

Spatial optical solitons are special types of solitons propagating in waveguides. Their stability under certain circumstances is a result of balance between nonlinearity and diffraction. They are an ideal tool in the field of nonlinear optics, in optical fibers, and for routing and switching purposes. These waves were discovered in the 1970s of the past century and found increasing interest since then. They originate from nonlinearities on the refractive index (Chen et al. 2012). Their counterpart, temporal solitons, in which nonlinearity balances dispersion, are used to generate ultra-short pulse in the range of picoseconds and femtoseconds and have various applications in communications technology.

Solitary waves may propagate in nonlinear multilayered media, along the interface of two solids, for example, along metal-dielectric interfaces, or between dielectric media under certain conditions (Lysak and Trofimov 2015). Experimental evidence for the existence of such surface waves was known since the early 1980s. A first report on the observation of discrete optical surface soliton at the interface between a nonlinear waveguide lattice and a continuous medium was given by Suntsov et al. (2006). Surface waves are governed by systems of partial differential equations known as “generalized Zakharov equations” (Lysak and Trofimov 2015; You and Ning 2017). Such equations have been studied extensively in the last decade. An example is given here below for reference (Khan et al. 2011):

$$iE_t + E_{xx} - 2\beta|E|^2 + 2FE = 0, \quad (10)$$

$$E_{tt} - E_{xx} + \left(|E|^2\right)_{xx} = 0. \quad (11)$$

Research on surface solitary waves has been growing in the past decade; applications are numerous, ranging from sensing to imaging. Broader perspectives in relatively new areas like nanophotonics are acquiring growing importance.

Conclusions

Solitons are found in solids with complex structure due to inherent nonlinearities. Under certain circumstances, these localized modes propagate with unchanged shape and velocity over appreciable distances as a result of balance between nonlinearity and dispersion effects. They are used in several important technological applications, the determination of material parameters, communications, and nondestructive testing of materials among others.

Solitons are bound to appear in models of continuous media once the proper nonlinearity has been identified, whether mechanical, electromagnetic, or other. They are shown to satisfy a system of partial differential equations relevant to the phenomenon under investigation. The mathematical form of the solution and its properties, combined with good experimentation, allow to decide on the ability of the used model to adequately describe reality.

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Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers

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Article Outline

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Introduction

In solitons theory, nonlinear partial differential equations (NLPDEs) are booming in many scientific fields such as ocean dynamics, plasma physics, fluid dynamics, hydrodynamics and theory of turbulence, optical fibers, chemical physics, chaos theory, and many others. Exact solutions of the nonlinear physical problems are significant and a vital topic in real life while the soliton-based algorithms are promising methods to evaluate the solutions of different nonlinear real-world problems (Kudryashov 2021; Khater et al. 2000, 2006a,b;

Helal and Seadawy 2009; Rizvi et al. 2021; Tariq et al. 2021a,b; Ahmed et al. 2021; Ali et al. 2021). Previously, many scientists devoted their attention to compute the solution of NLPDEs such as: non-local nonlinearity (Kudryashov 2021), variational-based approach (Khater et al. 2000), He's variational method (Khater et al. 2006a), direct algebraic technique (Khater et al. 2006b), bilinear approach (Helal and Seadawy 2009), HBM technique (Rizvi et al. 2021), extended Fan sub-equation technique (Tariq et al. 2021a), modified mathematical method (Ahmed et al. 2021), auxiliary equation method (Tariq et al. 2021b), Painleve analysis (Ali et al. 2021), asymptotic approach (Bildik and Deniz 2018), geometric singular perturbation (Ge and Du 2020), Lie symmetry analysis (Tian 2020), efficient integration process (Ghanbari et al. 2020), improved modified Kudryashov architectonic (Hyder and Barakat 2020), highly optical solitons (Kudryashov 2020a), nonlinearity and conservation (Ali et al. 2020), Hirota bilinear scheme (Foroutan et al. 2018), $(\frac{G'}{G})$ mechanism (Ismael et al. 2020), homotopy (Yu et al. 2019), extended mapping technique (Seadawy and Cheemaa 2019), non-perturbative technique (James et al. 2018), semi-inverse approach (He 2020), logistic function (Kudryashov 2020b), asymptotic technique (Bildik and Deniz 2018), extended and modified direct algebraic method and extended mapping method (Helal and Seadawy 2011, 2012; Helal et al. 2013, 2014, 2017; Zkan et al. 2021; Younas et al. 2020; Seadawy et al. 2019; Ahmed et al. 2019; Lu et al. 2018).

Many famous nonlinear equations are accomplished, for instance, KdV equation (Wang et al. 2016), Camassa Holm(CH)-equation (Ghaffar et al. 2020), Biswas model (Rizvi et al. 2020), the Schrodinger equation (Islam et al. 2017), Vakhnenko model (Seadawy et al. 2020), sine-Gordon equation (Dehghan et al. 2015), but here we will find the different wave form solutions including the lump, lump-one stripe, lump-two stripe, lump periodic, and periodic cross-kink form solutions of the generalized Burger Huxley (Alquran et al. 2021),

$$u_t - u_{xx} - \alpha u u_x - \beta u(1-u)(u-\gamma) = 0, \quad (1)$$

where $u = u(x, t)$ and α, β, γ are general real parameters. For the case of $\gamma = 1$ and $\alpha = 0$, Eq. (1) is reduced to the Huxley model which represents the propagation of nerve pulses in nerve fibers and wall motion in liquid crystals (Wang et al. 1990). For $\gamma = 1$ and $\beta = 0$, Eq. (1) is the Burger's equation which describes the field of wave propagation in nonlinear dissipative models. The Burger Huxley Eq. (1) describes the interaction of convection terms against diffusion transmission (Wang et al. 1990).

The content of this entry is arranged as follows: in the section “[Lump Solution](#),” the lump solutions for Eq. (1) will be obtained with some 3D, 2D, and contour shapes. The lump-one stripe, lump-two stripe solutions for the GBH-equation will be proposed in the sections “[Lump-One Stripe Solution](#)” and “[Lump-Two Stripe Solution](#),” correspondingly. In the section “[Lump Periodic Solution](#),” we will study the lump periodic solutions. The periodic cross-kink wave form solutions will be found in the section “[Periodic Cross-Kink Solutions](#).” In the section “[Figures Analysis](#),” we will accomplish the results and discussions with eclair literature. Finally some conclusions are given in the section “[Concluding Remarks](#).”

Lump Solution

For the exploration of lump solutions for Eq. (1), we assume the transformation (WaZhou et al. 2019; Wang 2018),

$$u = m + 2b(\ln f)_x, \quad (2)$$

which changes Eq. (1),

$$\begin{aligned} & -m^2\beta f^3 + m^3\beta f^3 + m\gamma\beta f^3 - m^3\gamma\beta f^3 \\ & -4bm\beta f^2 f_x + 6bm^2\beta f^2 f_x + 2bm\beta\gamma f^2 f_x \\ & + \dots + 6bff_{xx}f_{xx} - 4b^2\alpha f_x f_{xx} - 2bf^2 f_{xxx} = 0. \end{aligned} \quad (3)$$

Now f in Eq. (3) can be supposed as (Islam et al. 2017),

$$f = g^2 + h^2 + a_7, \quad (4)$$

where $g = a_1x + a_2t + a_3$ and $h = a_4x + a_5t + a_6$. While, $a_i (1 \leq i \leq 7)$ are any parameters to be measured. Now, putting f into Eq. (3) and associating the coefficients of the t and x implies us:

Set I When $a_5 = a_6 = a_7 = 0$, then we get the following values of the parameters

$$\begin{aligned} a_1 &= a_1, m = \frac{a_2(2a_1^2 + a_4^2)}{2\alpha a_1^3}, a_2 = a_2, \\ \alpha &= \alpha, \beta = 0, b = \frac{3(a_1^2 - a_4^2)}{\alpha(3a_1^2 - a_4^2)}, a_3 = a_3, a_4 = a_4. \end{aligned} \quad (5)$$

then,

$$\begin{aligned} u_1(x, t) &= \frac{a_2(2a_1^2 + a_4^2)}{2\alpha a_1^3} \\ &+ \frac{6(a_1^2 - a_4^2)}{(3a_1^2 - a_4^2)} \frac{(2a_4^2x + 2a_1(a_2t + a_1x))}{a_4^2x^2 + \alpha(a_2t + a_1x)^2}. \end{aligned} \quad (6)$$

Set II When $a_5 = a_6 = a_7 = 0$, then we get the following values of the parameters

$$\begin{aligned} a_1 &= \sqrt{\frac{-1}{5}}a_4, m = 1, \\ a_2 &= \frac{\frac{\sqrt{-5}}{15}(xb\gamma - 18xb - 9\gamma + 18)a_4}{b}, \\ \alpha &= \alpha, \beta = \frac{9 - 8xb}{12b^2}, b = b, a_3 = a_3, a_4 = a_4. \end{aligned} \quad (7)$$

then,

$$\begin{aligned} u_2(x, t) &= 1 \\ &+ 9 \frac{\left(2a_4^2x + \frac{2ia_4\left(\frac{ia_4x}{\sqrt{5}} + \frac{4ia_4tx\left(\frac{-90-27\gamma}{4}\right)}{27\sqrt{5}}\right)}{\sqrt{5}} \right)}{2\alpha\left(a_4^2x^2 + \frac{ia_4x}{\sqrt{5}} + \frac{4ia_4tx\left(\frac{-90-27\gamma}{4}\right)}{27\sqrt{5}}\right)^2}. \end{aligned} \quad (8)$$

Set III When $a_5 = a_6 = a_7 = 0$, then we get the following values of the parameters

$$\begin{aligned}
 m=0, a_1 &= \sqrt{\frac{-1}{5}}a_4, \alpha = \alpha, \beta = \frac{9-8\alpha b}{12b^2}, a_3 = a_3, & m = \gamma, a_1 &= \sqrt{\frac{-1}{5}}a_4, \\
 a_2 &= \frac{\sqrt{-5}(8\alpha b - 9)(\gamma + 1)a_4}{b}, a_4 = a_4. & a_2 &= -\frac{\sqrt{-5}(8\alpha b - 9)(\gamma + 1)a_4}{b}, \\
 & & \alpha &= \alpha, \beta = \frac{9-8\alpha b}{12b^2}, a_3 = a_3, a_4 = a_4.
 \end{aligned} \tag{9}$$

then,

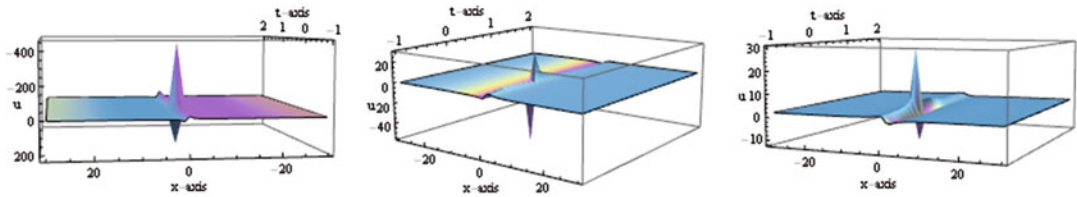
$$u_3(x, t) = 9 \frac{\left(2a_4^2 x + \frac{2ia_4 \left(\frac{ia_4 x}{\sqrt{5}} + \frac{4ia_4 t \alpha (\gamma + 1)}{3\sqrt{5}} \right)}{\sqrt{5}} \right)}{2\alpha \left(a_4^2 x^2 + \left(\frac{ia_4 x}{\sqrt{5}} + \frac{4ia_4 t \alpha (\gamma + 1)}{3\sqrt{5}} \right)^2 \right)}. \tag{10}$$

then,

$$u_4(x, t) = \gamma + \frac{9 \left(2a_4^2 x + \frac{2ia_4 \left(\frac{ia_4 x}{\sqrt{5}} + \frac{4ia_4 t \alpha (\gamma + 1)}{3\sqrt{5}} \right)}{\sqrt{5}} \right)}{2\alpha \left(a_4^2 x^2 + \left(\frac{ia_4 x}{\sqrt{5}} + \frac{4ia_4 t \alpha (\gamma + 1)}{3\sqrt{5}} \right)^2 \right)}. \tag{12}$$

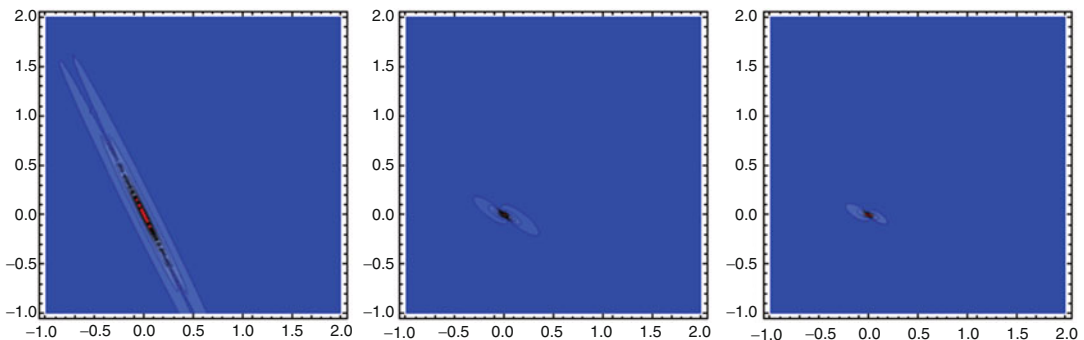
Set IV When $a_5 = a_6 = a_7 = 0$, then we get the following values of the parameters

See Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12.



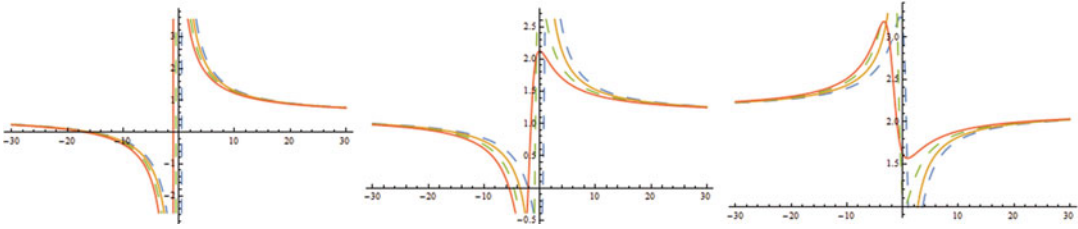
Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 1 The bright-dark lump solutions via the solution in Eq. (6) are presented with different parameters.

Three-dimensional profiles at (i) $a_1 = 1, a_2 = 10, a_4 = -2, \alpha = -1$, (ii) $a_1 = 20, a_2 = 30, a_4 = -2, \alpha = 2$, and (iii) $a_1 = 20, a_2 = 40, a_4 = -2, \alpha = 2$, respectively



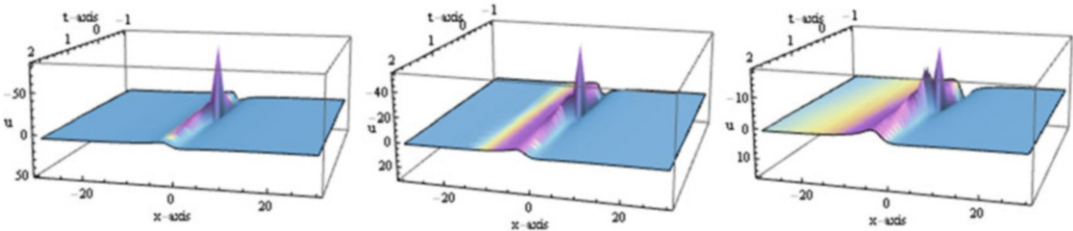
Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 2 The bright-dark lump solutions via the solution in Eq. (6) are plotted with appropriate parameters.

Contour profiles at (i) $a_1 = 1, a_2 = 10, a_4 = -2, \alpha = -1$, (ii) $a_1 = 20, a_2 = 30, a_4 = -2, \alpha = 2$, and (iii) $a_1 = 20, a_2 = 40, a_4 = -2, \alpha = 2$, respectively



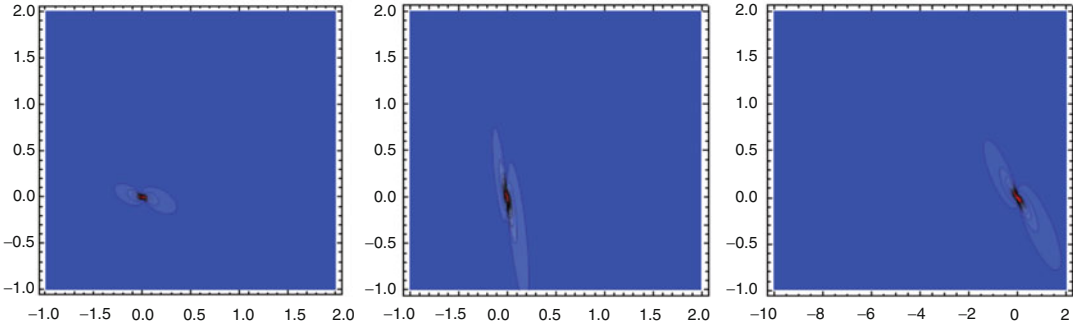
Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 3 The bright-dark lump solutions via the solution in Eq. (6) are expressed with suitable parameters.

Two-dimensional graphs at (i) $a_1 = 1, a_2 = 10, a_4 = -2, \alpha = -1$, (ii) $a_1 = 20, a_2 = 30, a_4 = -2, \alpha = 2$, and (iii) $a_1 = 20, a_2 = 40, a_4 = -2, \alpha = 2$, respectively



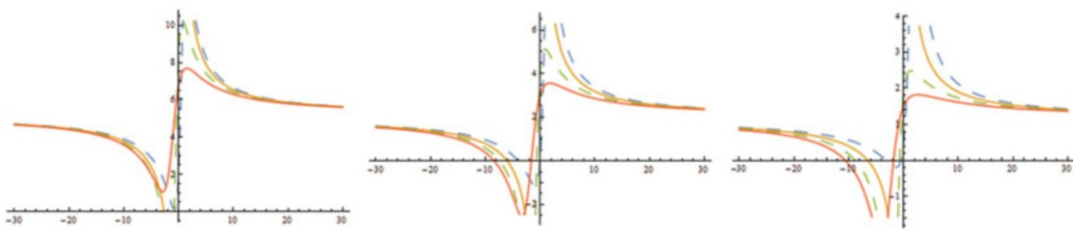
Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 4 The lump solutions via the solution in Eq. (6) are presented with different parameters. Three-

dimensional profiles at (i) $a_1 = 20, a_2 = 50, a_4 = -50, \alpha = 2$, (ii) $a_1 = 40, a_2 = 90, a_4 = -90, \alpha = 2$, and (iii) $a_1 = 40, a_2 = 100, a_4 = -100, \alpha = 9$, respectively



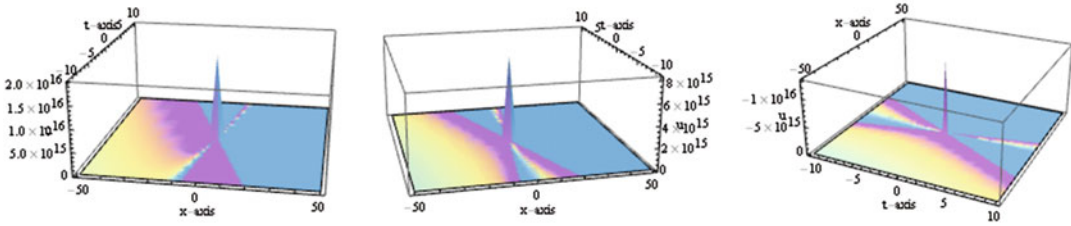
Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 5 The lump solutions via the solution in Eq. (6) are presented with various parameters. Contour

shapes at (i) $a_1 = 20, a_2 = 50, a_4 = -50, \alpha = 2$, (ii) $a_1 = 40, a_2 = 90, a_4 = -90, \alpha = 2$, and (iii) $a_1 = 40, a_2 = 100, a_4 = -100, \alpha = 9$, respectively



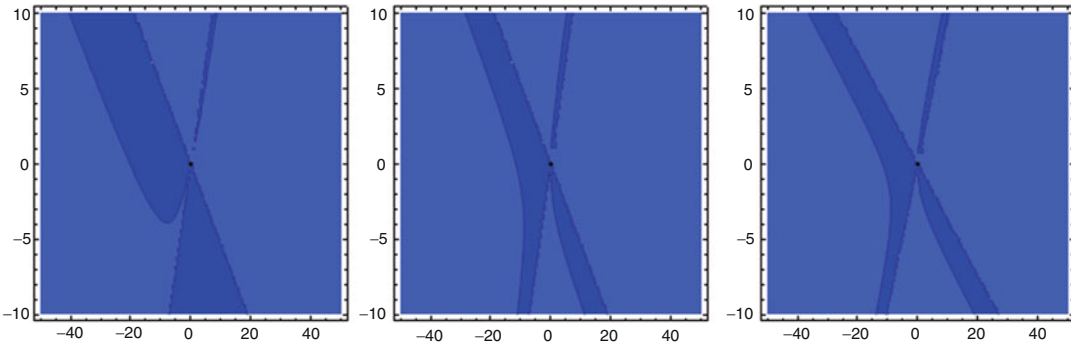
Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 6 The lump solutions through the solution in Eq. (6) are presented with various parameters. Two-

dimensional shapes at (i) $a_1 = 20, a_2 = 50, a_4 = -50, \alpha = 2$, (ii) $a_1 = 40, a_2 = 90, a_4 = -90, \alpha = 2$, and (iii) $a_1 = 40, a_2 = 100, a_4 = -100, \alpha = 9$, respectively

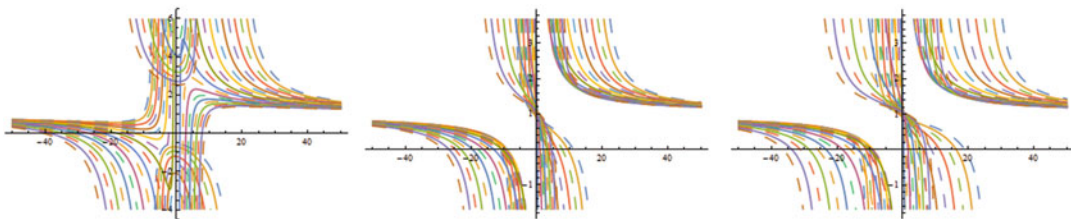


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 7 The lump solutions through the solution

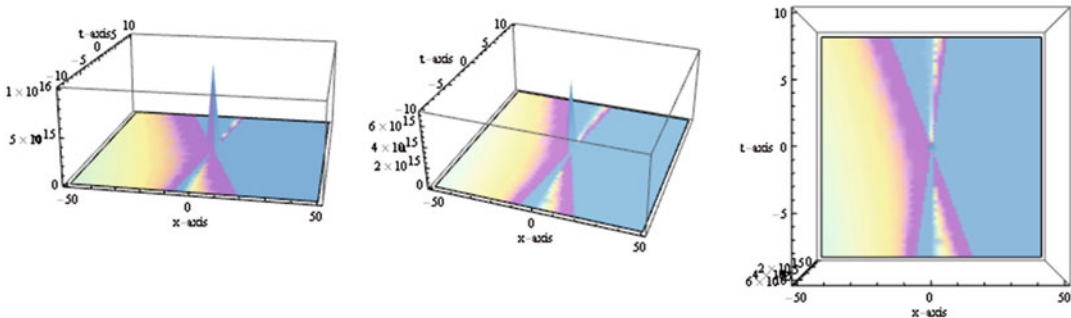
in Eq. (8) are presented with various parameters $\alpha = 1$, $\gamma = -1$. Three-dimensional shapes at (i) $a_4 = -1$, (ii) $a_4 = -10$, and (iii) $a_4 = -20$, respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 8 The contour shapes of lump solutions for Fig. 7 respectively

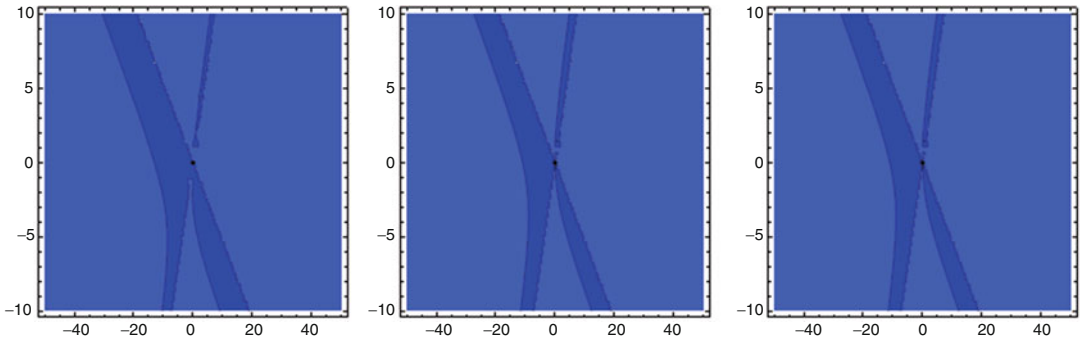


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 9 The 2D shapes of lump solutions for Fig. 7 respectively

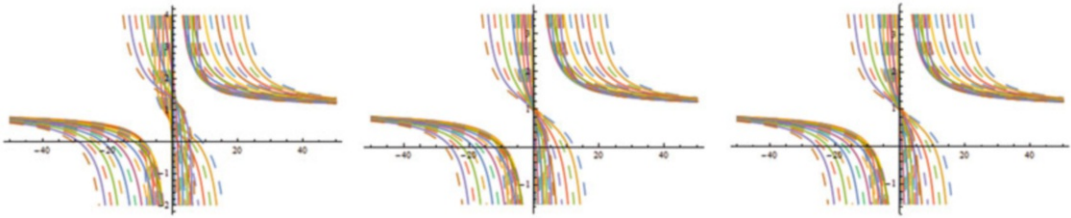


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 10 The lump solutions via the solution in

Eq. (8) are presented with different parameters $\alpha = 1$, $\gamma = -1$. Three-dimensional shapes at (i) $a_4 = 20$, (ii) $a_4 = 30$, and (iii) $a_4 = 40$, respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers,
Fig. 11 The contour shapes of lump solutions for Fig. 10 respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers,
Fig. 12 The 2D shapes of lump solutions for Fig. 10 respectively

Lump-One Stripe Solution

Now to investigate these solutions, we consider the f in Eq. (3) as (Wang 2018),

$$f = g^2 + h^2 + a_7 + b_1 \exp(f_1), \quad (13)$$

where $g = a_1x + a_2t + a_3$; $h = a_4x + a_5t + a_6$; and $f_1 = k_1x + k_2t$. However, $a_i (1 \leq i \leq 7)$, k_1 and k_2 are real constants to be found. After that, setting assumed f in Eq. (3) and separating the coefficients of t and x :

Set I Whenever $a_2 = a_3 = a_6 = a_7 = 0$, then we get the following values of the parameters

$$\begin{aligned} \gamma = \gamma, a_1 = a_1, a_5 = -\frac{4a_2b\beta(a_4^2\gamma + a_1^2\gamma - 3ma_1^2 - 3ma_4^2 + a_1^2 + a_4^2)}{a_1^2}, \\ \alpha = \alpha, \beta = \beta, b_1 = \frac{\gamma\beta bm}{-\gamma\beta m^2 + \gamma\beta m - bm^2 + \beta}. \end{aligned} \quad (14)$$

then,

$$u_5(x, t) = m + \frac{2b \left(2a_1^2x + 2a_4 \left(a_4x - \frac{4a_2b\beta t (a_4^2\gamma + a_1^2\gamma - 3ma_1^2 - 3ma_4^2 + a_1^2 + a_4^2)}{a_1^2} \right) + \frac{k_1\gamma\beta bme^{k_2t+k_1x}}{\beta - \gamma\beta m^2 + \gamma\beta m - bm^2} \right)}{a_1^2x^2 + \left(a_4x - \frac{4a_2b\beta t (a_4^2\gamma + a_1^2\gamma - 3ma_1^2 - 3ma_4^2 + a_1^2 + a_4^2)}{a_1^2} \right)^2 + \frac{k_1\gamma\beta bme^{k_2t+k_1x}}{\beta - \gamma\beta m^2 + \gamma\beta m - bm^2}}. \quad (15)$$

Set II Whenever $a_2 = a_3 = a_6 = a_7 = 0$, then we get the following values

$$\begin{aligned} a_1 &= \sqrt{-3}a_1, \\ a_5 &= -\frac{4a_2b(2\gamma^2\beta + 12m^2\beta - 10\gamma\beta m - 4m\beta + 2\beta\gamma)}{3(\gamma - 2m)}, \\ \alpha &= \alpha, \beta = \frac{2m\beta - \beta\gamma}{\gamma - 2m}, b_1 = b_1, k_1 = 0. \end{aligned} \quad (16)$$

then,

$$u_6(x, t) = m + \frac{2b \left(-6a_4^2x + 2a_4 \left(a_4x - \frac{4a_4bt \left(2\beta\gamma - 4m\beta - 10m\gamma\beta - \frac{(12m^2 + 2\gamma^2)(2m\beta - \beta\gamma)}{\gamma - 2m} \right)}{3\gamma - 6m} \right) \right)}{-3a_4^2x^2 + \left(a_4x - \frac{4a_4bt \left(2\beta\gamma - 4m\beta - 10m\gamma\beta - \frac{(12m^2 + 2\gamma^2)(2m\beta - \beta\gamma)}{\gamma - 2m} \right)}{3\gamma - 6m} \right)^2 + \frac{m\gamma\beta be^{k_2t}}{\beta m\gamma - bm^2 - \frac{2m\beta - \beta\gamma}{\gamma - 2m} - \beta\gamma m^2}} \quad (17)$$

See Figs. 13, 14, 15, and 16.

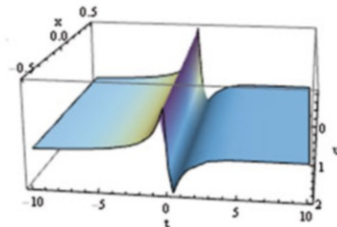
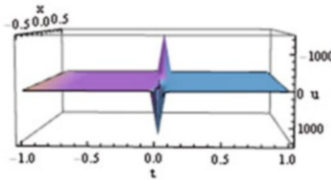
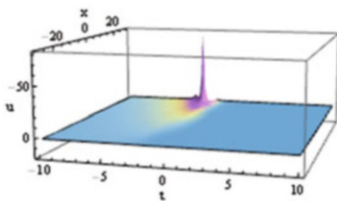
Lump-Two Stripe Solution

To investigate these solutions, we assume the f in Eq. (3) as (Li and Ma 2018),

$$f = F_1^2 + F_2^2 + k_7 + m_1 \exp(G_1) + m_2 \exp(G_2), \quad (18)$$

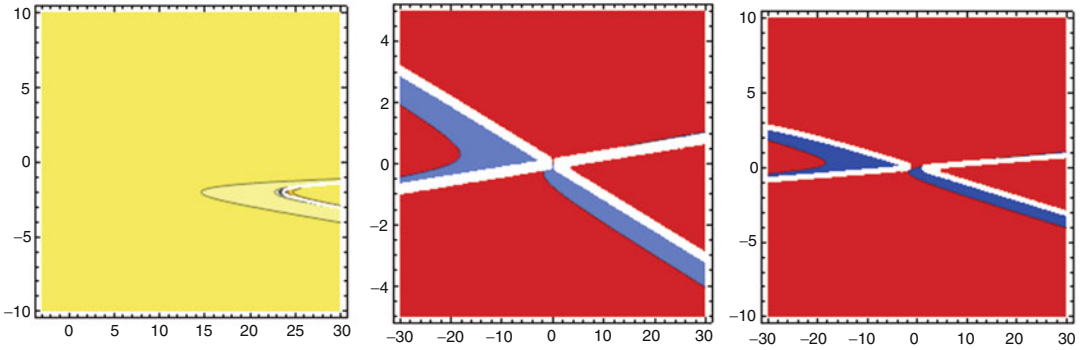
where $F_1 = k_1x + k_2t + k_3$; $F_2 = k_4x + k_5t + k_6$; $G_1 = b_1x + b_2t$; and $G_2 = b_3x + b_4t$. However, $k_i (1 \leq i \leq 7)$, b_1 , b_2 , m_1 , m_2 are some real values to be found. After that, setting assumed f in Eq. (3) and separating the coefficients of t and x :

Set I When $k_3 = a_6 = a_7 = a_8 = b_1 = 0$, then we get the following values of the parameters

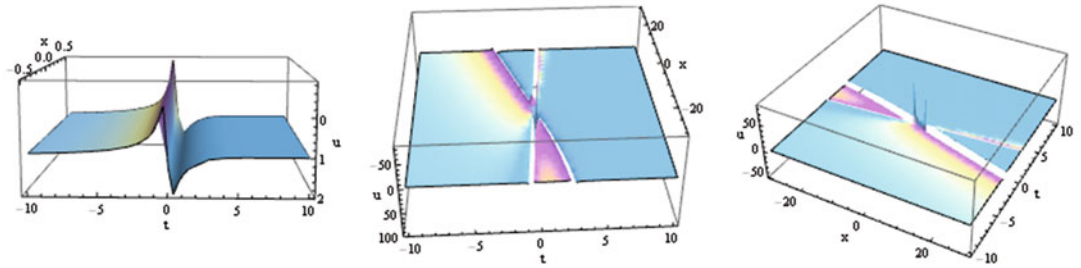


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 13 The lump one-stripe solutions via the solution in Eq. (17) are presented with different parameters

$\alpha = 3$, $\beta = -2$, $m = 1$, $b = 5$, $k_2 = 2$, and $\gamma = 3$. Three-dimensional shapes at (i) $a_4 = 0.01$, (ii) $a_4 = 0.05$, and (iii) $a_4 = 1$, respectively

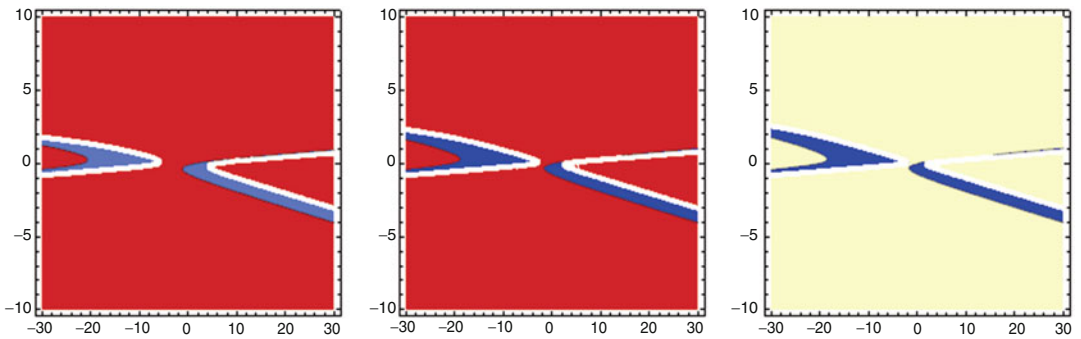


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 14 The contour shapes of lump solutions for Fig. 13 respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 15 The lump one-stripe solutions via the solution in Eq. (17) are presented with different parameters

$\alpha = 3, \beta = -2, m = 1, b = 5, k_2 = 2$, and $\gamma = 3$. Three-dimensional shapes at (i) $a_4 = -0.25$, (ii) $a_4 = -0.50$, and (iii) $a_4 = -0.75$, respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 16 The contour shapes of lump solutions for Fig. 15 respectively

$$k_1 = -\frac{\sqrt{-1}}{50} \frac{k_2^2}{b\alpha k b_3}, k_2 = k_2, k_4 = \frac{k_2^2}{50\alpha k b_3 b}, k_5 = 0, b_2 = b_2, m_2 = m_2, \alpha = \alpha. \quad (19)$$

then,

$$u_7(x, t) = m + \frac{2b \left(m_2 b_3 e^{b_4 t + b_3 x} + \frac{k_2^4 x}{1250\alpha^2 b^2 b_3^2 k^2} - \frac{ik_2^2 \left(k_2 t - \frac{ik_2^2 x}{50\alpha b b_3 k} \right)}{25\alpha b b_3 k} \right)}{m_2 b_3 e^{b_4 t + b_3 x} + \left(k_2 t - \frac{ik_2^2 x}{50\alpha b b_3 k} \right)^2 + \frac{k_2^4 x}{2500\alpha^2 b^2 b_3^2 k^2}}. \quad (20)$$

Set II Whenever $k_3 = a_6 = a_7 = a_8 = b_1 = 0$, then we get the following values

$$k_1 = k_1, k_2 = -\frac{50\alpha b k k_1 k_4^3}{10abk_1^4 + 20abk_1^2 k_4^2 - 9k_1^2 - 18k_1^2 k_4^2 - 9k_4^2}, \quad (21)$$

$$k_4 = k_4, b_2 = b_2, k_5 = \frac{50\alpha k_4^4 b k k_1}{20abk_1^2 k_4^2 - 9k_1^2 + 10abk_1^4 - 18k_1^2 k_4^2 - 9k_4^2}.$$

then,

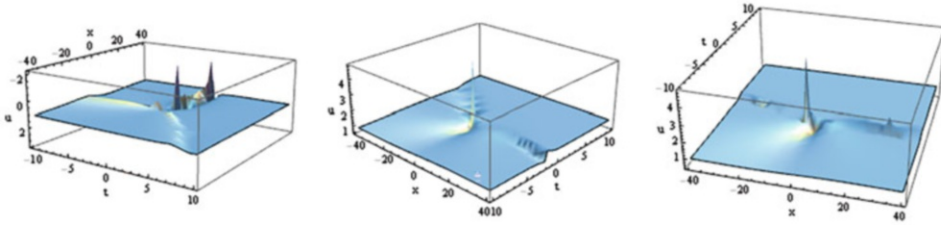
$$u_8(x, t) = m + \frac{2b(m_2 b_3 e^{b_4 t + b_3 x} + 2k_1(k_1 x - \chi_1) + 2k_4(k_4 x - \chi_2))}{m_1 e^{b_4 t} + m_2 e^{b_4 t + b_3 x} + (k_1 x - \chi_1)^2 + (k_4 x - \chi_2)^2}, \quad (22)$$

where $\chi_1 = \frac{50\alpha b k k_1 k_4^3 t}{-9k_1^4 + 10abk_1^4 - 9k_4^2 - 18k_1^2 k_4^2 + 20abk_1^2 k_4^2}$
 $\chi_2 = \frac{50k_4^4 a b k k_1 t}{20abk_1^2 k_4^2 - 9k_1^2 + 10abk_1^4 - 18k_1^2 k_4^2 - 9k_4^2}.$

See Figs. 17, 18, 19, 20, 21, and 22.

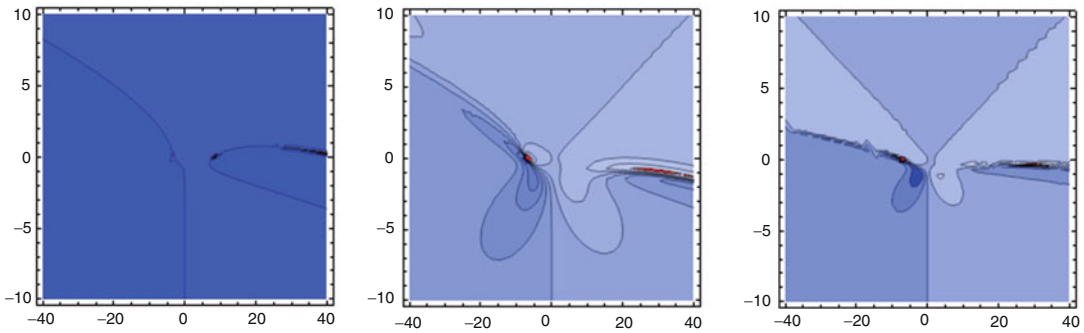
and Lump Periodic Solution

To seek these solutions, first we assume the f in Eq. (3) as (Wu et al. 2018),

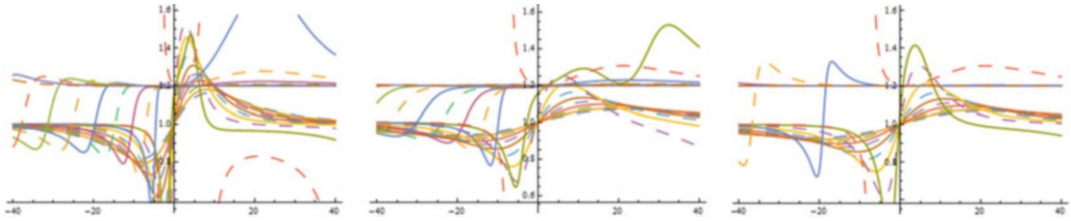


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 17 The lump two-stripe solutions via the solution in Eq. (20) are presented with different parameters

$\alpha = 0.1, m = 1, \beta = -2, b_3 = -1, k = 3, m_2 = 10, b = 1, k_2 = -1$, and $\gamma = 3$. Three-dimensional shapes at (i) $\alpha = 0.1, b_4 = 1$, (ii) $\alpha = 0.25, b_4 = 1$, and (iii) $b_4 = 3, \alpha = 0.25$, respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 18 The contour shapes of solutions for Fig. 17 respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 19 The 2D graphs of solutions for Fig. 17 respectively

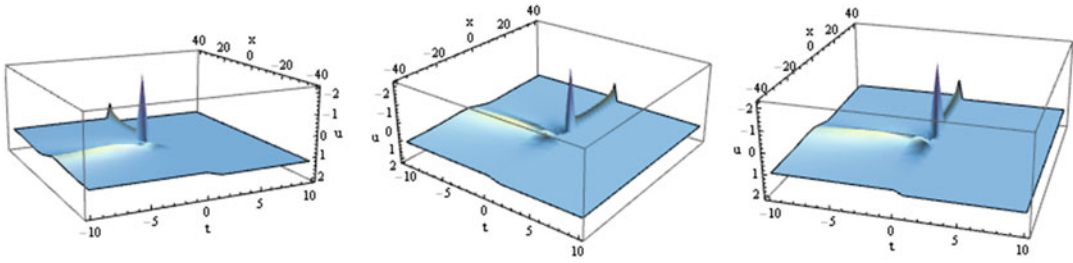
$$f = g^2 + h^2 + a_7 + b_1 \cos(f_1), \quad (23)$$

where $g = a_1x + a_2t + a_3$; $h = a_4x + a_5t + a_6$; and $f_1 = k_1x + p_1t$. However, $a_i (1 \leq i \leq 7)$, b_1, k_1, p_1 , are any constants. After that, setting assumed f in Eq. (3) and separating the coefficients of t, x and cosine function:

Set I

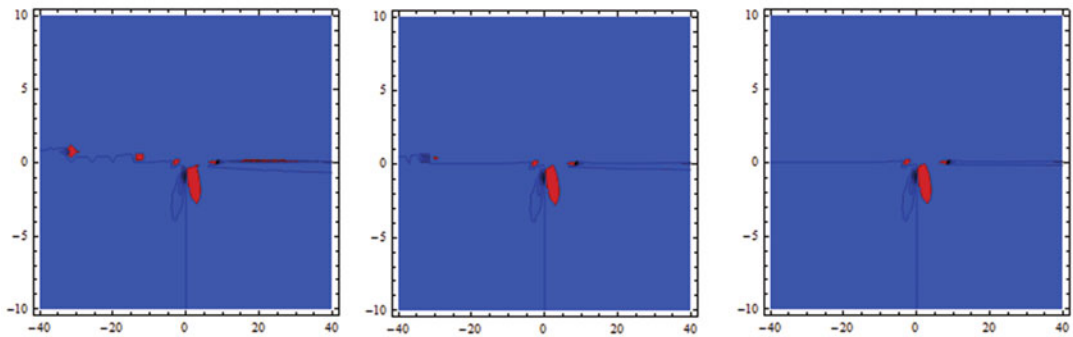
$$\begin{aligned} a_1 &= -\frac{-a_4a_5}{a_2}, a_2 = a_2, a_3 = a_3, a_4 \\ &= a_4, a_5 = a_5, \alpha = \alpha, \beta = \beta, b_1 \\ &= b_1, k_1 = k_1. \end{aligned} \quad (24)$$

then,

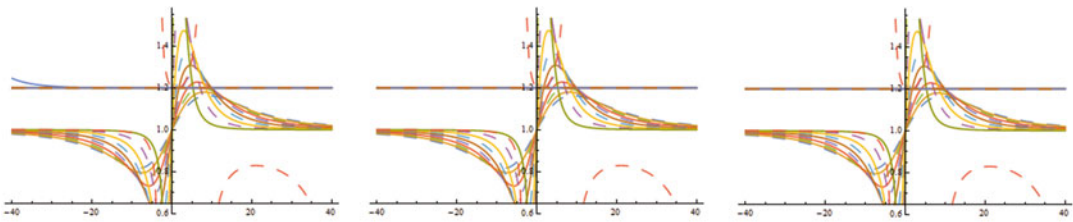


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 20 The lump two-stripe solutions via the solution in Eq. (20) are expressed with various parameters

$\alpha = 0.1, m = 1, \beta = 2, b_3 = -1, k = 3, m_2 = 10, b = 1, k_2 = -1$, and $\gamma = 3$. Three-dimensional shapes at (i) $\alpha = 0.1, b_4 = 10$, (ii) $\alpha = 0.25, b_4 = 20$, and (iii) $b_4 = 50, \alpha = 0.25$, respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 21 The contour shapes of solutions for Fig. 20 respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 22 The 2D graphs of solutions for Fig. 20 respectively

$$u_9(x, t) = m + \frac{2b \left(2a_4(a_6 + a_5t + a_4x) - \frac{2a_4a_5 \left(a_3 + a_2t - \frac{-a_4a_5}{a_2} \right)}{a_2} - b_1k_1 \sin(p_1t + k_1x) \right)}{a_7 + (a_6 + a_5t + a_4x)^2 + \left(a_3 + a_2t - \frac{-a_4a_5}{a_2} \right)^2 + b_1 \cos(p_1t + k_1x)}. \quad (25)$$

Set II

then,

$$a_1 = a_1, a_2 = a_2,$$

$$k_1 = \frac{\sqrt{(4\beta\alpha b - 2\beta)(2\gamma m - 3m^2 - \gamma + 2m)}}{2\alpha b - 1},$$

$$a_3 = a_3, a_4 = a_4, a_5 = a_5, \alpha = \alpha, \beta = \beta, b_1 = b_1.$$

(26)

$$u_{10}(x, t) = m$$

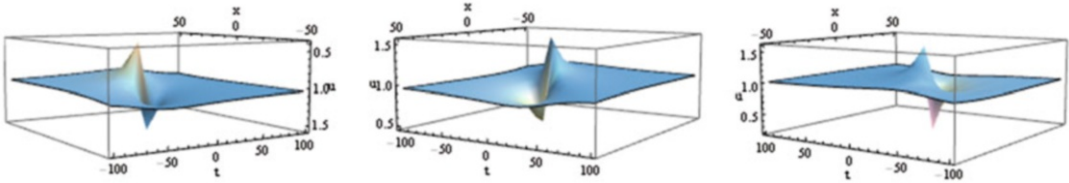
$$+ \frac{2b \left(2a_1(a_3 + a_2t + a_1x) + 2a_4(a_6 + a_5t + a_4x) - \frac{b_1 \sqrt{(4\beta\alpha b - 2\beta)(2\gamma m - 3m^2 - \gamma + 2m)} \sin(\Pi)}{2\alpha b - 1} \right)}{a_7 + (a_3 + a_2t + a_1x)^2 + (a_6 + a_5t + a_4x)^2 + b_1 \cos(\Pi)}, \quad (27)$$

where $\Pi = p_1 t + \frac{\sqrt{(4\beta\alpha b - 2\beta)(2\gamma m - 3m^2 - \gamma + 2m)}}{2\alpha b - 1}$.

Periodic Cross-Kink Solutions

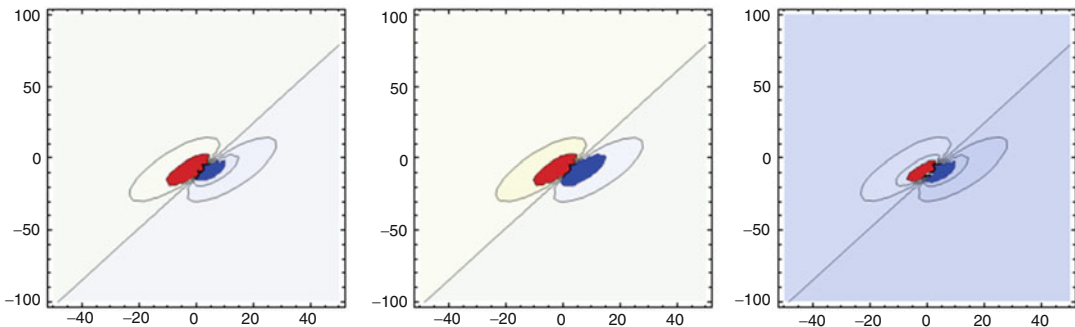
To find these specific solutions, first we consider f in Eq. (3) as,

See Figs. 23, 24, 25, 26, 27, and 28.

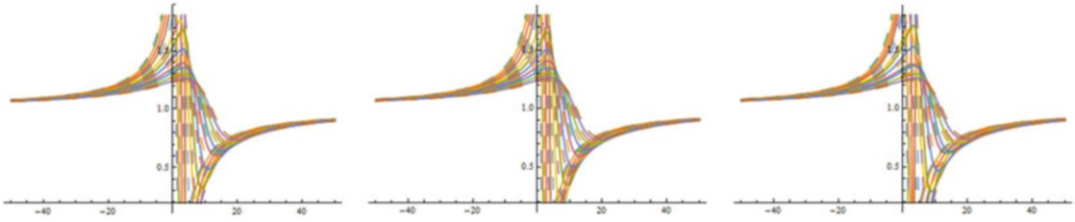


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 23 The graphs of the solution in Eq. (27) are anticipated with appropriate parameters $\alpha = 0.1, m = 1$,

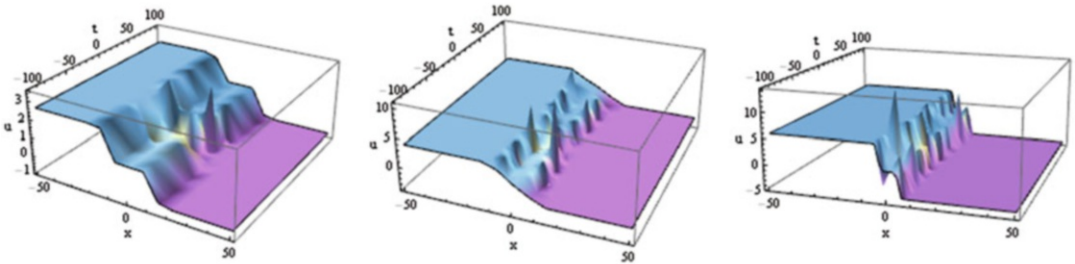
$\beta = 0.5, \gamma = 1, b = -1, a_1 = -1, a_2 = 1, a_3 = 10, a_4 = -1, a_5 = 0.1, a_6 = 3, a_7 = 0.5$, and $p_1 = -2$. Three-dimensional shapes at (i) $b_1 = 1$, (ii) $b_1 = 4$, and (iii) $b_1 = 8$, respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 24 The contour shapes of solutions for Fig. 23 respectively

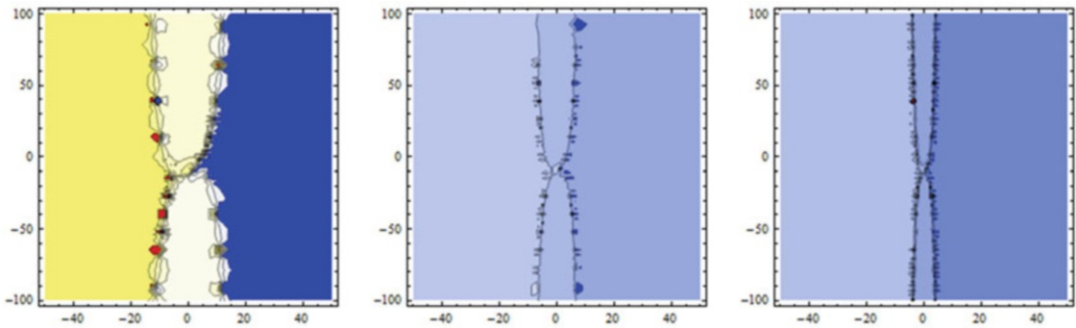


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 25 The 2D graphs of solutions for Fig. 23 respectively

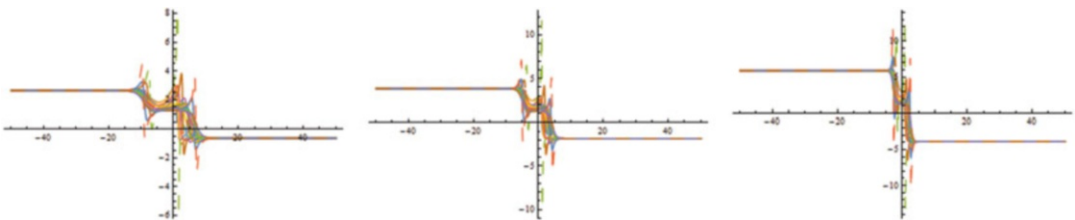


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 26 The lump-periodic graphs of the solution in Eq. (27) are anticipated with appropriate parameters $\alpha =$

$0.1, m = 1, \beta = 0.5, b = -1, b_1 = 1, a_1 = -1, a_2 = 1, a_3 = 10, a_4 = -1, a_5 = 0.1, a_6 = 3, a_7 = 0.5,$ and $p_1 = -2$. Three-dimensional shapes at (i) $\gamma = 1$, (ii) $\gamma = 4$, and (iii) $\gamma = 20$, respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 27 The contour shapes of solutions for Fig. 23 respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 28 The 2D profiles of solutions for Fig. 26 respectively

$$f = e^{-G_1} + m_1 e^{G_1} + m_2 \cos(G_2) + m_3 \cosh(G_3) + m_4, \quad (28)$$

where $G_1 = a_1x + a_2t + a_3$; $G_2 = a_4x + a_5t + a_6$; and $G_3 = a_7x + a_8t + a_9$. However, $a_i (1 \leq i \leq 9)$, $m_i (1 \leq i \leq 4)$ are any parameters. Afterwards, inserting assumed f in Eq. (3) and separating the coefficients of the sine, cosine, hyperbolic, and exponential functions gives us:

$$\begin{aligned} \text{Set I} \\ a_1 &= 0, \\ a_2 &= 2a_7^2\alpha b - 6\beta m^2 + 4\beta m + 4m\beta\gamma - 2\beta\gamma, \\ a_3 &= a_3, a_4 = \sqrt{-1}a_7, a_5 = a_5, a_6 = a_6, \\ \alpha &= \alpha, \beta = \beta, m_1 = 0, m_2 = m_2. \end{aligned} \quad (29)$$

then,

$$u_{11}(x, t) = m + \frac{2b(-ia_7m_2 \sin(a_6x + a_5t + ia_7x) + a_7m_3 \sinh(a_9x + a_8t + a_7x))}{e^{-a_3-t(2a_7^2\alpha b - 6\beta m^2 + 4\beta m + 4m\beta\gamma - 2\beta\gamma)} + m_4 + m_2 \cos(a_6x + a_5t + ia_7x) + m_3 \cosh(a_9x + a_8t + a_7x)}. \quad (30)$$

Set II

$$a_1 = 0, a_2 = 0, a_3 = a_3,$$

$$a_4 = \sqrt{-1}a_7, a_5 = a_5,$$

$$a_6 = a_6,$$

$$\alpha = \frac{2a_7^2\alpha b - 6\beta m^2 + 4\beta m + 4m\beta\gamma - 2\beta\gamma}{a_7^2b},$$

$$\beta = \beta, m_1 = m_1.$$

(31)

then,

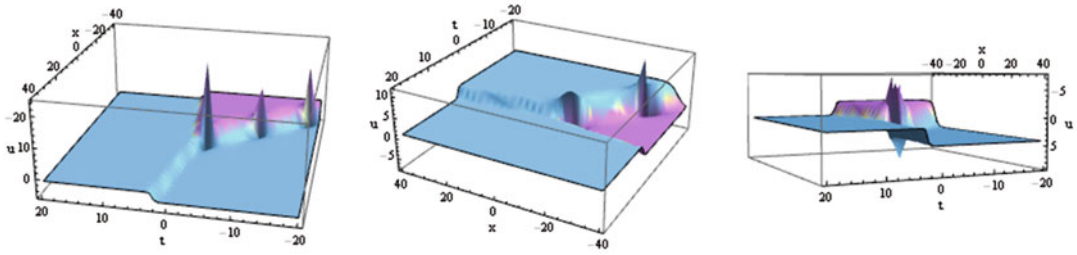
$$u_{12}(x, t) = m + \frac{2b(-ia_7m_2 \sin(a_6x + a_5t + ia_7x) + a_7m_3 \sinh(a_9x + a_8t + a_7x))}{e^{-a_3} + m_4 + m_2 \cos(a_6x + a_5t + ia_7x) + m_3 \cosh(a_9x + a_8t + a_7x)}. \quad (32)$$

See Figs. 29, 30, 31, 32, 33, and 34.

Figures Analysis

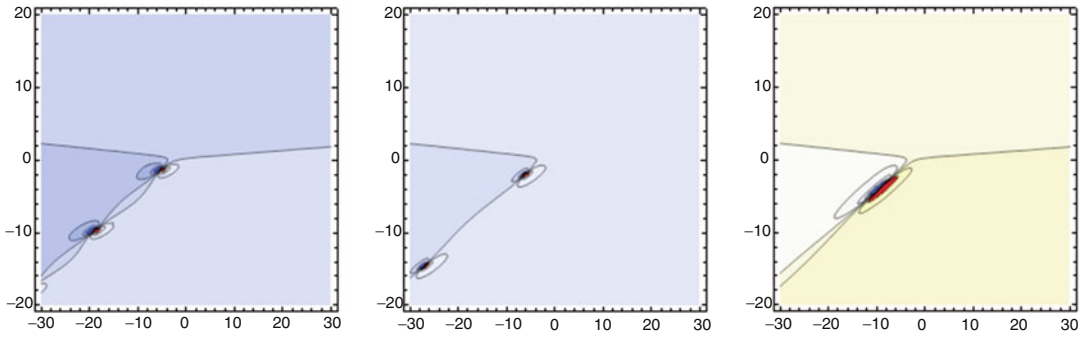
In this section, we present a detailed comparison of our accomplished results with the earlier literature. Many mathematicians have used different schemes for exact solutions of the GBH-equation. Specially, Wang et al. applied nonlinear transformations (Wang et al. 1990), Yefimova et al. got

exact solutions for GBH-equation via Cole-Hopf transformation (Yefimova and Kudryashov 2004), Wazwaz used tanh method to get wave form solutions for GBH-model (Wazwaz 2005), Wang et al. achieved new explicit solutions for proposed model via $(\frac{G'}{G})$ -expansion architectonic (Wang et al. 2013), Hashim et al. applied the Adomian decomposition approach (Hashim et al. 2006), Manafian et al. used $(\frac{G'}{G})$ -expansion approach (Manafian and Lakestani 2015), Batiha et al. utilized VIM to find exact solutions for concerning

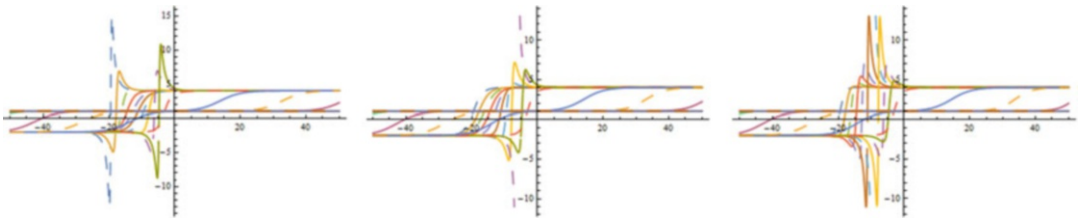


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 29 The periodic cross-kink solutions via the solution in Eq. (32) are presented with different

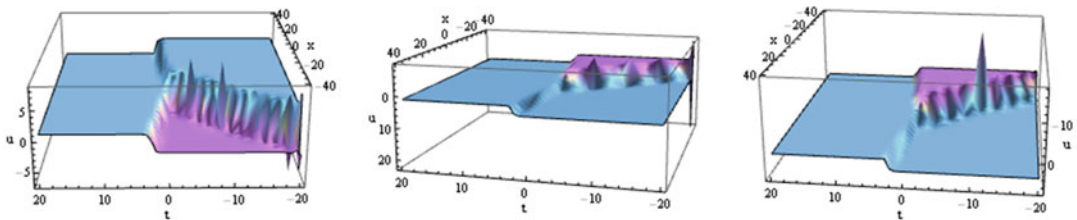
parameters $a_3 = 1$, $a_6 = -1$, $a_7 = 0.3$, $a_8 = -1$, $a_9 = 1$, $m_2 = -1$, $m_3 = 2$, $m_4 = 0.1$, $\alpha = 3$, $\beta = -2$, $\gamma = 3$, $m = 1$, $b = 5$, and $k_2 = 2$. Three-dimensional profiles at (i) $a_5 = -0.25$, (ii) $a_5 = -0.50$, and (iii) $a_5 = -0.75$, respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 30 The contour shapes of solutions for Fig. 29 respectively

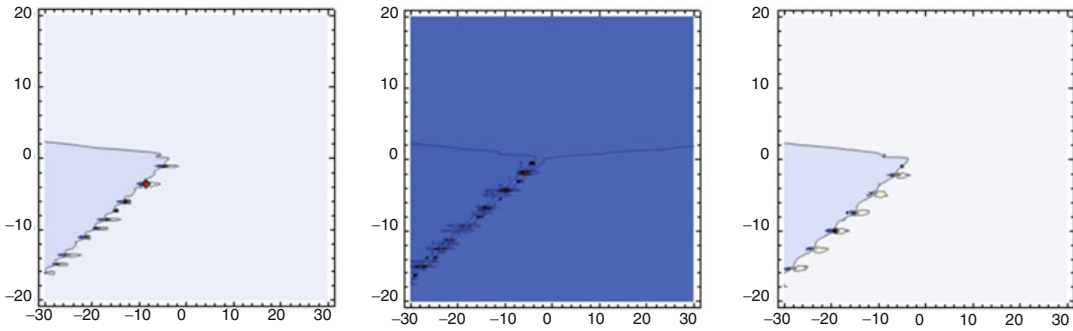


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 31 The 2D profiles of solutions for Fig. 29 respectively

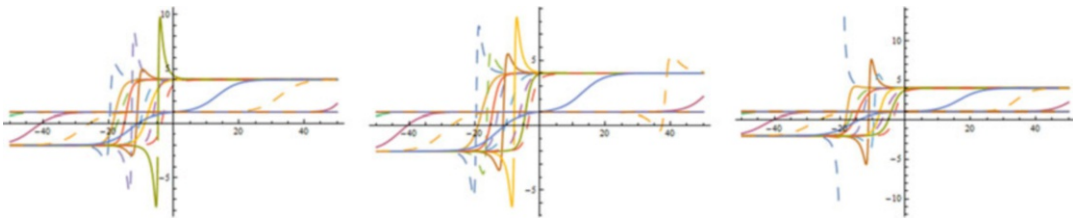


Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers, Fig. 32 The periodic cross-kink solutions via the solution in Eq. (32) are expressed with various

parameters $a_3 = 1$, $a_6 = -1$, $a_7 = 0.3$, $a_8 = -1$, $a_9 = 1$, $m_2 = -1$, $m_3 = 2$, $m_4 = 0.1$, $\alpha = 3$, $\beta = -2$, $\gamma = 3$, $m = 1$, $b = 5$, $k_2 = 2$. Three-dimensional profiles at (i) $a_5 = 55$, (ii) $a_5 = 10$, and (iii) $a_5 = 20$, respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers,
Fig. 33 The contour shapes of solutions for Fig. 32 respectively



Applications of Lump and Interaction Soliton Solutions to the Model of Liquid Crystals and Nerve Fibers,
Fig. 34 The 2D profiles of solutions for Fig. 32 respectively

model (Batiha et al. 2008), Celik used Haar wavelet technique for solving GBH-equation (Celik 2012), Darvishi et al. applied spectral collocation method (Darvishi et al. 2008), I. Celik utilized Chebyshev wavelet collocation method for GBH-equation (Celik 2016), Wang applied variational principle with fractal derivative (Wang 2021), and Marwan et al. used Kudryashov-expansion scheme to find solitary solutions for proposed model (Alquran et al. 2021). But here, the purpose of the work is to find the lump, lump-one stripe, lump-two stripe, and periodic solutions for GBH-equation with the aid of appropriate transformation technique. These solutions have many applications in the fields of science, for instance, oceanographic engineering, capillary flow, physics, biology, finance, liquid crystals, and nonlinear optics (Alquran et al. 2021; Wang et al. 1990; WaZhou et al. 2019; Wang 2018; Li and Ma 2018; Wu et al. 2018). In Fig. 1, the bright-dark lump solutions via the solution in Eq. (6) are presented with different parameters. Three-dimensional profiles at (i) $a_1 = 1, a_2 = 10, a_4 = -2, \alpha = -1$, (ii) $a_1 = 20, a_2 = 30, a_4 = -2$,

$\alpha = 2$, and (iii) $a_1 = 20, a_2 = 40, a_4 = -2, \alpha = 2$, respectively. Figures 2 and 3 show the corresponding contour and 2D profiles for Fig. 1, respectively. Similarly in Fig. 4, the lump solutions via the solution in Eq. (6) are expressed with various parameters. Three-dimensional graphs at (i) $a_1 = 20, a_2 = 50, a_4 = -50, \alpha = 2$, (ii) $a_1 = 40, a_2 = 90, a_4 = -90, \alpha = 2$, and (iii) $a_1 = 40, a_2 = 100, a_4 = -100, \alpha = 9$, respectively. Figures 5 and 6 show the corresponding contour and 2D profiles for Fig. 4, respectively. In Fig. 7, the lump solutions through the solution in Eq. (8) are presented with various parameters $\alpha = 1, \gamma = -1$. Three-dimensional shapes at (i) $a_4 = -1$, (ii) $a_4 = -10$, and (iii) $a_4 = -20$, respectively. Figures 8 and 9 interpret the contour and 2D profiles for Fig. 7, respectively. In Fig. 10, the lump solutions via the solution in Eq. (8) are presented with suitable constants $\alpha = 1, \gamma = -1$. Three-dimensional shapes at (i) $a_4 = 20$, (ii) $a_4 = 30$, and (iii) $a_4 = 40$, respectively. Figures 11 and 12 interpret the contour and 2D graphs for Fig. 10, respectively. In Fig. 13, the lump one-stripe solutions via the solution in Eq. (17) are accomplished

with parameters $\alpha = 3$, $\beta = -2$, $m = 1$, $b = 5$, $k_2 = 2$, and $\gamma = 3$. Three-dimensional graphs at (i) $a_4 = 0.01$, (ii) $a_4 = 0.05$, and (iii) $a_4 = 1$, respectively. Figure 14 interprets the contour graphs for Fig. 13 respectively. In Fig. 15, the lump one-stripe solutions via the solution in Eq. (17) are accomplished with constants $\alpha = 3$, $\beta = -2$, $m = 1$, $b = 5$, $k_2 = 2$, and $\gamma = 3$. Three-dimensional graphs at (i) $a_4 = -0.25$, (ii) $a_4 = -0.50$, and (iii) $a_4 = 0.75$, respectively. Figure 16 shows the contour graphs for Fig. 15 correspondingly. In Fig. 17, the lump two-stripe solutions via the solution in Eq. (20) are presented with parameters $\alpha = 0.1$, $m = 1$, $\beta = -2$, $b_3 = -1$, $k = 3$, $m_2 = 10$, $b = 1$, $k_2 = -1$, and $\gamma = 3$. Three-dimensional shapes at (i) $\alpha = 0.1$, $b_4 = 1$, (ii) $\alpha = 0.25$, $b_4 = 1$, and (iii) $b_4 = 3$, $\alpha = 0.25$, respectively. Figures 18 and 19 show the contour and 2D graphs for Fig. 17, respectively. Similarly in Fig. 20, the lump two-stripe solutions via the solution in Eq. (20) are expressed with various parameters $\alpha = 0.1$, $m = 1$, $\beta = -2$, $b_3 = -1$, $k = 3$, $m_2 = 10$, $b = 1$, $k_2 = -1$, and $\gamma = 3$. Three-dimensional shapes at (i) $\alpha = 0.1$, $b_4 = 10$, (ii) $\alpha = 0.25$, $b_4 = 20$, and (iii) $b_4 = 50$, $\alpha = 0.25$, successively. Figures 21 and 22 show the contour and 2D graphs for Fig. 20, respectively. In Fig. 23, the graphs of the solution in Eq. (27) are anticipated with appropriate parameters $\alpha = 0.1$, $m = 1$, $\beta = 0.5$, $\gamma = 1$, $b = -1$, $a_1 = -1$, $a_2 = 1$, $a_3 = 10$, $a_4 = -1$, $a_5 = 0.1$, $a_6 = 3$, $a_7 = 0.5$, and $p_1 = -2$. Three-dimensional shapes at (i) $b_1 = 1$, (ii) $b_1 = 4$, and (iii) $b_1 = 8$, respectively. Figures 24 and 25 show the contour and 2D graphs for Fig. 23, respectively. On the same manners, in Fig. 26, the lump-periodic graphs of the solution in Eq. (27) are anticipated with parameters $\alpha = 0.1$, $m = 1$, $\beta = 0.5$, $b = -1$, $b_1 = 1$, $a_1 = -1$, $a_2 = 1$, $a_3 = 10$, $a_4 = -1$, $a_5 = 0.1$, $a_6 = 3$, $a_7 = 0.5$, and $p_1 = -2$. Three-dimensional shapes at (i) $\gamma = 1$, (ii) $\gamma = 4$, and (iii) $\gamma = 20$, respectively. Figures 27 and 28 present the concerning contour and 2D profiles for Fig. 26, successively. In Fig. 29, the periodic cross-kink wave solutions via the solution in Eq. (32) are presented with different parameters $a_3 = 1$, $a_6 = -1$, $a_7 = 0.3$, $a_8 = -1$, $a_9 = 1$, $m_2 = -1$, $m_3 = 2$, $m_4 = 0.1$, $\alpha = 3$, $\beta = -2$, $\gamma = 3$, $m = 1$, $b = 5$, and $k_2 = 2$. Three-dimensional profiles at (i) $a_5 = -0.25$, (ii) $a_5 = -0.50$, and

(iii) $a_5 = -0.75$, respectively. Figures 30 and 31 present the concerning contour and 2D profiles for Fig. 29, successively. Similarly in Fig. 32, the periodic cross-kink solutions through solution in Eq. (32) are expressed with various values of $a_3 = 1$, $a_6 = -1$, $a_7 = 0.3$, $a_8 = -1$, $a_9 = 1$, $m_2 = -1$, $m_3 = 2$, $m_4 = 0.1$, $\alpha = 3$, $\beta = -2$, $\gamma = 3$, $m = 1$, $b = 5$, and $k_2 = 2$. Three-dimensional profiles at (i) $a_5 = 55$, (ii) $a_5 = 10$, and (iii) $a_5 = 20$, respectively. Finally, Figs. 33 and 34 show the contour and 2D profiles for Fig. 32, successively.

Concluding Remarks

This entry retrieved some novel lump, lump-one stripe, lump-two stripe, lump periodic, and periodic cross-kink solutions for the generalized Burger Huxley equation via appropriate transformation technique. Moreover, we analyzed a class of analytical rational lump-type solutions in detail, which are generated from positive quadratic polynomial function and rationally localized in many directions in the space, based upon bilinear form. Associated three-dimensional, two-dimensional, and contour plots along with suitable choices of the involved parameters are given to anticipate the characteristics of the solutions.

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Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation

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Article Outline

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Introduction

Fractional calculus emerge in nonlinear sciences having so many applications in various physical systems (Dalir and Bashour 2010). Nonlinear fractional partial differential equations (NLPDEs) have so many uses in many areas of science like hydrodynamics, kinematics, fluid mechanics, optical fibers, biology, plasma and solid-state physics, etc. Majority of the solutions for NLPDEs in mathematical physics are analytical solution. These solutions have been studied in literature by many integration schemes (Khater et al. 2000, 2006a,b; Younas et al. 2021; Akram et al. 2021; Helal and Seadawy 2009, 2011, 2012; Baleanu

et al. 2012; Zhang et al. 2010; Chen et al. 2020). Most of the fractional models have been studied in literature through different schemes like sin-cosine technique (Korkmaz et al. 2018), Kudryashov method (Akbari 2017), extended trial equation scheme (Gepreel 2016), tanh approach (Malfliet 2004), $\frac{G'}{G}$ -expansion architectonic (Helal et al. 2013), and many more are used to find the analytic solutions for NLPDEs (Ullah et al. 2021; Helal et al. 2014, 2017; Seadawy et al. 2019, 2018; Ali et al. 2018; Arshad et al. 2017; Ahmed et al. 2019; Cheemaa et al. 2018, 2019; Ozkan et al. 2020; Seadawy and Cheemaa 2020; Rizvi et al. 2020).

In the last few years, a lot of practical and analytical work has been done on NLPDEs to understand the wave properties of nonlinear systems. Lump waves are rationally localized solutions in all directions. Lump two-kink solution is the special kind of solution for nonlinear Schrödinger equation (NLSE). This kind of solution is generally used for several essential applications including the generation of super-continuum pulses or pulse compression. Another very renowned wave phenomenon is rogue (killer or monster) waves appear mostly in ocean. These waves have so many applications in plasma physics, optical fiber, and super fluid (Elboree 2019). Song and Xue retrieved first- and second-order rogue waves of NLSE with time-dependent linear potential function and also explained the nonlinear dynamic behavior in detail (Song and Xue 2016). Ilhan and Manafian obtained the periodic and periodic cross-kink wave solutions with the aid of Hirota bilinear technique (Ilhan and Manafian 2019). Ren et al. retrieved the lump soliton and lump-periodic soliton solutions of NLPDE (Ren et al. 2019). Chen and Xiu obtained the lump solution of generalized Bogoyavlenskii–Konopelchenko equation by using log transformation (Chen and Ma 2018). Gepreel and Nofal discussed the wave solutions of some NLPDEs by using extended trial equation technique (Gepreel and Nofal 2015). The governing equation for TFBE is given as follows:

$$4D_t^\beta \Omega + \Omega_{xxy} - \Omega^2 \Omega_y - 4\Omega_x \Psi = 0, \quad (1)$$

$$\Psi_x = \Omega \Omega_y,$$

where $t > 0$ and $0 < \beta \leq 1$, in which $\Omega(x, y, t)$ and $\Psi(\Omega(x, y, t))$ are unknown operation. Conformable fractional derivative is defined as,

$$D_t^\beta(p(t)) = \lim_{h \rightarrow 0} \frac{p(t + ht^{1-\beta}) - p(t)}{h}. \quad (2)$$

Equation (2) consists of following properties (Zheng et al. 2019).

1. $D_t^\beta(p(t) \pm q(t)) = D_t^\beta(p(t)) \pm D_t^\beta(q(t))$
2. $D_t^\beta(p(t)q(t)) = D_t^\beta(p(t))q(t) \pm D_t^\beta(q(t))p(t)$
3. $D_t^\beta(p(t)/q(t)) = \frac{D_t^\beta(p(t))q(t) - D_t^\beta(q(t))p(t)}{q(t)^2}$
4. $D_t^\beta(p(t)) = t^{1-\beta} D_t^\beta(p(t))$

We use the following conformal transformation,

$$\tau = \frac{t^\beta}{\beta}. \quad (3) \quad \text{and}$$

Using Eq. (3) into Eq. (1) we get,

$$4\Omega_\tau + \Omega_{xxy} - \Omega^2 \Omega_y - 4\Omega_x \Psi = 0, \quad (4)$$

$$\Psi_x - \Omega \Omega_y = 0.$$

This entry is organized in the following manner: in the section “**Rogue Waves**,” we will study rogue waves; in the section “**Periodic Cross-Kink Wave Solution**,” we will discuss PCKW solution; in the section “**Lump Two-Kink**,” we will obtain lump with two kink solution; in the section “**Periodic Wave**,” we analyze periodic waves; in the section “**Figures Analysis**,” we will discuss our results in details; and finally in the section “**Conclusion**,” we will give concluding remarks.

Rogue Waves

In this section, we retrieve the solution of rogue waves, which contain the sum of quadratic and

hyperbolic trigonometry functions. By using following transformation,

$$\Omega(x, y, t) = 2b(\ln g)_y, \quad (5)$$

$$\Psi(x, y, t) = 2(\ln f)_{xx}.$$

Now substituting Eq. (5) into Eq. (4) we get,

$$4f^2 g^2 g_\tau g_y - 4b^2 f^2 g_y^4 - 4f^2 g^3 g_y \tau$$

$$+ 4b^2 f^2 g g_y^2 g_{yy} + 8g^2 g_y f_x^2 g_x + 6f^2 g_y^2 g_x^2$$

$$- 2f^2 g g_{yy} g_x^2 - 8g^3 f_x^2 g_x y - 8f^2 g g_y g_x g_{xy}$$

$$+ 2f^2 g^2 g_x y^2 + 2f^2 g^2 g_x g_x y - 8f g^2 g_y g_x f_{xx}$$

$$+ 8f g^3 g_x y f_{xx} - 2f^2 g g_y^2 g_{xx} + f^2 g^2 g_{yy} g_{xx}$$

$$+ 2f^2 g^2 g_y g_{xx} y - f^2 g^3 g_{xx} y y$$

$$= 0 \quad (6)$$

$$-2b^2 f^3 g_y^3 + 2b^2 f^3 g g_y g_{yy} - 2g^3 f_x^3$$

$$+ 3f g^3 + f_x f_{xx} - f^2 g^3 f_{xxx} = 0 \quad (7)$$

For rogue waves, we assume the following function g and f :

$$g = \xi_1^2 + \xi_2^2 + \cosh(\eta) + a_9, \quad (8)$$

$$f = \xi_1^2 + \xi_2^2 + \cosh(\eta) + a_{10}$$

where

$$\xi_1 = a_4 + a_3 \tau + a_2 y + a_1 x,$$

$$\xi_2 = a_8 + a_7 \tau + a_6 y + a_5 x, \quad \text{and}$$

$$\eta = b_1 x + b_2 y + b_3 \tau.$$

However, $a_i (1 \leq i \leq 10)$ and $b_i (i = 1, 2, 3)$ are real constants to be determined. Put Eq. (8) into Eqs. (6) and (7). By putting all the coefficient of the x, y, τ to be zero, we retrieve some equations which give values of coefficient such as:

Set 1

$$\begin{aligned}
a_1 &= 0, a_2 = \sqrt{-3}a_6, a_3 = 0, a_4 = a_4, a_5 = a_5, \\
a_6 &= a_6, a_7 = a_7, \\
a_8 &= \frac{-1}{3}b^2\sqrt{-3}a_4, \\
a_9 &= a_9, \\
a_{10} &= \frac{1}{9} \frac{48a_6^2 - 12a_4^2b_2^2 - 3a_9b_2^2 - 24a_4^2b^2b_2^2 + 28b^4a_4^2b_2^2}{b_2^2}, \\
b_1 &= 0, b_2 = b_2, b_3 = b_3
\end{aligned} \tag{9}$$

Substituting Eq. (9) into Eq. (8) and then by using Eq. (5) and Eq. (3) to retrieve the rogue wave solution of Eq. (1),

$$\begin{aligned}
\Omega &= \frac{2b \left(2i\sqrt{3}a_6\delta_1 + 2a_6\delta_2 + \sinh \left(yb_2 + \frac{t^\beta b_3}{\beta} \right) b_2 \right)}{\cosh \left(yb_2 + \frac{t^\beta b_3}{\beta} \right) + \delta_1^2 + \delta_2^2 + a_9}, \\
\Psi &= \frac{2 \left(-4a_5^2\delta_2^2 + 2a_5^2 \left(\cosh \left(yb_2 + \frac{t^\beta b_3}{\beta} \right) + \delta_1^2 + \delta_2^2 + \epsilon_1 \right) \right)}{\left(\cosh \left(yb_2 + \frac{t^\beta b_3}{\beta} \right) + \delta_1^2 + \delta_2^2 + \epsilon_1 \right)^2}
\end{aligned} \tag{10}$$

where $\delta_1 = a_4 + i\sqrt{3}ya_6$, $\delta_2 = \frac{-ib^2a_4}{\sqrt{3}} + xa_5 + ya_6 + \frac{t^\beta a_7}{\beta}$, and $\epsilon_1 = \frac{48a_6^2 - 12a_4^2b_2^2 - 24b^2a_4^2b_2^2 + 28b^4a_4^2b_2^2 - 3a_9b_2^2}{9b_2^2}$. Now, we discuss some graphical representation of the above solution.

$$\begin{aligned}
g &= e^{-\xi_1} + b_1 e^{\xi_1} + b_2 \cos(\xi_2) + b_3 \cosh(\xi_3) + a_{11} \\
f &= e^{-\xi_1} + k_1 e^{\xi_1} + k_2 \cos(\xi_2) + k_3 \cosh(\xi_3) + a_{12}
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\xi_1 &= a_4 + a_3\tau + a_2y + a_1x, \\
\xi_2 &= a_7\tau + a_6y + a_5x \quad \text{and} \\
\xi_3 &= a_8x + a_9y + a_{10}\tau
\end{aligned}$$

Periodic Cross-Kink Wave Solution

In this section, we discuss the periodic cross-kink solution which contain exponential, trigonometry, and hyperbolic trigonometry function with free parameters, and we obtain the periodic cross-kink wave solution for Eq. (1). Now we assume that,

where $a_i (i = 1, 2, \dots, 12)$, $b_i (i = 1, 2, 3)$, and $k_i (i = 1, 2, 3)$ are real parameters. Substituting Eq. (11) into Eq. (6) and Eq. (7), and by removing all the coefficient of exp, cosh, cos, sin, sinh to be

zero, we obtain some system of equations. By **Set 1** solving that system of equations, we retrieve the values of the coefficient which are given below:

$$\begin{aligned}
 a_1 = 0, a_2 &= \frac{\sqrt{-\frac{2b^2a_6^3b_2^2 + 3a_5^3b_2k_2 + 2a_5^3k_2^2}{2a_6b_2^2 - 12a_6b_2k_2}}}{b}, \\
 a_3 = 0, a_4 &= a_4, a_5 = a_5, a_6 = a_6 \\
 a_7 &= \frac{1}{2} \frac{2a_6b_2^3a_5^2 - a_5^3b_2^2k_2 - 13a_6b_2^2a_5^2k_2 + 8k_2b^2a_6^3b_2^2 - 8a_5^3b_2k_2^2 + 6a_6b_2k_2^2a_5^2 + 4a_5^3k_2^3}{(b_2(-b_2 + 6k_2)(b_2 + k_2))}, \\
 a_8 &= a_8, a_9 = 0, a_{10} = a_{10}, a_{11} = a_{11}, \\
 a_{12} &= a_{12}, b_1 = 0, b_2 = b_2, \\
 b_3 &= \frac{-2b_2k_3(a_8a_6a_5 - 2a_8^3 + a_{10})}{(k_2(6a_8a_5^2 - a_8a_6a_5 - 2a_8^3 + 2a_{10}))}, \\
 k_1 &= 0, k_2 = k_2, k_3 = k_3,
 \end{aligned} \tag{12}$$

Put Eq. (12) into Eq. (11) and also substituting Eq. (11) into Eq. (5) along with Eq. (3), we retrieve the periodic cross-kink solution for Eq. (1).

$$\begin{aligned}
 \Omega &= \frac{2b \left(-\sin(\theta_1)a_6b_2 - \frac{e^{-a_4 - \frac{y\theta_2}{b}}}{b} \theta_2 \right)}{e^{-a_4 - \frac{y\theta_2}{b}} + a_{11} + \cos(\theta_1)b_2 - \frac{2 \cosh(\theta_3)(a_5a_6a_8 - 2a_8^3 + a_{10})b_2k_3}{(6a_5^2a_8 - a_5a_6a_8 - 2a_8^3 + 2a_{10})k_2}} \\
 \Psi &= \frac{\left(2 \left(-\sin(\theta_1)a_5k_2 + \sinh(\theta_3)a_8k_3 \right)^2 + \left(e^{-a_4 - \frac{y\theta_2}{b}} + a_{12} + \cos(\theta_1)k_2 \cosh(\theta_3)k_3 \right) \left(-\cos(\theta_1)a_5^2k_2 + \cosh(\theta_3)a_8^2k_3 \right) \right)}{\left(e^{-a_4 - \frac{y\theta_2}{b}} + a_{12} + \cos(\theta_1)k_2 + \cosh(\theta_3)k_3 \right)^2}
 \end{aligned} \tag{13}$$

where $\theta_1 = xa_5 + ya_6 + t^\beta (2a_5^2a_6b_2^3 - a_5^3b_2^2k_2 - 13a_5^2a_6b_2^2k_2 + 8b^2a_6^3b_2^2k_2 - 8a_5^3b_2k_2^2 + 6a_5^2a_6b_2k_2^2 + 4a_5^3k_2^3)$

$$\theta_2 = \sqrt{\frac{2b^2a_6^3b_2^2 - 3a_5^3b_2k_2 - 2a_5^3k_2^2}{2a_6b_2^2 - 12a_6b_2k_2}}, \quad \theta_3 = xa_8 + \frac{t^\beta a_{10}}{\beta}.$$

Therefore we retrieve a solution of Eq. (1) and

discuss some graphical representation of the above solution that are given below:

Lump Two-Kink

In this section, we discuss lump two-kink which contain quadratic and exponential function. We

retrieve the lump two-kink solution of the Eq. (1). Now we assume that

$$\begin{aligned} g &= \zeta_1^2 + \zeta_2^2 + e^{G_1} + e^{G_2} + a_9 \\ f &= \zeta_1^2 + \zeta_2^2 + e^{G_1} + e^{G_2} + a_{10} \end{aligned} \quad (14)$$

where

$$\begin{aligned} \zeta_1 &= a_4 + a_3\tau + a_2y + a_1x, \\ \zeta_2 &= a_8 + a_7\tau + a_6y + a_5x \\ G_1 &= b_1x + b_2y + b_3\tau, \quad \text{and} \\ G_2 &= b_4x + b_5y + b_6\tau \end{aligned}$$

where $a_i (1 \leq i \leq 10)$ and $b_i (1 \leq i \leq 6)$ are real constants to be set up. Using Eq. (14) into Eq. (6) and Eq. (7), and by putting all the coefficient of x , y , τ , and $\exp$ to be zero, we retrieve some system of equations. By solving that system of equations,

we obtain the values of the constant which are given below:

Set 1

$$\begin{aligned} a_1 &= 0, a_2 = a_2, a_3 = 0, a_4 = a_4, a_5 = a_5, \\ a_6 &= \frac{1}{5}i\sqrt{5}a_2, a_7 = a_7, \\ a_8 &= \frac{1}{5}ia_4\sqrt{5}, \\ a_9 &= a_9, a_{10} = \frac{-1 - a_2^2 - 10a_2a_4b_2 + 15a_4^2b_2^2}{b_2^2}, \\ b_1 &= 0, b_2 = b_2, b_3 = \frac{1}{2}b^2b_2^3, \\ b_4 &= b_4, b_5 = b_5, b_6 = b_6, m_1 = m_1, m_2 = m_2 \end{aligned} \quad (15)$$

Put Eq. (15) into Eq. (14) and substituting Eq. (14) into Eq. (5) along with Eq. (3), we get the following solutions

$$\begin{aligned} \Omega &= \frac{2b \left(2a_2(ya_2 + a_4) + \frac{2ia_2\zeta_1}{\sqrt{5}} + e^{yb_2 + \frac{b^2i\beta b_2^3}{2\beta}}b_2m_1 + e^{xb_4 + yb_5 + \frac{i\beta b_6}{\beta}}b_5m_2 \right)}{(ya_2 + a_4)^2 + \zeta_1^2 + a_9 + e^{yb_2 + \frac{b^2i\beta b_2^3}{2\beta}}m_1 + e^{xb_4 + yb_5 + \frac{i\beta b_6}{\beta}}m_2} \\ \Psi &= \frac{\left(2 \left(- \left(2a_5\zeta_1 + e^{xb_4 + yb_5 + \frac{i\beta b_6}{\beta}}b_4m_2 \right)^2 + \left((ya_2 + a_4)^2 + \zeta_1^2 - \zeta_2 + e^{yb_2 + \frac{b^2i\beta b_2^3}{2\beta}}m_1 + e^{xb_4 + yb_5 + \frac{i\beta b_6}{\beta}}m_2 \right) \zeta_3 \right) \right)}{\left((ya_2 + a_4)^2 + \zeta_1^2 - \zeta_2 + e^{yb_2 + \frac{b^2i\beta b_2^3}{2\beta}}m_1 + e^{xb_4 + yb_5 + \frac{i\beta b_6}{\beta}}m_2 \right)^2}, \end{aligned} \quad (16)$$

where $\zeta_1 = \frac{iy a_2}{\sqrt{5}} + \frac{ia_4}{\sqrt{5}} + xa_5 + \frac{i\beta a_7}{\beta}$, $\zeta_2 = \frac{-a_2^2 - 10a_2a_4b_2 + 15a_4^2b_2^2}{5b_2^2}$, and $\zeta_3 = 2a_5^2 + e^{xb_4 + yb_5 + \frac{i\beta b_6}{\beta}}b_4^2m_2$. Therefore we obtain a solution of Eq. (1). Now we study some graphical representation of the above solution.

Periodic Wave

For periodic wave of Eq. (1), we assume

$$\begin{aligned} g &= \zeta_1^2 + \zeta_2^2 + z_1 \cos(\eta) + a_9 \\ f &= \zeta_1^2 + \zeta_2^2 + z_1 \cos(\eta) + a_{10} \end{aligned} \quad (17)$$

where

$$\begin{aligned} \zeta_1 &= a_4 + a_3\tau + a_2y + a_1x, \\ \zeta_2 &= a_8 + a_7\tau + a_6y + a_5x \quad \text{and} \\ \eta &= n_1x + n_2y + n_3\tau. \end{aligned}$$

$a_i (i = 1, \dots, 10)$ and $n_i (i = 1, 2, 3)$ are real constants to be measured. Put Eq. (17) into

Eq. (6) and Eq. (7), and by removing the coefficient of $\cos$, $\sin$, x , y , τ , we retrieve values of the parameters which are given below:

Set 1

$$\begin{aligned} a_1 &= 0, \\ a_2 &= \sqrt{-5a_6}, a_3 = 0, a_4 = a_4, a_5 = a_5, a_6 = a_6, \\ a_7 &= \frac{-1}{10} \frac{n_2^2 (10b^2 a_6^2 - a_5^2) z_1}{a_6(z_1 + z_2)}, \\ a_8 &= \frac{-1}{5} \sqrt{-5a_4}, a_9 = 0, \end{aligned}$$

$$\begin{aligned} a_{10} &= -\frac{2(5b^2 a_6^2 - 8a_5^2) a_4^2}{10b^2 a_6^2 - a_5^2}, n_1 = 0, n_2 = n_2, \\ n_3 &= \frac{-1}{20} \frac{(10b^2 a_6^2 - a_5^2) n_2}{a_4^2} \end{aligned} \quad (18)$$

Put these values in Eq. (17) and also substituting Eq. (17) into Eq. (5) with (3), we get the following solutions

$$\begin{aligned} \Omega &= \frac{2b(2i\sqrt{5}a_6(a_4 + i\sqrt{5}ya_6) - \sin(\vartheta_1)n_2z_1 + 2a_6\mu_1)}{(a_4 + i\sqrt{5}ya_6)^2 + \cos(\vartheta_1)z_1 + \mu_1^2} \\ \Psi &= \frac{2(-4a_5^2\mu_1^2 + 2a_5^2((a_4 + i\sqrt{5}ya_6)^2 - \mu_2 + \cos(\vartheta_1)z_2 + \mu_1^2))}{((a_4 + i\sqrt{5}ya_6)^2 - \mu_2 + \cos(\vartheta_1)z_2 + \mu_1^2)^2} \end{aligned} \quad (19)$$

where $\vartheta_1 = yn_2 - \frac{t^\beta(-a_5^2 + 10b^2a_6^2)n_2}{20\beta a_4^2}$, $\mu_1 = -\frac{ia_4}{\sqrt{5}} + xa_5 + ya_6 - \frac{t^\beta(-a_5^2 + 10b^2a_6^2)n_2^2z_1}{10\beta a_6(z_1 + z_2)}$, and $\mu_2 = \frac{2a_4^2(-8a_5^2 + 5b^2a_6^2)}{-a_5^2 + 10b^2a_6^2}$. Therefore we obtain a periodic wave solution of Eq. (1), and the graphical representation are given below:

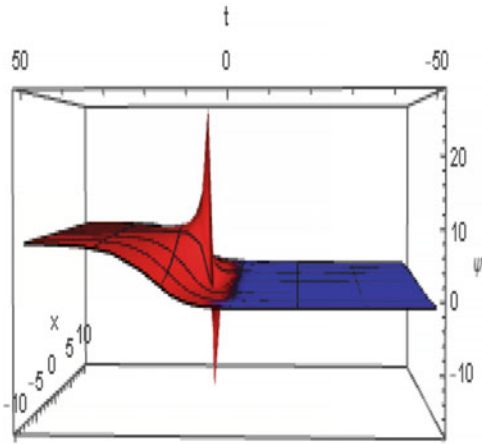
Figures Analysis

In this section, we will study the physical explanation of the graphs. Salmi et al. obtained the analytic solution of TFBE by using the first integral architectonic and also found the generalized trigonometric and hyperbolic solutions (Eslami et al. 2017). Hater et al. retrieved the elliptic and solitary wave solutions for Bogoyavlenskii equation by using the Khater method (Akuamoah and Seadawy 2020). However, we obtained some new types of solutions for our governing model.

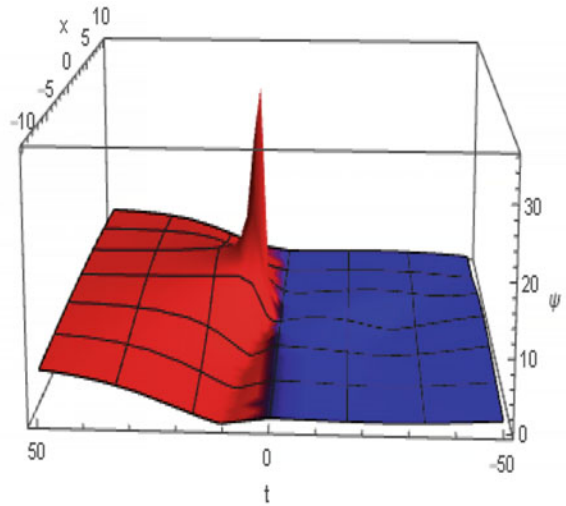
Rogue waves have been observed in liquid helium, in nonlinear optics, and in microwave cavities other than water. The graphics of rogue wave solution of Eq. (1) are shown in Figs. 1, 2, 3, 4, 5, and 6. We study the graphs at different moments $\beta = 0.5, \beta = 0.3, \beta = 0.8$, and Fig. 1a–c presents the 3D plots of Ω at different variations, and we observe the bright wave and also a dark wave. Figure 2 and 3 represent the contour and 2D plots of Ω . The solution of Ψ shows same properties at variation $\beta = 0.5, \beta = 0.8, \beta = 0.9$ in Figs. 4, 5, and 6. Graphs of periodic cross-kink solution of Eq. (1) are presented in Figs. 7, 8, 9, 10, 11, and 12. We observe different graphs when we assume various assumption $\beta = 0.9, \beta = 0.5$, and $\beta = 0.7$. Figure 7a–c represent the 3D plots of Ω in which we study bright waves. Figures 8 and 9 represent the contour and 2D plots of Ω . The solution of Ψ shows same properties at variation $\beta = 0.1, \beta = 0.4$, and $\beta = 0.7$ in Figs. 10, 11, and 12. Now we study the graphical representation of lump two-kink of Eq. (1) which are given in Figs. 13, 14,

15, 16, 17, and 18. Figure 13a represent one bright wave at $y = 0$, and Fig. 13b, c presents the bright solution at the height of crust at $y = 2, y = -2$. In Fig. 13d, e, no lump solution occur at $y = 15, y = -15$. Figures 14 and 15 represent the contour and 2D plots of Ω . We discuss the graphs of Ψ , Fig. 16a

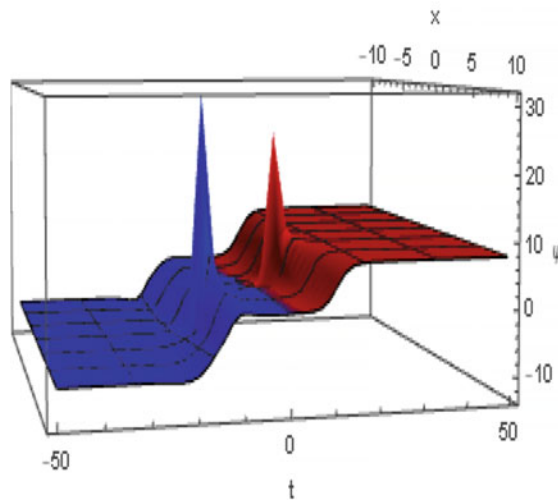
represents the two bright wave and one dark solution at $y = 0$, and Fig. 16b show that bright solution at $y = 2$. Figure 16c, e no solution are occur and Fig. 16d the graph are discontinues. Figures 17 and 18 represent the contour and 2D plots of Fig. 16. The graphical representation of periodic wave



(a) $\beta = 0.5$



(b) $\beta = 0.3$



(c) $\beta = 0.8$

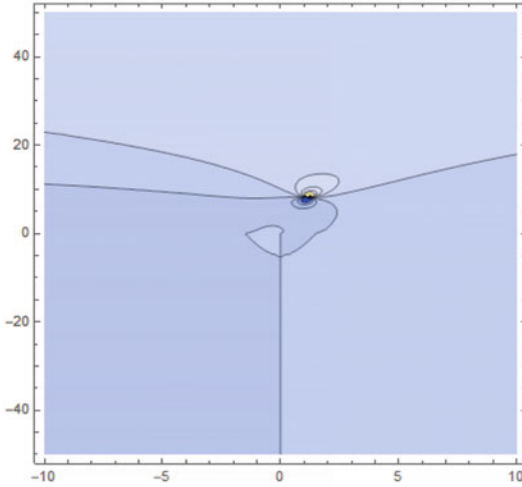
Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation,

Fig. 1 Plot 3D of $\Omega(x, y, t)$ in Eq. (10), for $b = 2, a_6 = 7, a_5 = 5, a_7 = -1, a_9 = 4, a_4 = 6, b_2 = 2, b_3 = 1, y = 0$ respectively

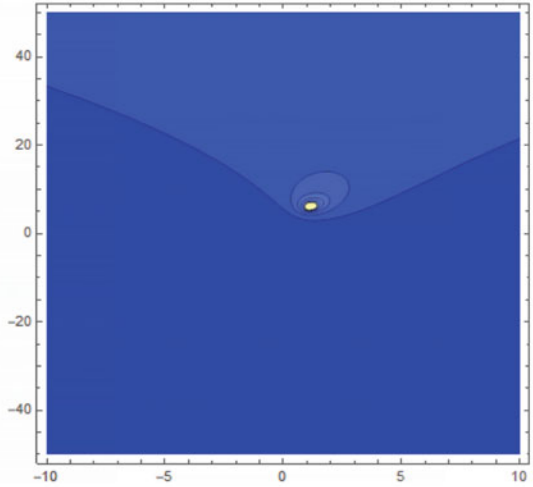
solution of Eq. (1) are shown in Figs. 19, 20, 21, 22, 23, and 24 in which we present 3D plots at different variation $\beta = 0.7$, $\beta = 0.8$, and $\beta = 0.9$. Figure 19a–c show the bright solutions at the trough, and Figs. 20 and 21 present the contour or 2D graphs of Fig. 19 and the solution of Ψ present the same properties at variation $\beta = 0.2$, $\beta = 0.4$, and $\beta = 0.9$.

Conclusion

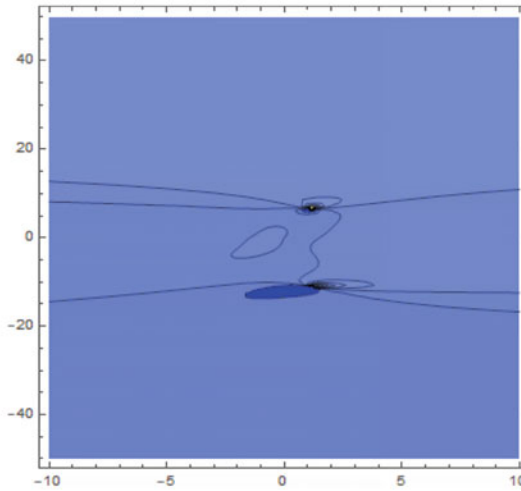
In this entry, we studied TFBE model for various lump, rogue wave, and interaction solutions. To the best of our knowledge, no one discussed these solutions ever. These solutions have so many application in various areas of sciences. We used distinct transformations to obtain these



(a) $\beta = 0.5$

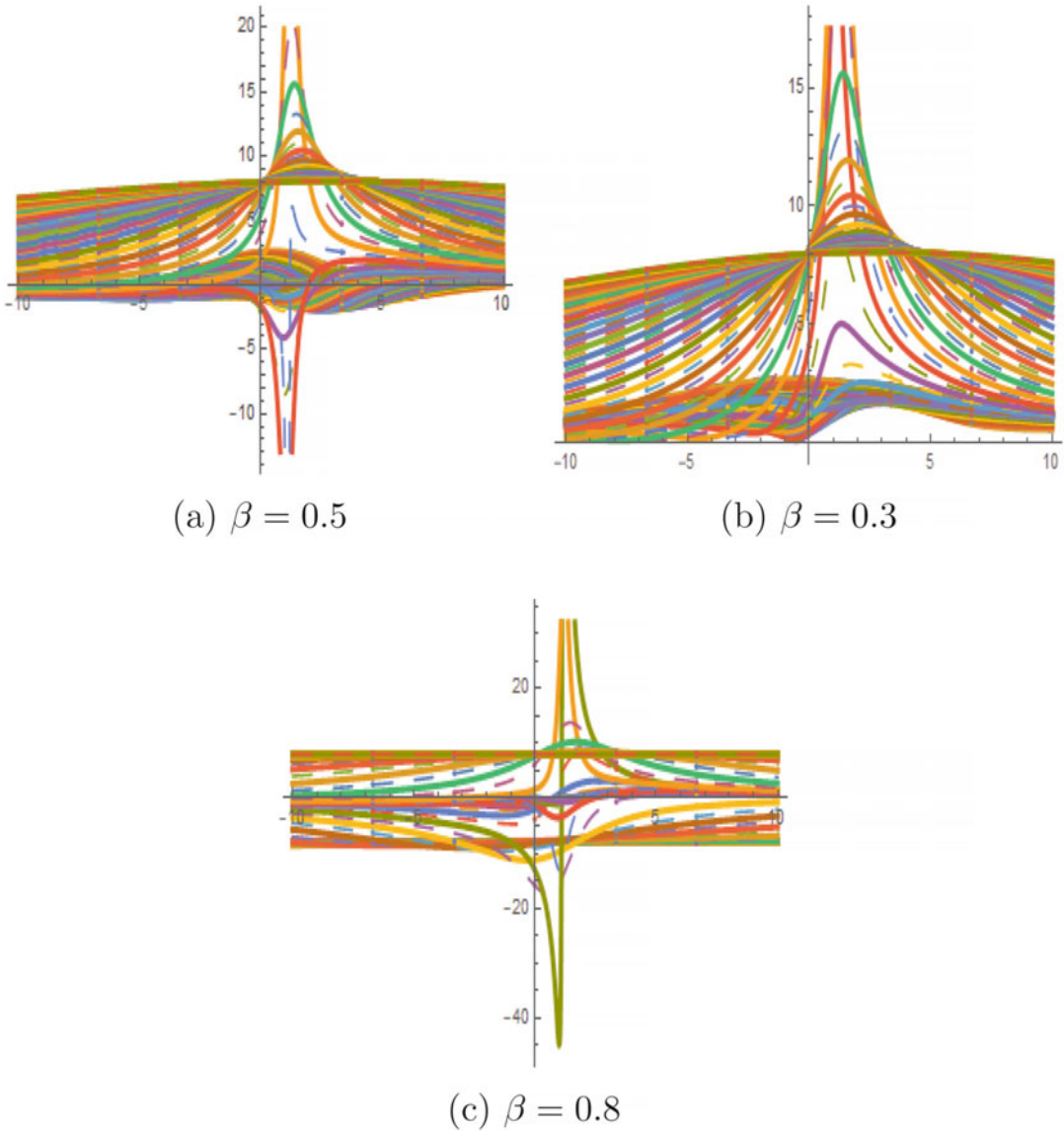


(b) $\beta = 0.3$



(c) $\beta = 0.8$

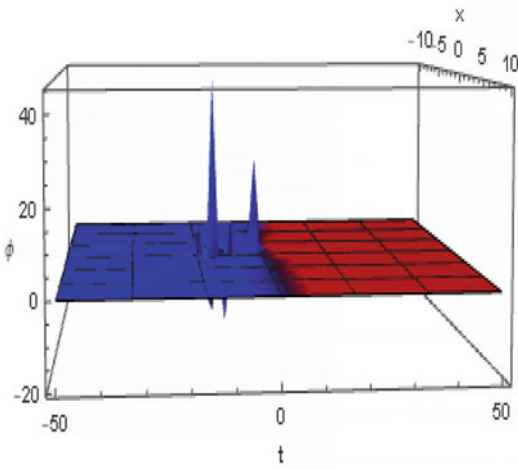
Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 2 Corresponding contour plots of Fig. 1



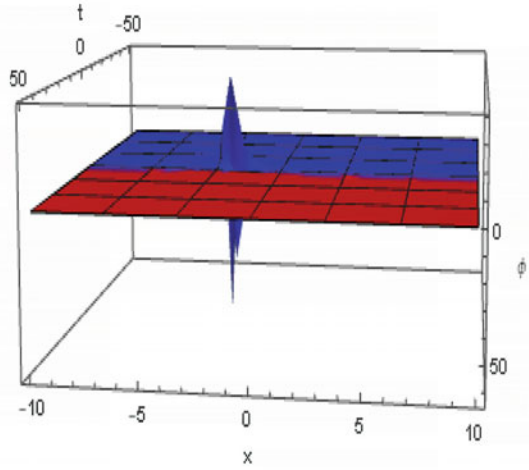
Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 3 Corresponding 2D plots of Fig. 1

solutions. Over and above board, we discussed the graphical representation of the different kind of solutions for our governing model. For proper understanding, each solution have been shown in the form of 3D, contour, and 2D plots. We established the better and significant solutions for our model in terms of rogue waves, periodic,

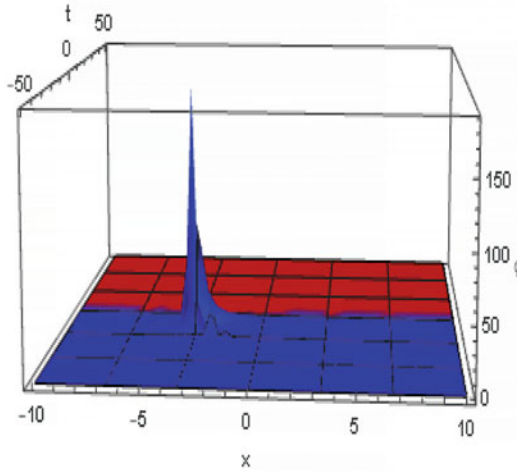
periodic cross-kink, and lump two-kink solution at different free parameters. These free parameters are very important such as we put different values of these parameters to find different solutions. All computations have been found using Mathematica and also verified by putting these values on given equations.



(a) $\beta = 0.5$



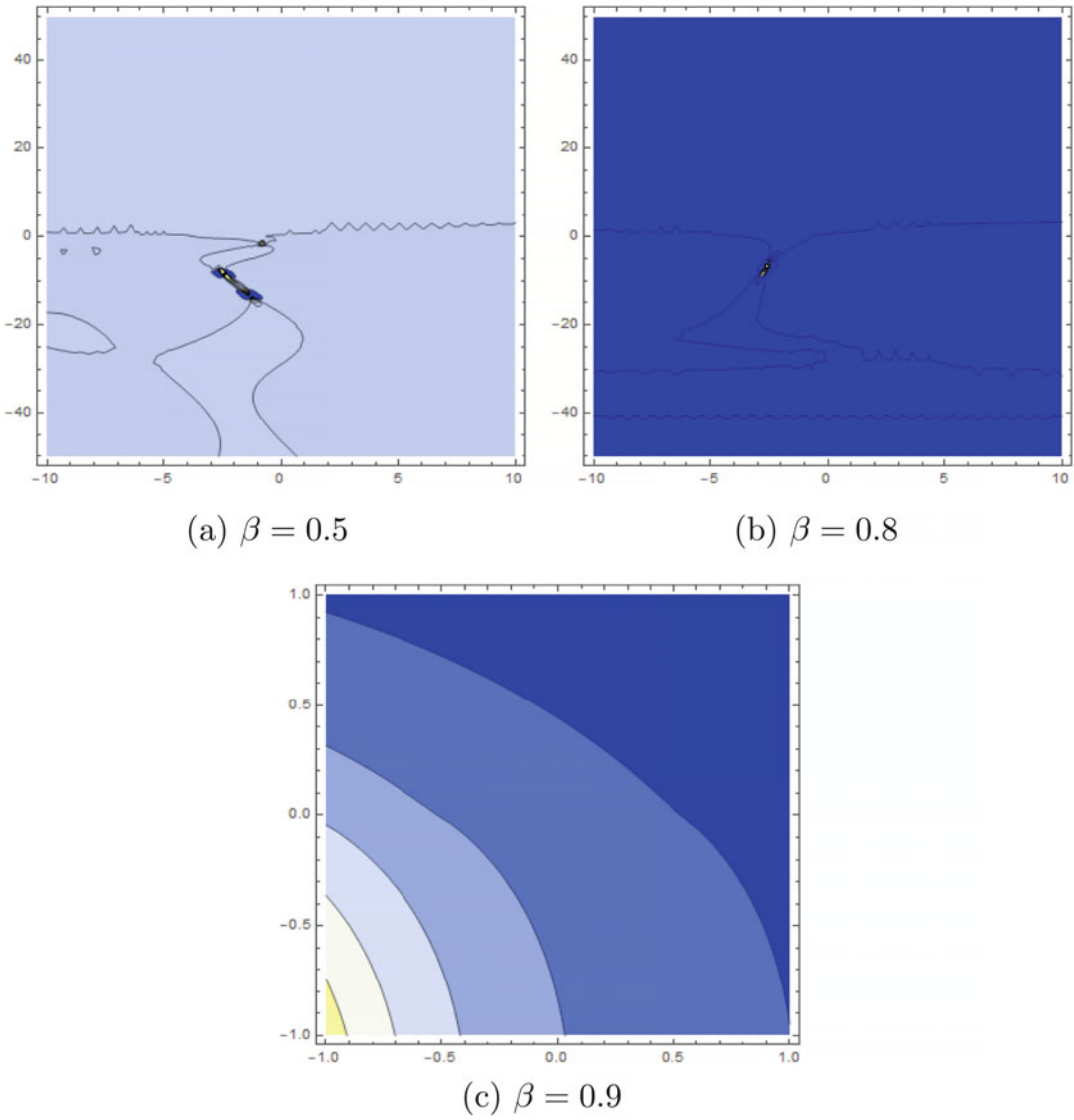
(b) $\beta = 0.8$



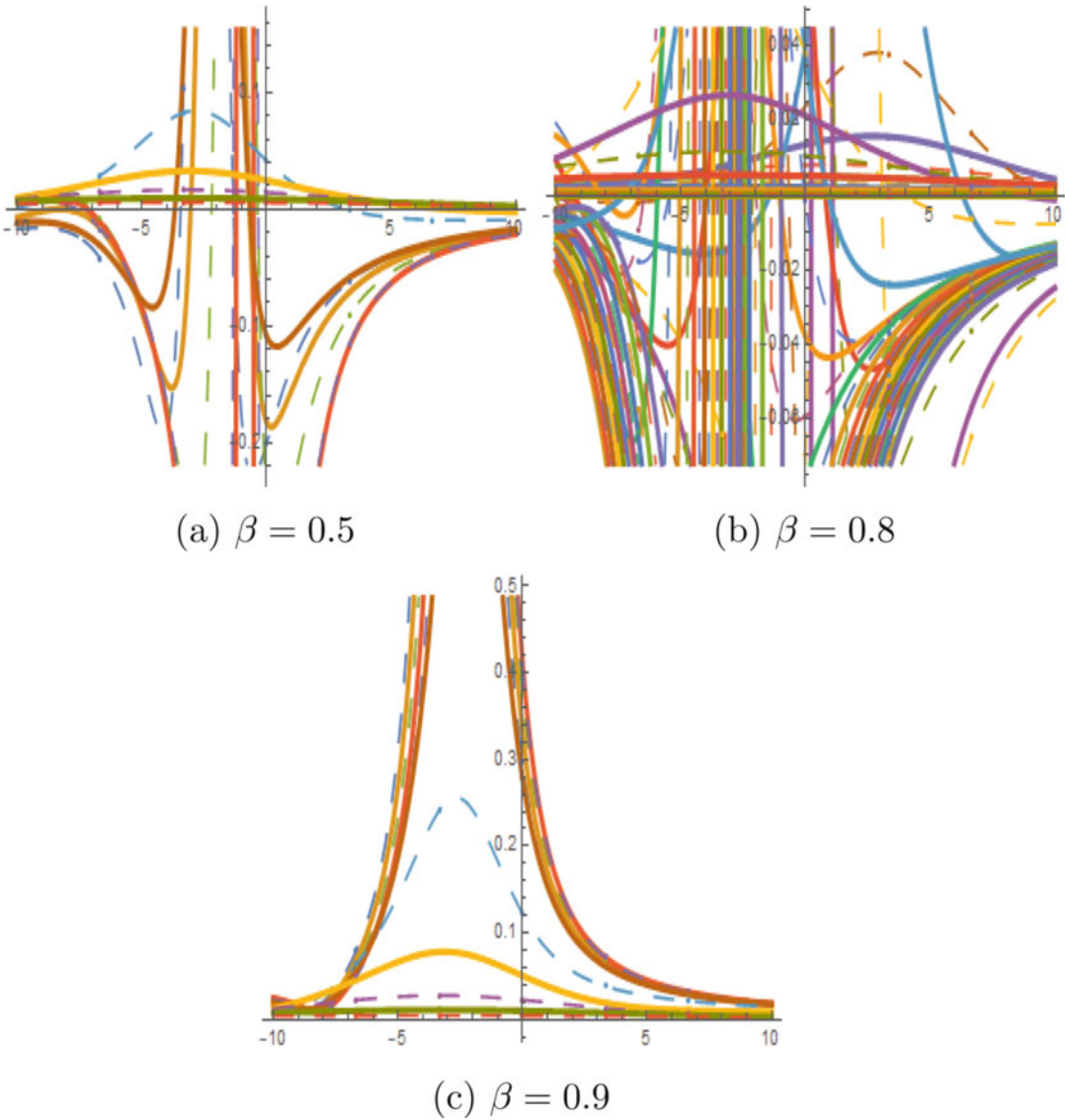
(c) $\beta = 0.9$

Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation,

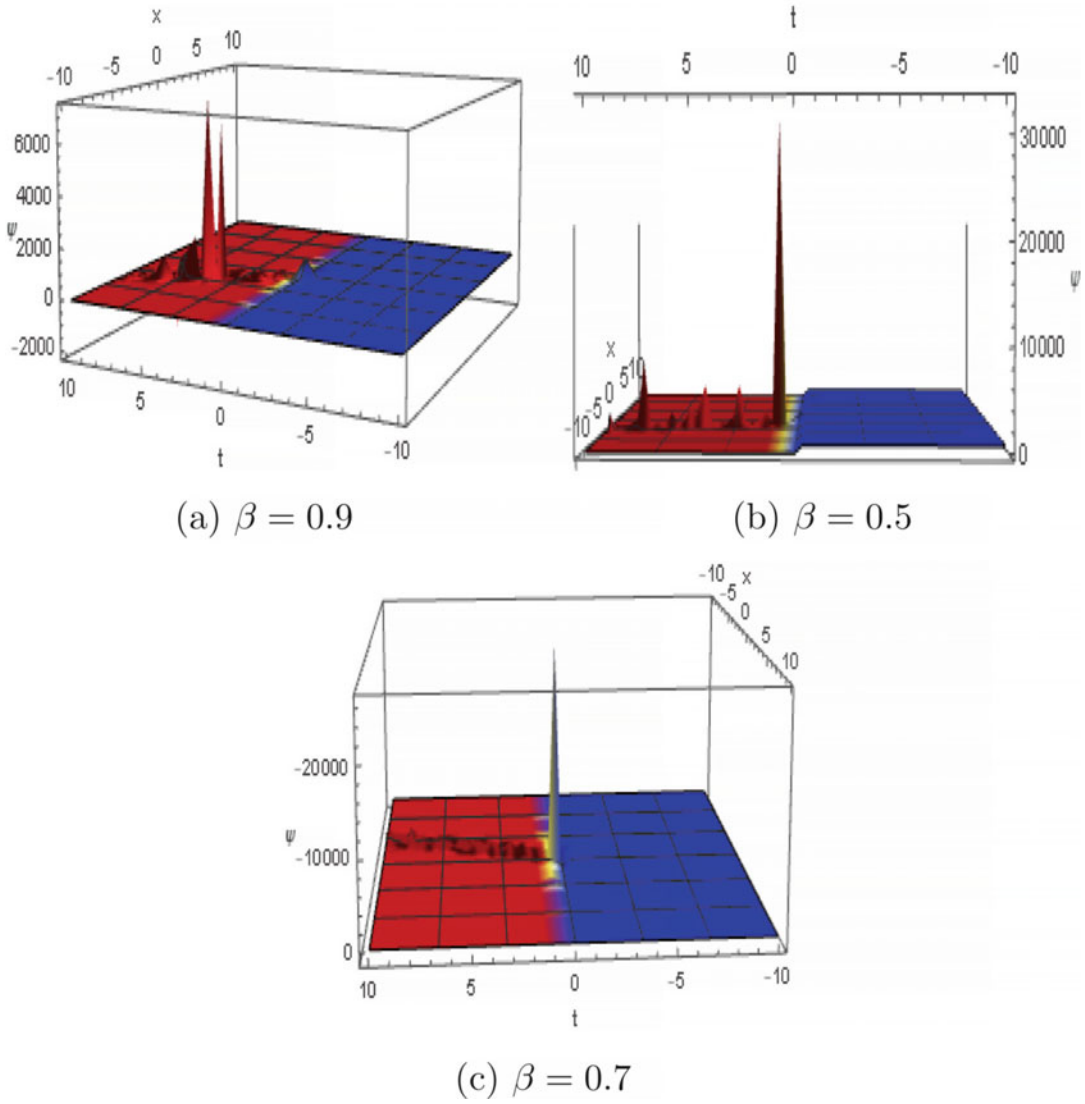
Fig. 4 Plot 3D of $\Psi(x, y, t)$ in Eq. (10), for $b = 2$, $a_6 = 7$, $a_5 = 5$, $a_7 = -1$, $a_9 = 4$, $a_4 = 6$, $b_2 = 2$, $b_3 = 1$, $y = 3$ respectively



Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 5 Corresponding contour plots of Fig. 4

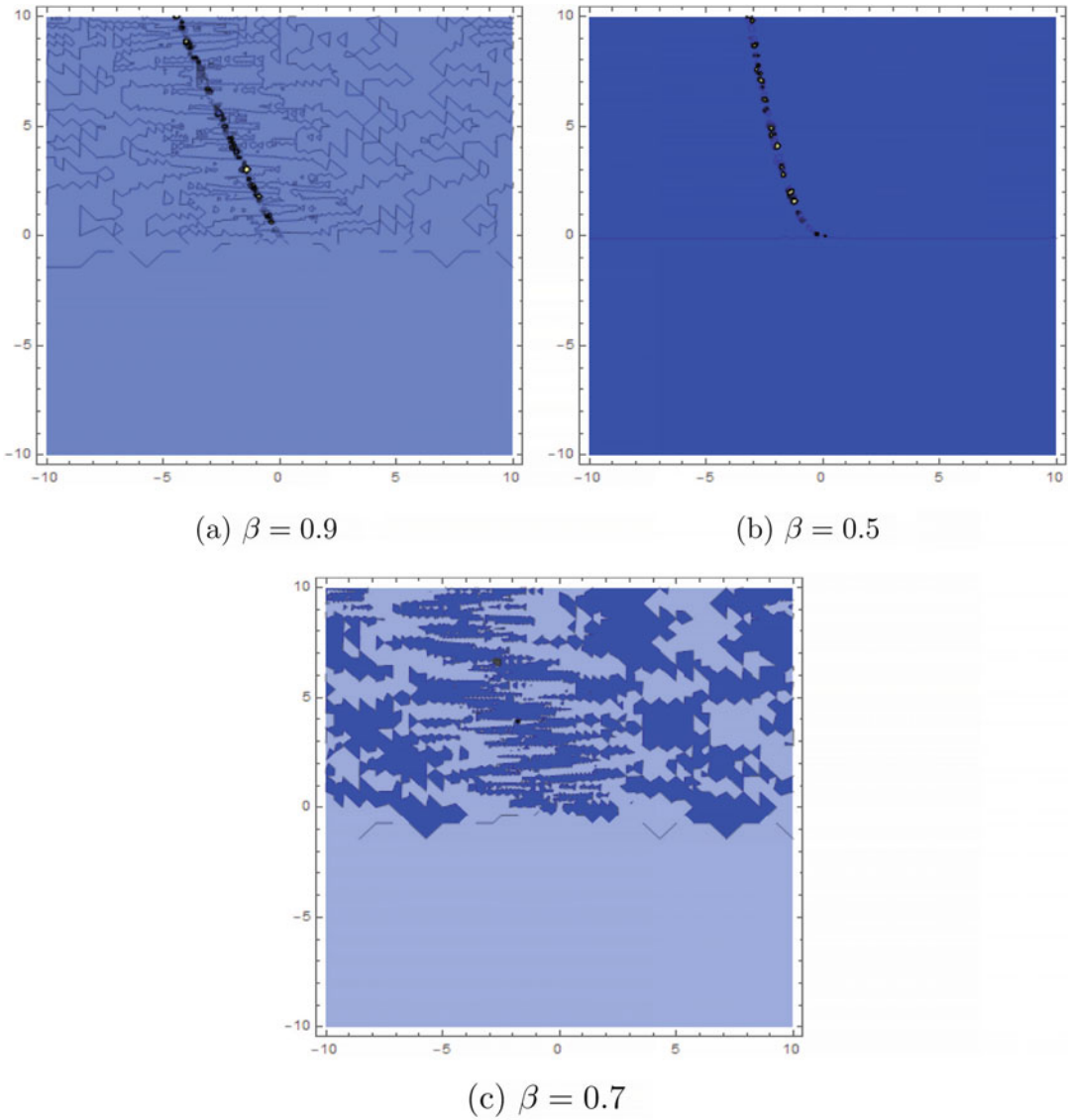


Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 6 Corresponding 2D plots of Fig. 4

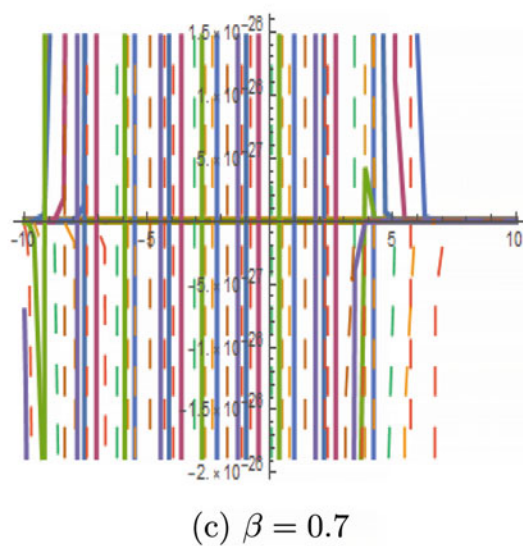
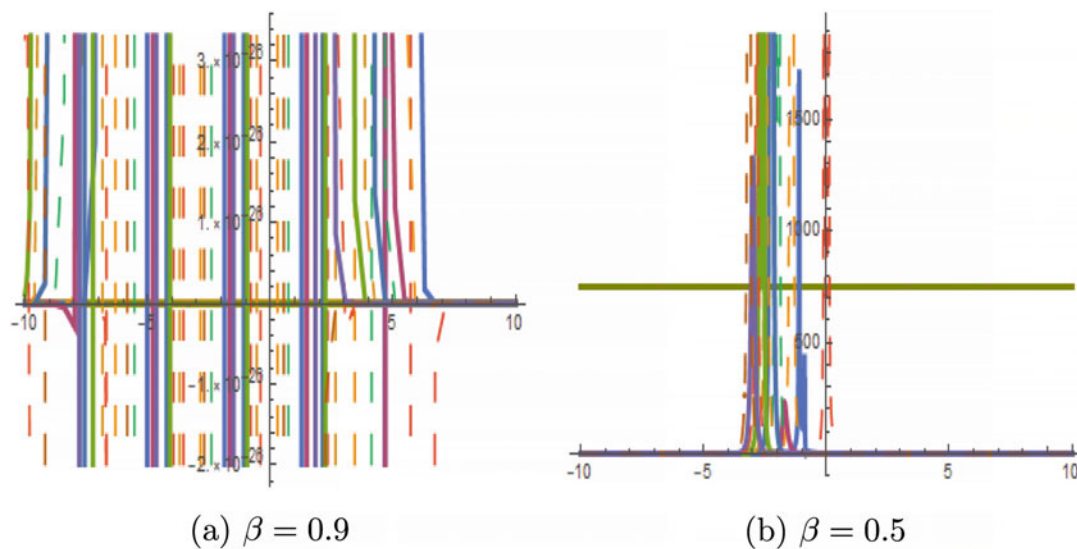


Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation,

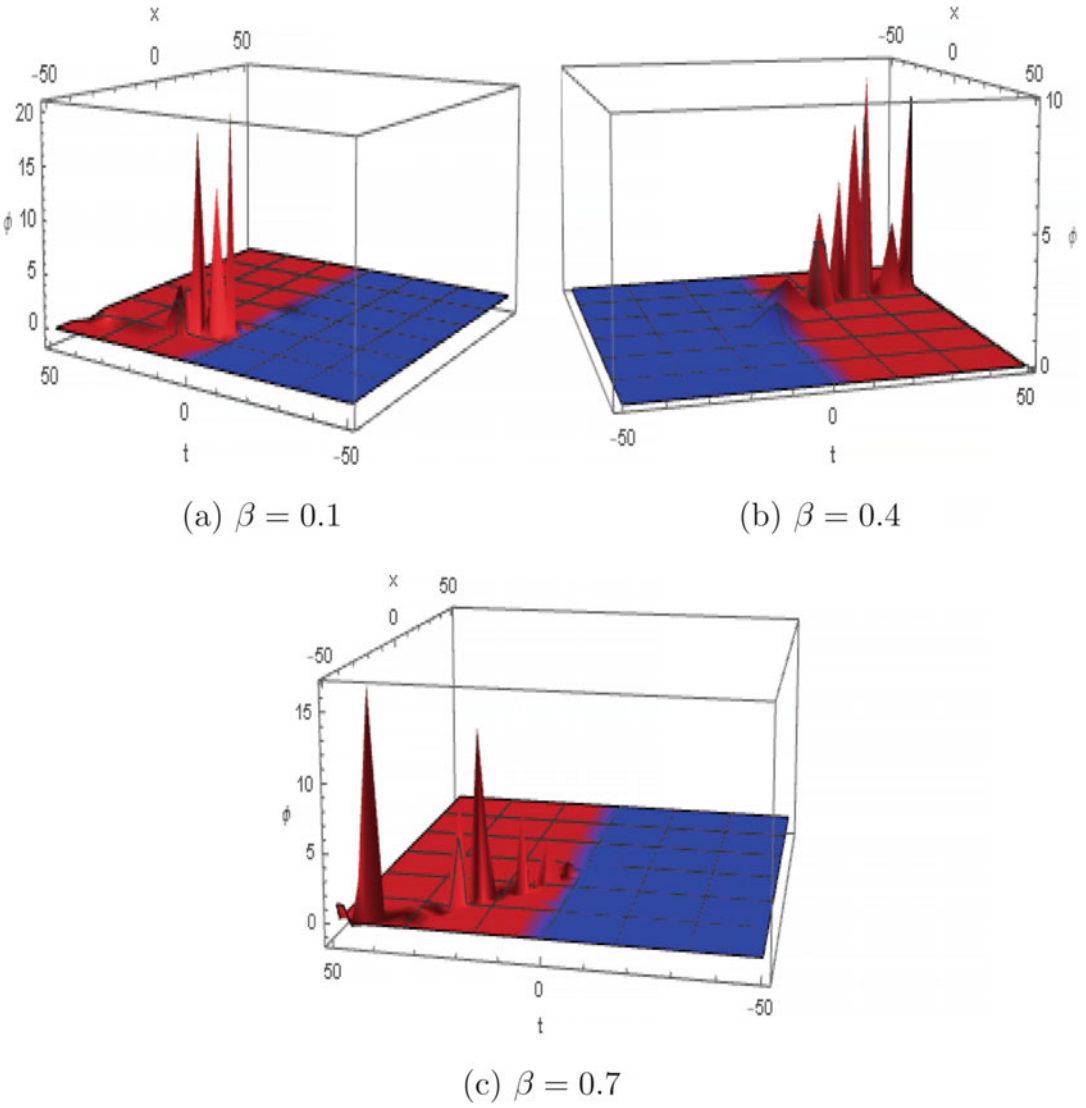
Fig. 7 Plot 3D of $\Omega(x, y, t)$ in Eq. (13), for $b = 25$, $a_4 = 10$, $a_7 = -5$, $b_2 = 10$, $a_8 = 10$, $a_6 = 15$, $b_3 = 2$, $k_2 = 5$, $a_5 = -1$, $a_{11} = 15$, $a_{10} = 5$, $k_3 = 3$, $y = 10$ respectively



Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 8 Corresponding contour plots of Fig. 7

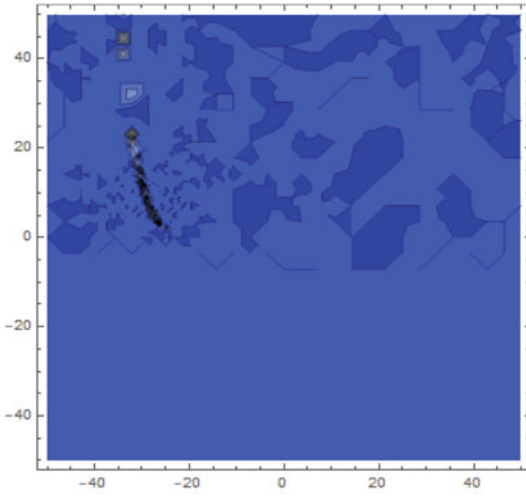


Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 9 Corresponding 2D plots of Fig. 7

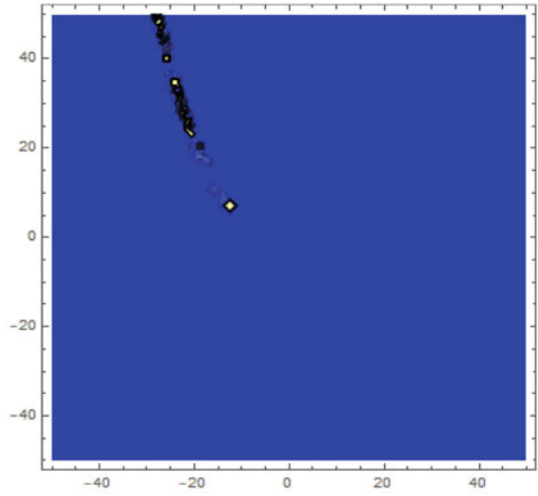


Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation,

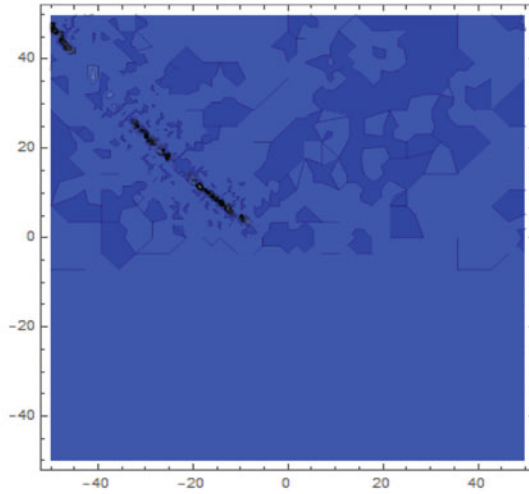
Fig. 10 Plot 3D of $\Psi(x, y, t)$ in Eq. (13), for $b = 2, a_7 = 1, b_2 = 3, a_6 = 10, a_8 = 3, a_4 = 10, a_{12} = 9, k_3 = 5, b_3 = 2, k_1 = 4, k_2 = 6, a_{10} = 7, a_5 = 3, y = 0$ respectively



(a) $\beta = 0.1$

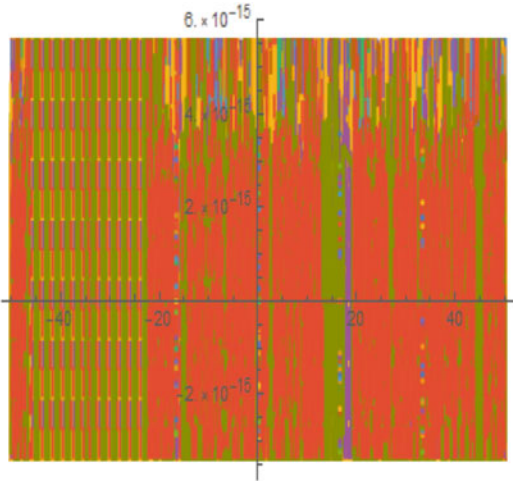


(b) $\beta = 0.4$

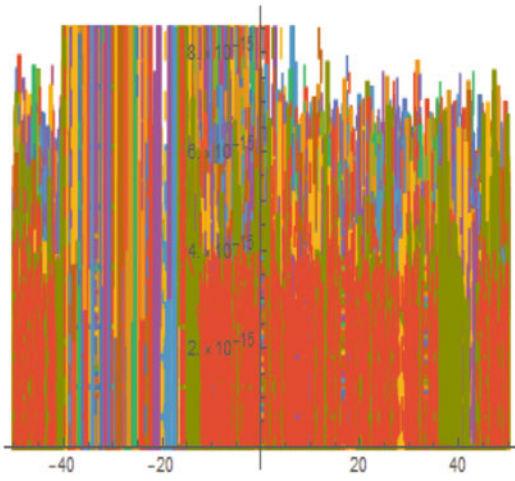


(c) $\beta = 0.7$

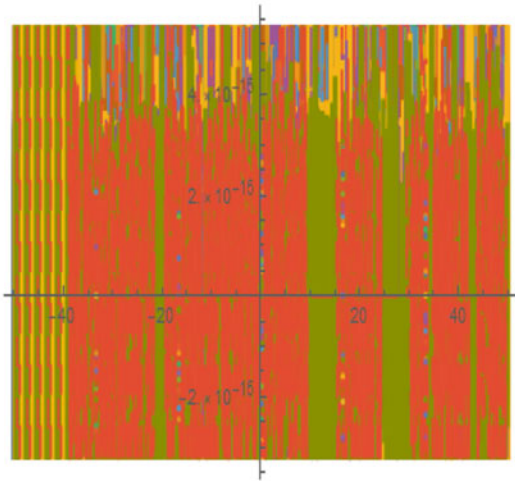
Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 11 Corresponding contour plots of Fig. 10



(a) $\beta = 0.1$

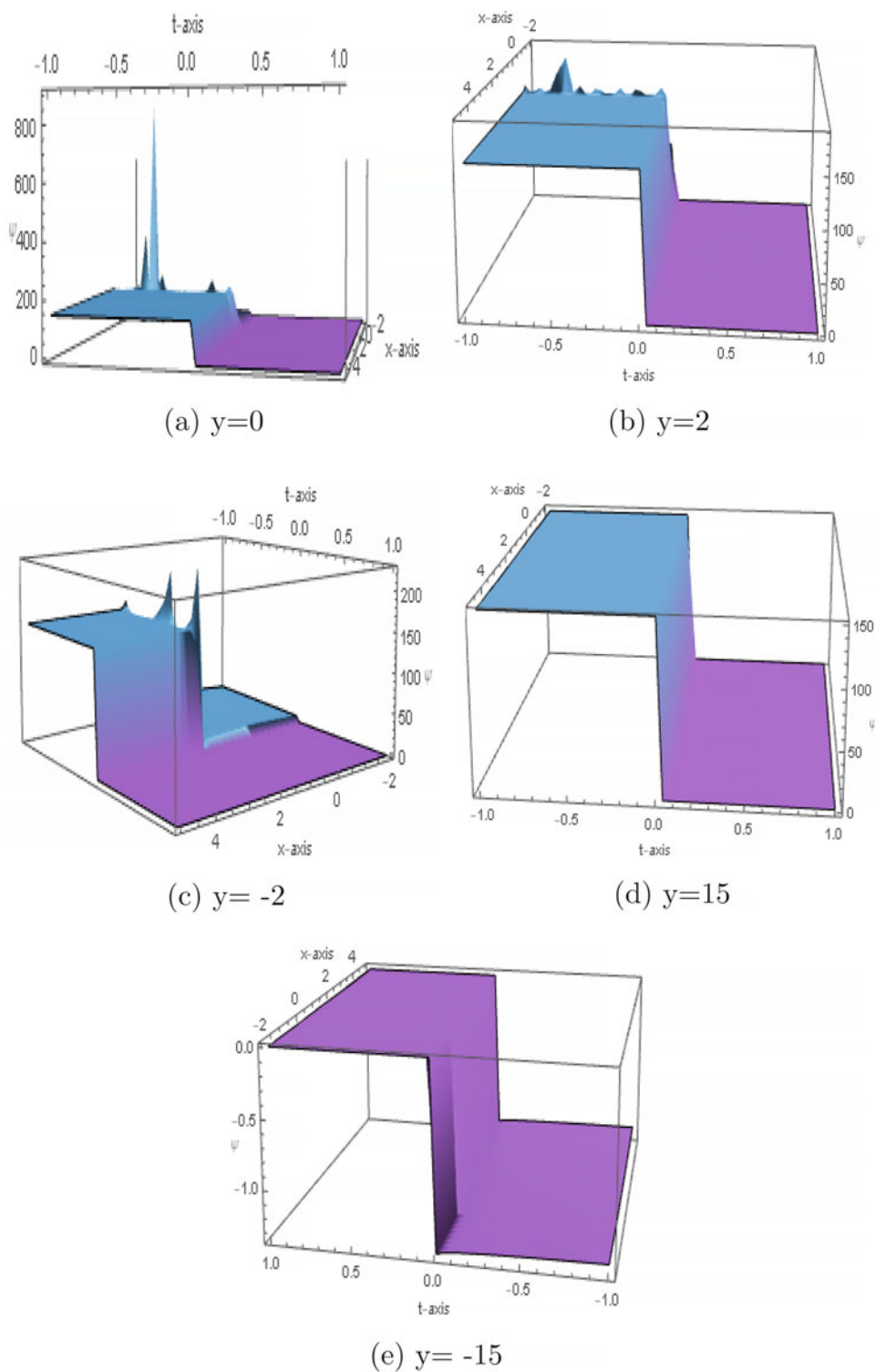


(b) $\beta = 0.4$



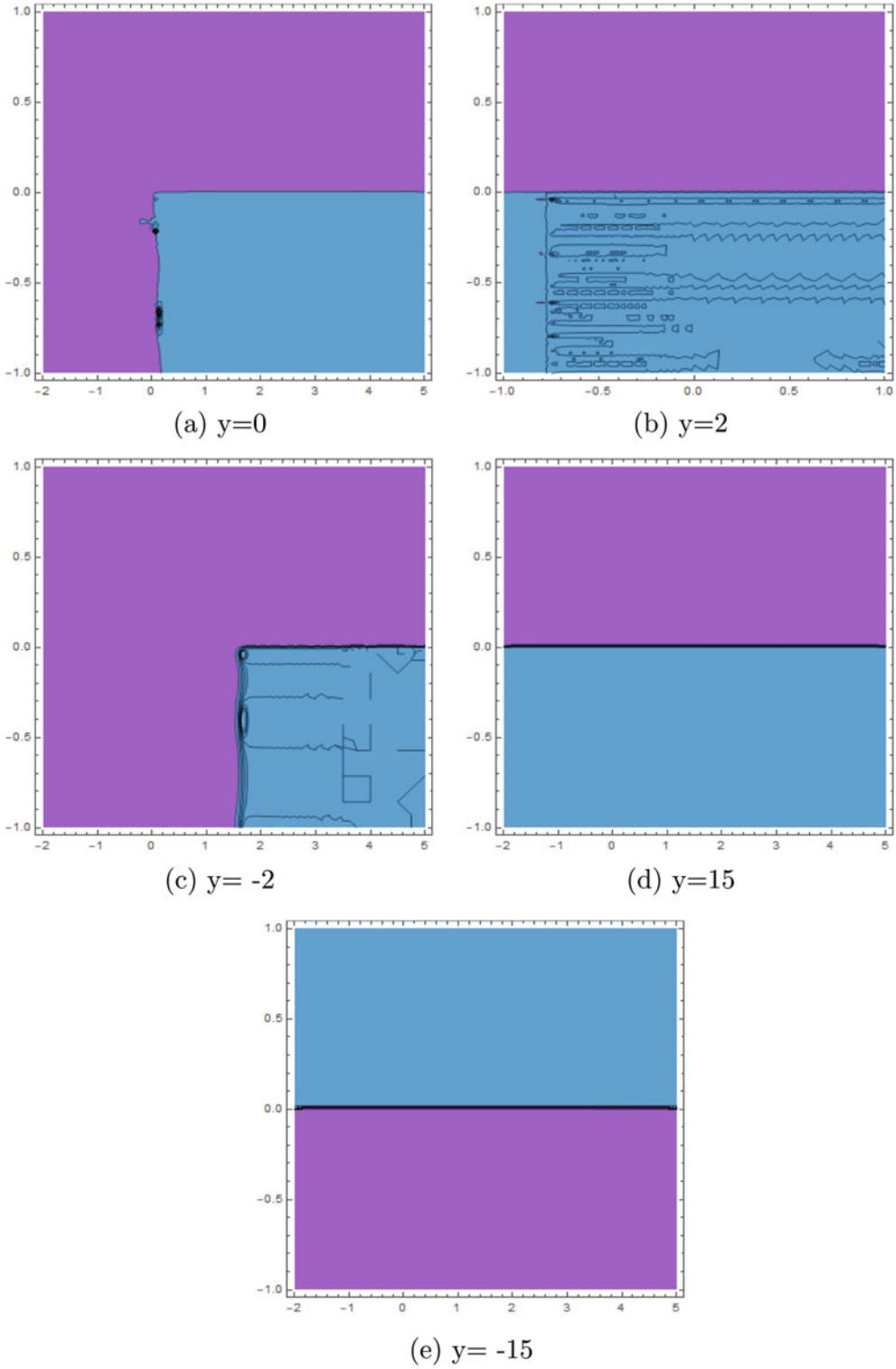
(c) $\beta = 0.7$

Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 12 Corresponding 2D plots of Fig. 10

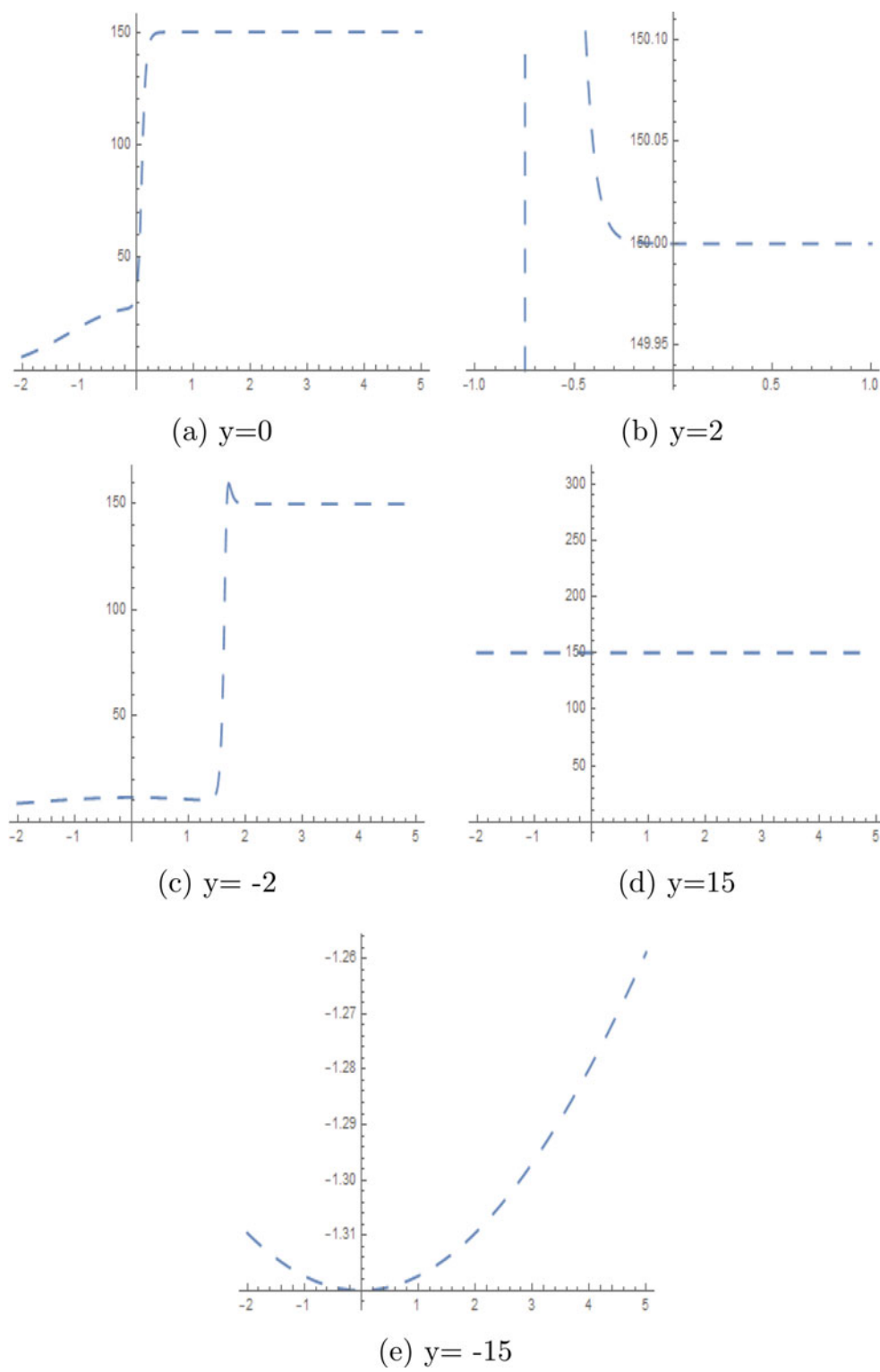


Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation,

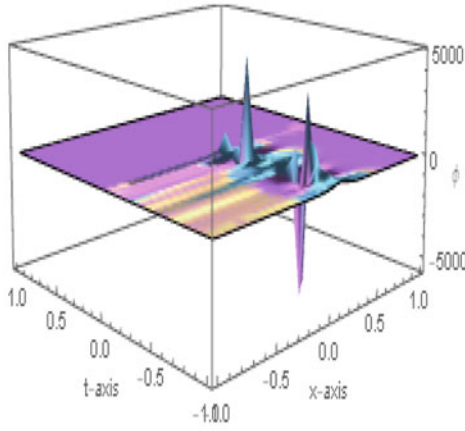
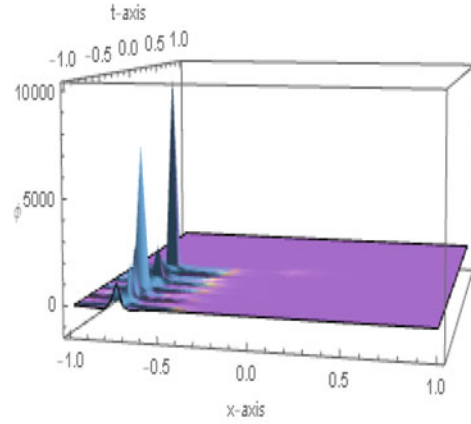
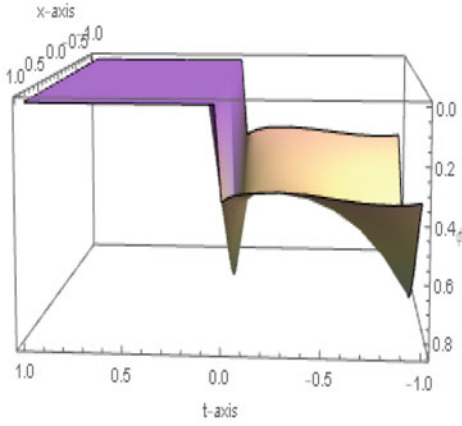
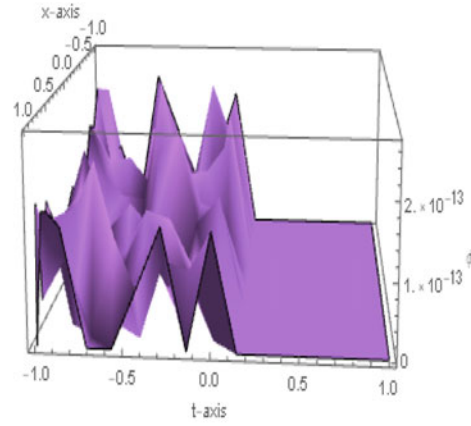
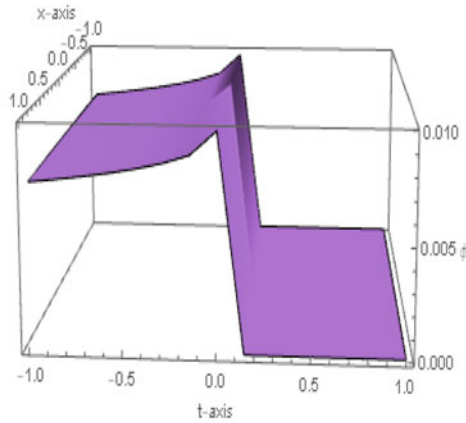
Fig. 13 Plot 3D of $\Omega(x, y, t)$ in Eq. (16), for $b = 5$, $a_2 = 7.5$, $a_5 = 4$, $a_4 = 2$, $b_2 = 7$, $b_3 = -1$, $a_7 = 6$, $a_9 = 5.5$, $m_1 = 25$, $b_4 = 20$, $b_5 = 15$, $b_6 = 7.5$, $m_2 = 10$, $\beta = 0.5$ respectively



Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 14 Corresponding contour plots of Fig. 13

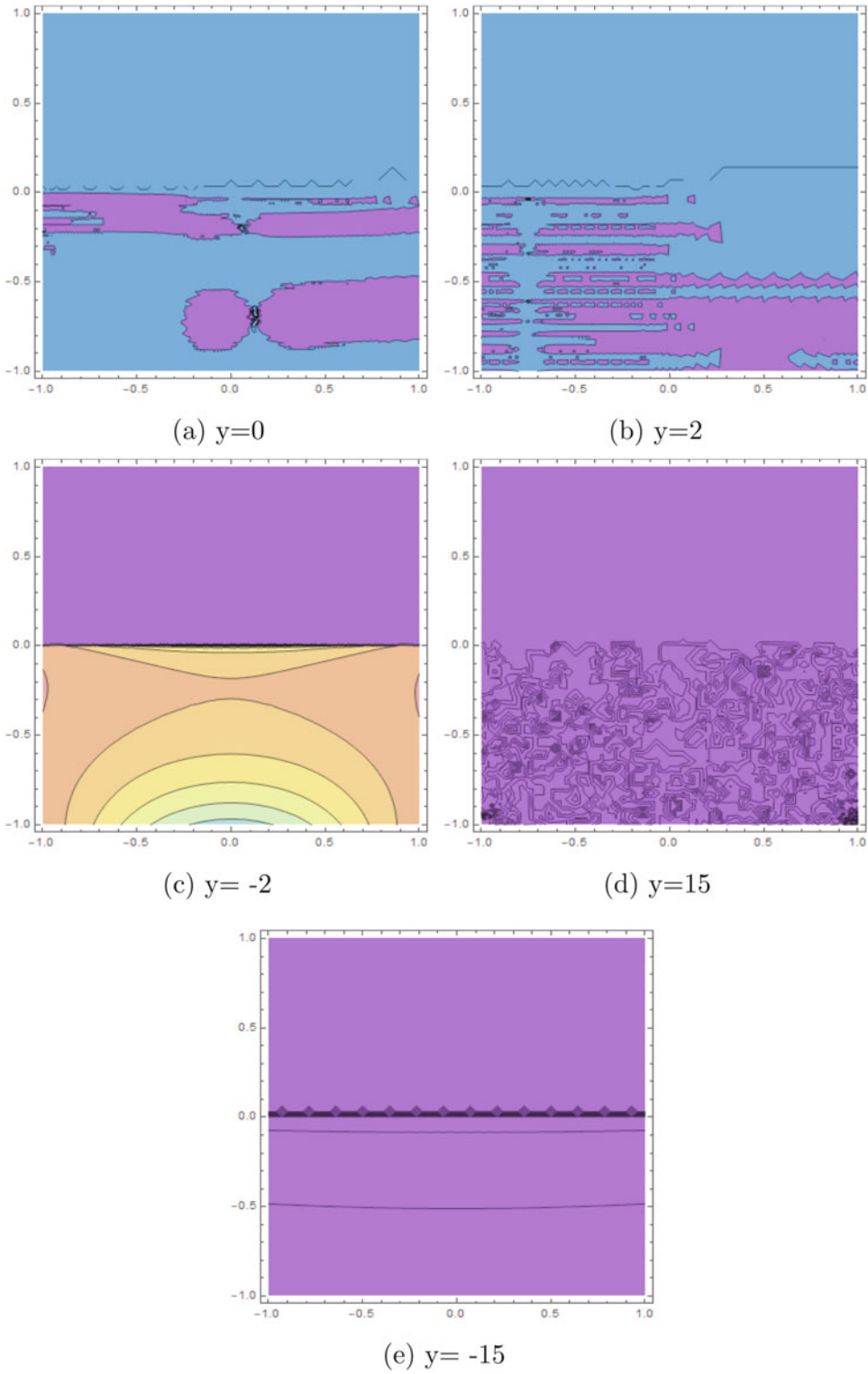


Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 15 Corresponding 2D plots of Fig. 13

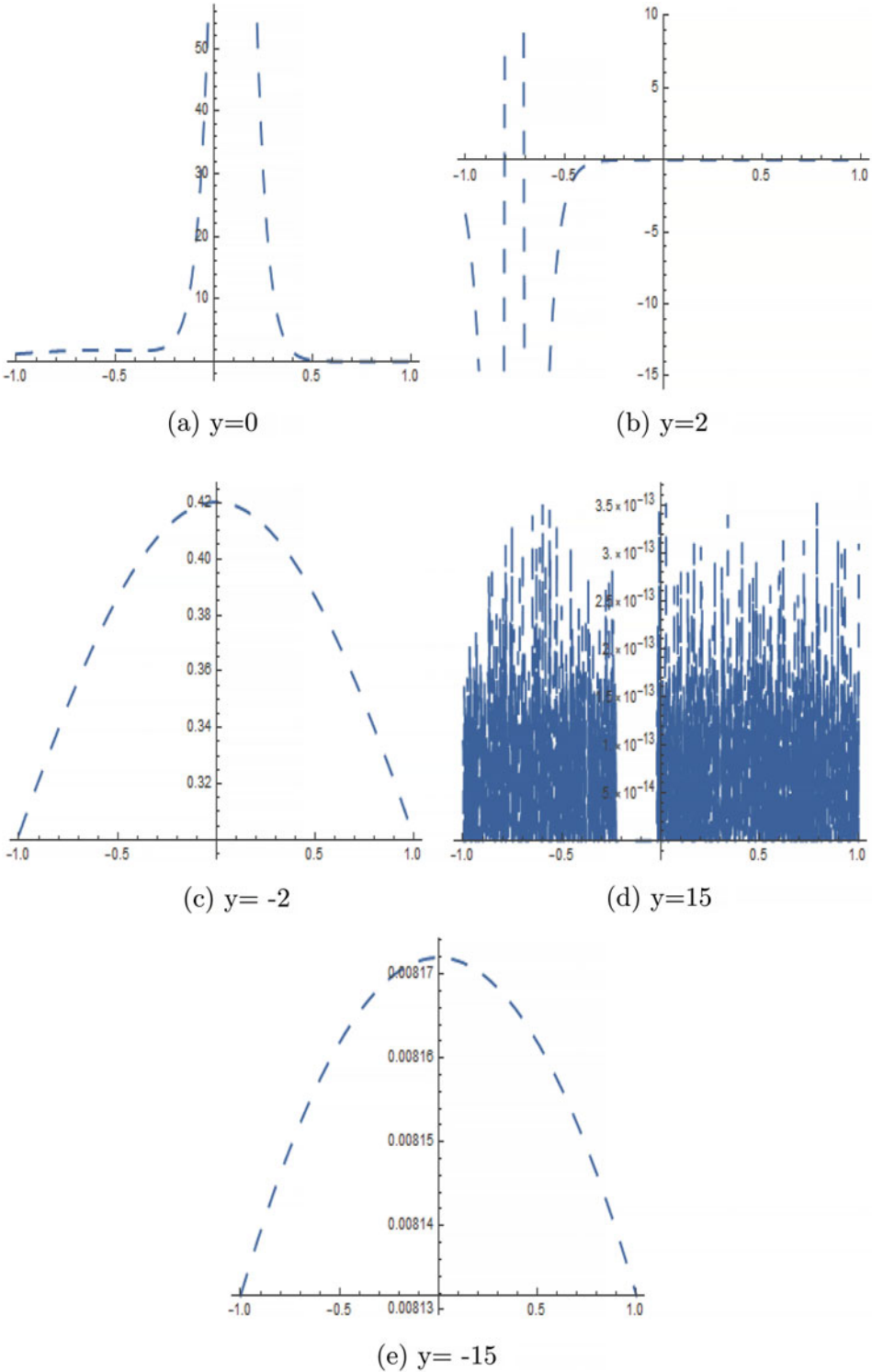
(a) $y=0$ (b) $y=2$ (c) $y=-2$ (d) $y=15$ (e) $y=-15$

Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation,

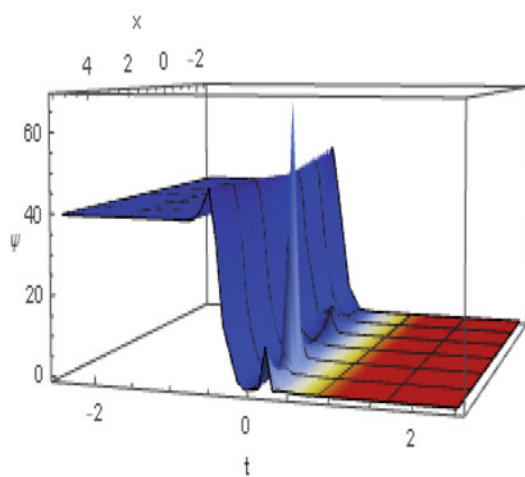
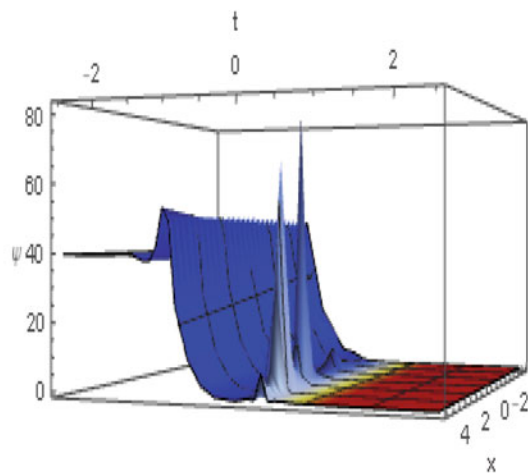
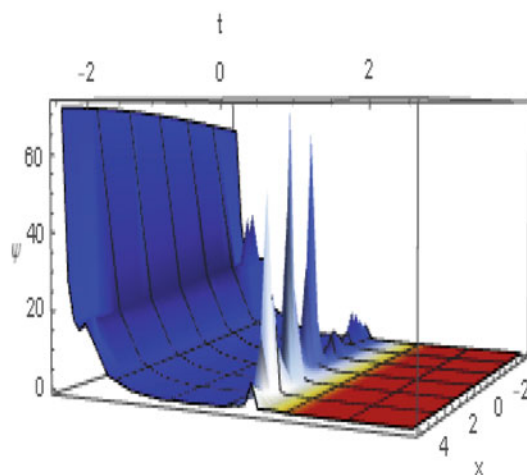
Fig. 16 Plot 3D of $\Psi(x, y, t)$ in Eq. (16), for $b = 5$, $a_2 = 7.5$, $a_5 = 4$, $a_4 = 2$, $b_2 = 7$, $b_3 = -1$, $a_7 = 6$, $a_9 = 5.5$, $m_1 = 25$, $b_4 = 20$, $b_5 = 15$, $b_6 = 7.5$, $m_2 = 10$, $\beta = 0.5$ respectively



Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 17 Corresponding contour plots of Fig. 16

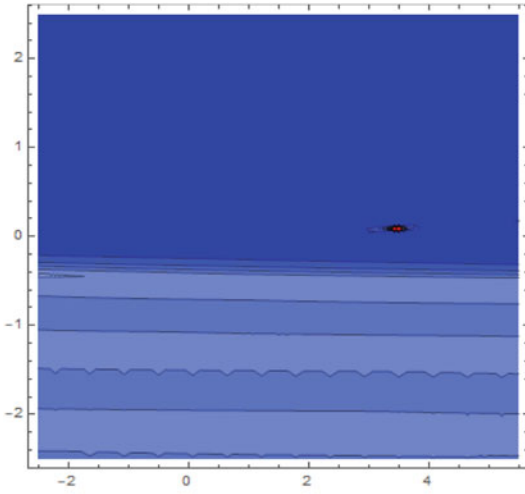


Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 18 Corresponding 2D plots of Fig. 16

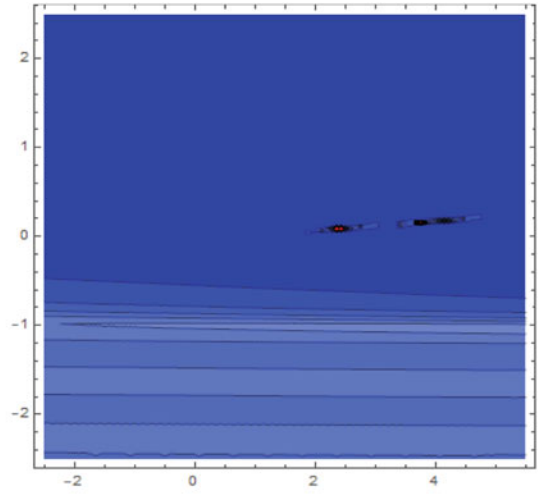
(a) $\beta = 0.7$ (b) $\beta = 0.8$ (c) $\beta = 0.9$

Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation,

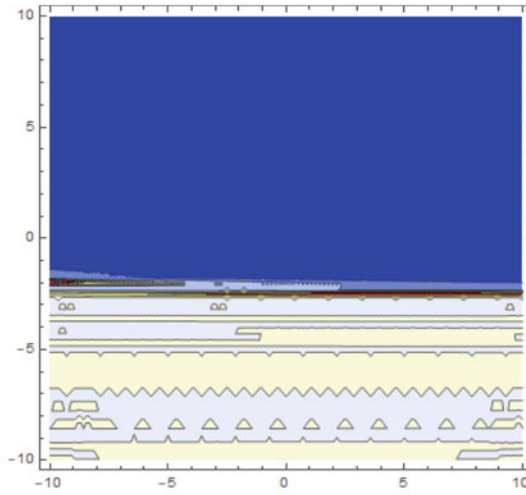
Fig. 19 Plot 3D of $\Omega(x, y, t)$ in Eq. (19), for $b = 2, a_7 = 2, a_6 = 1, a_5 = 3.5, n_2 = 10, n_1 = 2, n_3 = 4, z_2 = 5, z_1 = 1, a_{10} = 2, a_4 = 1, y = 0$ respectively



(a) $\beta = 0.7$

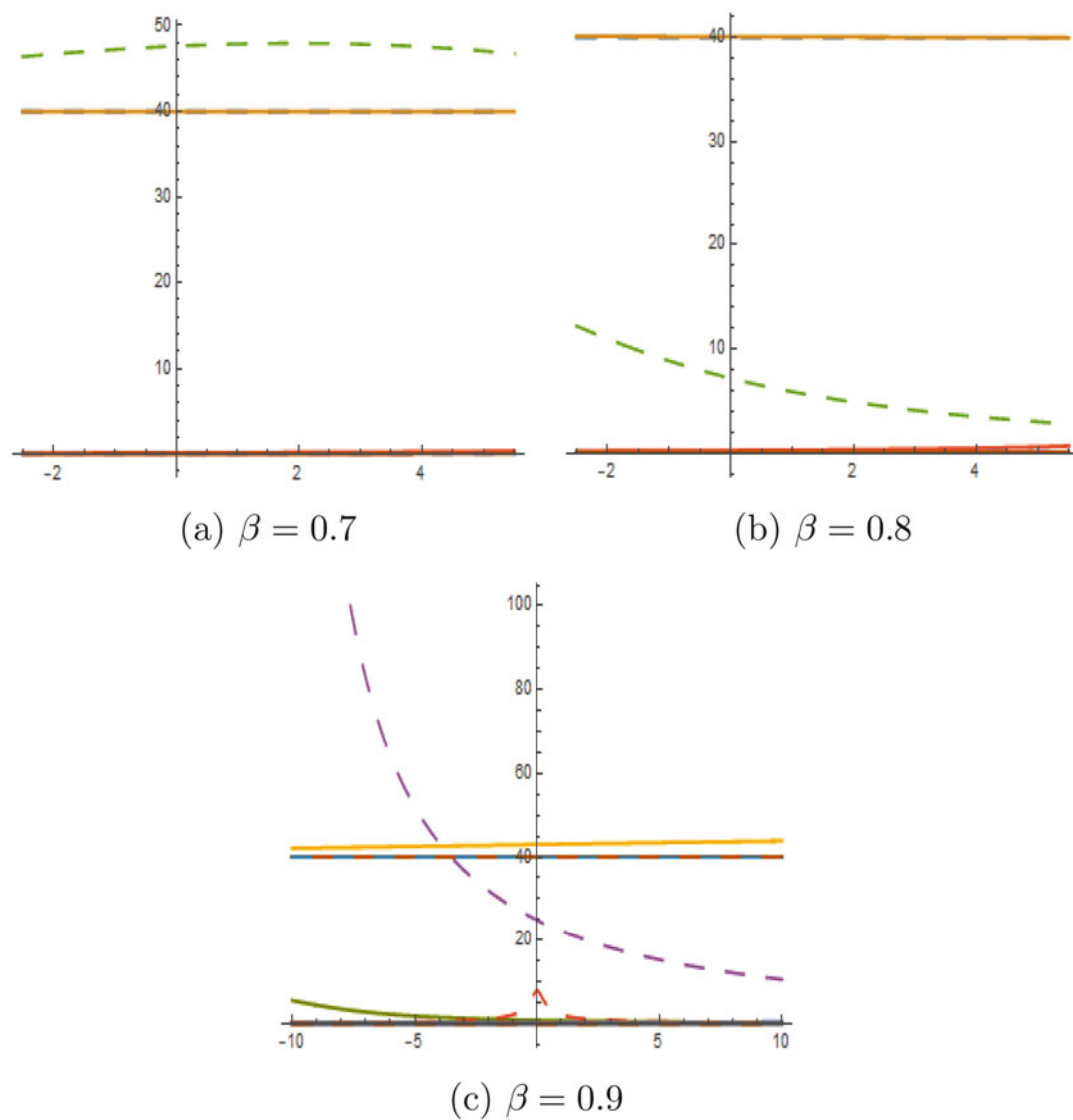


(b) $\beta = 0.8$

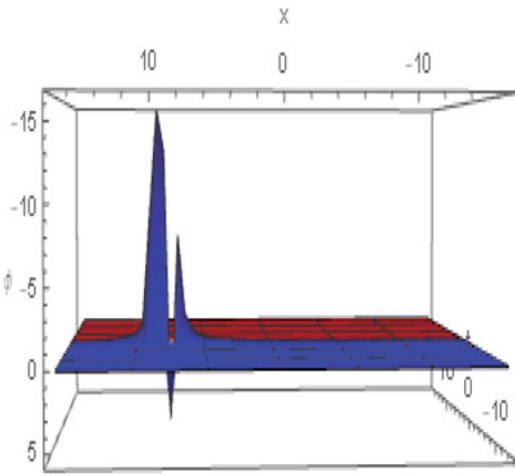


(c) $\beta = 0.9$

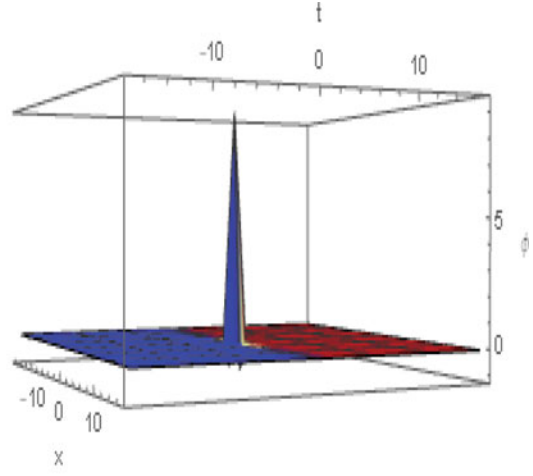
Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 20 Corresponding contour plots of Fig. 19



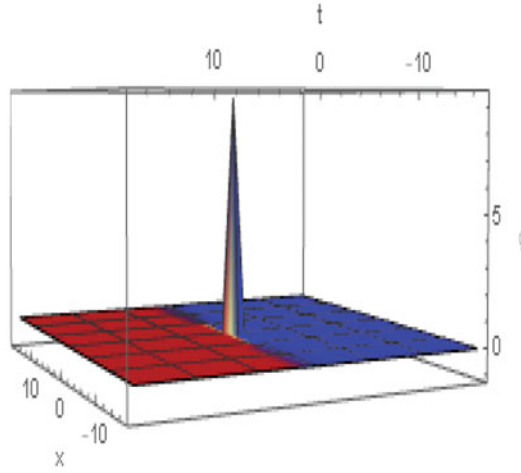
Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 21 Corresponding 2D plots of Fig. 19



(a) $\beta = 0.2$



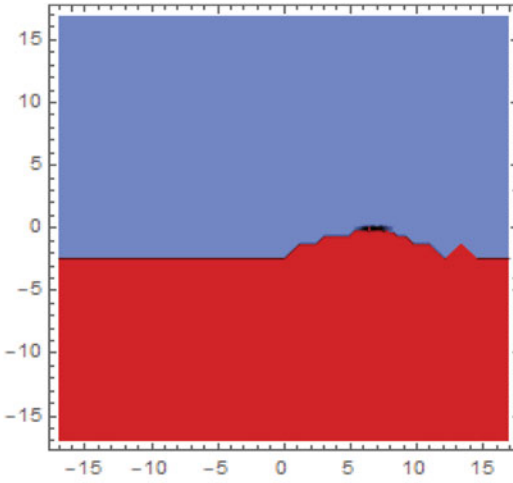
(b) $\beta = 0.4$



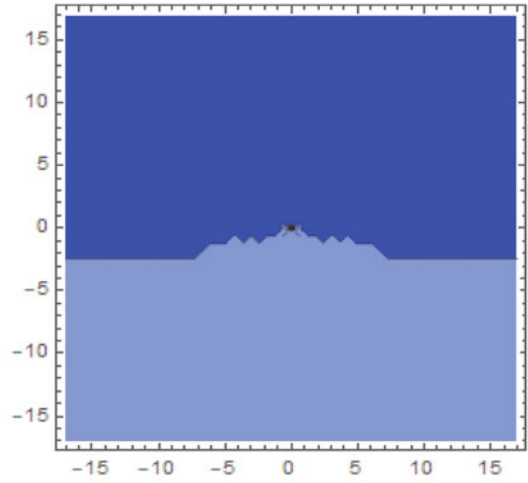
(c) $\beta = 0.9$

Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation,

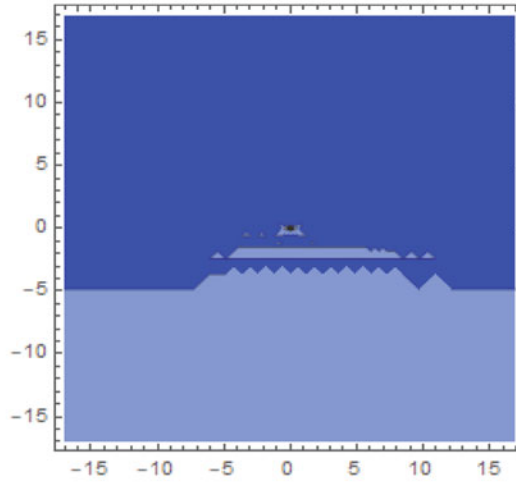
Fig. 22 Plot 3D of $\Psi(x, y, t)$ in Eq. (19), for $b = 5, a_7 = 2, a_6 = 1, a_5 = 3.5, n_2 = 35, n_1 = 2, n_3 = 10, z_2 = 5, z_1 = 1, a_{10} = 2, a_4 = 1, y = 0$ respectively



(a) $\beta = 0.2$

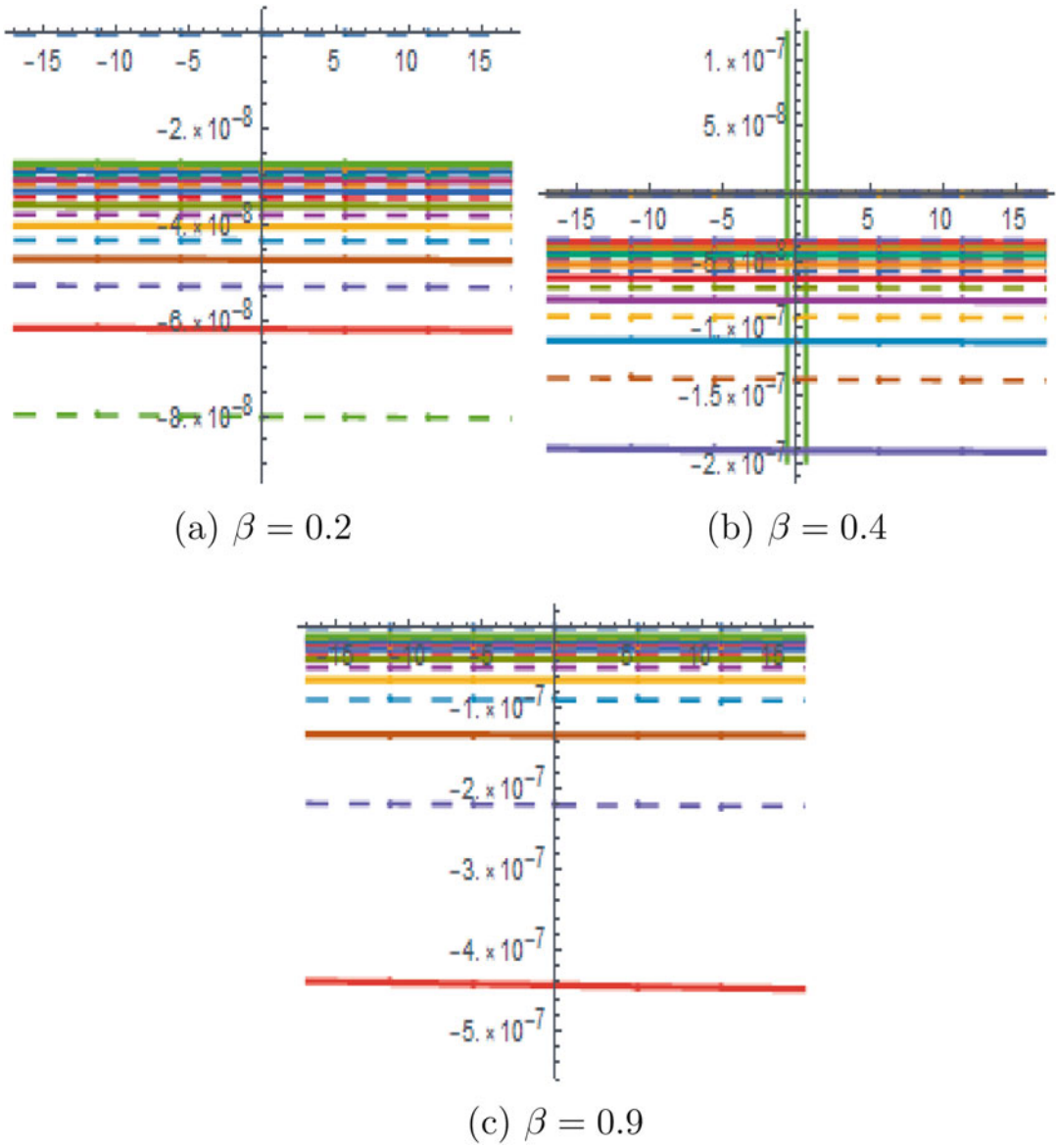


(b) $\beta = 0.4$



(c) $\beta = 0.9$

Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 23 Corresponding contour plots of Fig. 22



Periodic Cross-Kink, Rogue-Waves, and Lump Interaction Soliton Solutions with Kink and Periodic Waves for Fractional Bogoyavlenskii Equation, Fig. 24 Corresponding 2D plots of Fig. 22

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Double Tchebyshev Spectral Tau Algorithm for Solving KdV Equation, with Soliton Application

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Article Outline

Introduction
Case Study
Error Estimate
Applications
Concluding Remarks
Bibliography

Introduction

Numerical methods for differential equations can be classified into the local and global categories. The finite-difference and finite-element methods are based on local arguments, whereas the spectral method is global in character. In practice, finite-element methods are particularly well suited to problems in complex geometries, whereas spectral methods can provide superior accuracy, at the expense of domain flexibility. There are many numerical approaches, such as finite-elements and spectral-elements, which combine advantages of both the global and local methods (see, Shen et al. 2011). However, in this entry, our attention will be restricted to the global spectral methods.

Spectral methods, in the context of numerical schemes for differential equations, belong to the family of weighted residual methods (WRMs), which are traditionally regarded as the foundation of many numerical methods such as finite element, spectral, finite volume, and boundary element (cf. Finlayson 2013). WRMs represent a particular group of approximation techniques, in

which the residuals (or errors) are minimized in a certain way and thereby leading to specific methods including Galerkin, Petrov-Galerkin, collocation, and tau formulations.

Spectral methods is the name given to a numerical approach for the solution of differential equations. In this approach, the solution to the equation is approximated by a truncated series of special functions which are the eigenfunctions of some differential operators. Spectral methods may be viewed as development of the class of discretization schemes for differential equations known generally as the method of weighted residuals. The key elements of the method of weighted residuals are as follows:

- (i) The trial functions
- (ii) The test functions

The trial functions are also called the expansion or the approximating functions, while the test functions are known as weight functions. The trial functions are used as the basis functions for a truncated series expansion of the solution. The test functions are used to ensure that the differential equation is satisfied as closely as possible by the truncated series expansion. This is achieved by minimizing the residual, that is, the error in the differential equation produced by using the truncated expansion instead of the exact solution, with respect to a suitable norm. The residual satisfies a suitable orthogonality condition with respect to each of the test functions.

The main feature of the spectral methods is how to choose various orthogonal systems of infinitely differentiable global functions as trial functions. These choices of trial functions distinguish spectral methods from finite-element and finite-difference methods.

The test functions choices distinguish between the three most commonly used spectral schemes, namely, Galerkin, collocation, and tau methods. In Galerkin approach, the test functions are the same as the trial functions. They are, therefore, infinitely smooth functions which individually satisfy the boundary conditions. The differential equation is

enforced by requiring that the integral of the residual times each test function be zero.

In the collocation approach, the test functions are the Dirac delta functions centered at special collocation points. This approach requires the differential equation to be satisfied exactly at the collocation points.

Spectral tau methods are similar to Galerkin methods in the way that the differential equation is enforced. However, none of the test functions need to satisfy the boundary conditions. Hence, a supplementary set of equations are needed to apply the boundary conditions.

The Galerkin approach is perhaps the most esthetically pleasing of the methods of the weighted residuals, since the trial functions and the test functions are the same and the physical problem can be discretized in terms of a variational principle. Finite-element methods use this approach. Moreover, the first serious application of spectral methods to differential equations was through a Galerkin method (see Gottlieb and Orszag 1977).

The tau approach is a modification of the Galerkin method that is applicable to problems with nonperiodic boundary conditions. It may be viewed as a special case of the so-called Petrov-Galerkin method.

The first unifying mathematical assessment of the theory of spectral methods was contained in the monograph by Gottlieb and Orszag (1977). Since then, the theory has been extended to cover a large variety of problems, such as variable-coefficient and nonlinear problems.

In 2012, Canuto et al., in their excellent book, presented a unified theory of the mathematical analysis of spectral methods and applied it to many of the algorithms in current use. They paid special attention to those algorithmic details which are essential for the successful implementation of spectral methods.

The subject of orthogonal polynomials is a classical one whose origins can be traced to Legendre's work on planetary motion. With important applications to physics and to probability and statistics and other branches of mathematics, the subject flourished through the first third of the twentieth century. After the publication of Szego's well-known treatise on the subject, interest in orthogonal polynomials to wane as mathematicians turned their

attention to increasingly greater abstraction. Perhaps as a secondary effect of the computer revolution and the heightened activity in approximation theory and numerical analysis, interest in orthogonal polynomials has revived in recent years.

The main objective of this introductory section is to give some important definitions and concepts of orthogonal polynomials, spectral methods, and KdV equation.

Orthogonal Polynomials

Assume that P_N is the space of all polynomials of degree less than or equal to N and $\{p_k; k = 0, 1, 2, \dots\}$ is a system of algebraic polynomials (with degree of p_k equals k) which are mutually orthogonal over the interval (a, b) with respect to a weight function $w(x)$, i.e.,

$$\int_a^b w(x)p_n(x)p_m(x)dx = 0, \quad m \neq n. \quad (1)$$

The classical Weierstrass theorem implies that such a system is complete in the space $L_w^2(a, b)$. This is the space of functions v such that the norm

$$\|v\|_w^2 = \int_a^b v^2(x)w(x)dx, \quad (2)$$

is finite. The associated inner product is

$$(u, v)_w = \int_a^b u(x)v(x)w(x)dx. \quad (3)$$

The formal series of a function $u \in L_w^2(a, b)$ in terms of the system $\{p_k\}$ is

$$\sum_{k=0}^{\infty} \hat{u}_k p_k(x),$$

where the expansion coefficients $\hat{u}_k$ are defined as

$$\hat{u}_k = \frac{1}{h_k} \int_a^b u(x)p_k(x)w(x)dx, \quad (4)$$

where

$$h_k = \|p_k\|_w^2 = \int_a^b w(x)p_k^2 dx.$$

For an integer $N > 0$, the truncated series of $u(x)$ of order N is the polynomial

$$p_N u = \sum_{k=0}^N \hat{u}_k p_k(x). \quad (5)$$

Due to Eq. (1), p_N is the orthogonal projection of u upon P_N in the inner product (4), i.e.,

$$(p_N u, v)_w = (u, v)_w, \quad \forall v \in P_N. \quad (6)$$

The completeness of the system $\{p_k\}$ is equivalent to the property that for all $u \in L_w^2(a, b)$,

$$\|u - p_N u\| \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (7)$$

General Properties of Orthogonal Polynomials

All classical orthogonal polynomials have a number of properties in common. For instance,

- (i) $p_n(x)$ satisfies a second-order ordinary differential equation of the form

$$A(x) \frac{d^2 y}{dx^2} + B(x) \frac{dy}{dx} + \lambda_n y = 0 \quad (8)$$

where $A(x)$ and $B(x)$ are independent of n and λ_n is a constant depending only on n .

- (ii) $p_n(x)$ has the following generalized Rodrigue's formula

$$p_n(x) = \frac{1}{k_n w(x)} \frac{d^n}{dx^n} [w(x) \{g(x)\}^n], \quad (9)$$

where k_n is a constant and $g(x)$ is a polynomial in x independent of n .

- (iii) Any three consecutive orthogonal polynomials satisfy a recurrence formula of the form

$$p_{n+1}(x) = (A_n x + B_n) p_n(x) - C_n p_{n-1}(x), \quad (10)$$

$$n = 1, 2, \dots,$$

where A_n, B_n , and C_n are constants or functions of n only, $A_n > 0$, $C_n > 0$.

For more details and more properties of orthogonal polynomials, see Abramowitz and Stegun (1972), Andrews et al. (1999), and Luke (1969).

Classical Jacobi Polynomials

The classical Jacobi polynomials, associated with the real parameters $(\alpha > -1, \beta > -1)$ (see, Abramowitz and Stegun 1972; Andrews et al. 1999), are a sequence of polynomials $P_n^{(\alpha, \beta)}(x)$ ($n = 0, 1, 2, \dots$), each, respectively, of degree n . It is more convenient to introduce the normalized orthogonal polynomials, $R_n^{(\alpha, \beta)}(x) = \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)}$. This means that

$$R_n^{(\alpha, \beta)}(x) = \frac{n! \Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)} P_n^{(\alpha, \beta)}(x). \quad (11)$$

Now, it can be easily shown that the polynomials $R_n^{(\alpha, \beta)}(x)$ may be generated using the recurrence relation

$$\begin{aligned} 2(n + \lambda)(n + \alpha + 1)(2n + \lambda - 1) R_{n+1}^{(\alpha, \beta)}(x) \\ = (2n + \lambda - 1)_3 x R_n^{(\alpha, \beta)}(x) \\ + (\alpha^2 - \beta^2)(2n + \lambda) R_n^{(\alpha, \beta)}(x) \\ - 2n(n + \beta)(2n + \lambda + 1) R_{n-1}^{(\alpha, \beta)}(x), \\ n = 1, 2, \dots, \end{aligned}$$

starting from $R_0^{(\alpha, \beta)}(x) = 1$ and $R_1^{(\alpha, \beta)}(x) = \frac{1}{2(\alpha + 1)} [\alpha - \beta + (\lambda + 1)x]$, or obtained from Rodrigue's formula

$$\begin{aligned} R_n^{(\alpha, \beta)}(x) &= \left(\frac{-1}{2}\right)^n \frac{\Gamma(\alpha + 1)}{\Gamma(n + \alpha + 1)} \\ &\quad (1 - x)^{-\alpha} (1 + x)^{-\beta} D^n \\ &\quad \left[(1 - x)^{\alpha + n} (1 + x)^{\beta + n} \right], \end{aligned}$$

with $\lambda = \alpha + \beta + 1$, $(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)}$, $D = \frac{d}{dx}$, and are satisfying the orthogonality relation

$$\begin{aligned} \int_{-1}^1 (1 - x)^\alpha (1 + x)^\beta R_m^{(\alpha, \beta)}(x) R_n^{(\alpha, \beta)}(x) dx \\ = \begin{cases} 0, & m \neq n, \\ h_n^{\alpha, \beta}, & m = n, \end{cases} \quad (12) \end{aligned}$$

where

$$h_n^{\alpha, \beta} = \frac{2^\lambda n! \Gamma(n + \beta + 1) [\Gamma(\alpha + 1)]^2}{(2n + \lambda) \Gamma(n + \lambda) \Gamma(n + \alpha + 1)}. \quad (13)$$

These polynomials are eigenfunctions of the following singular Sturm-Liouville equation:

$$(1-x^2)\phi''(x) + [\beta - \alpha - (\lambda + 1)x]\phi'(x) + n(n + \lambda)\phi(x) = 0. \quad (14)$$

Of these polynomials, the most commonly used are the ultraspherical polynomials $C_n^{(\alpha)}(x)$, the Chebyshev polynomials of the first, second, third, and fourth kinds $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$, and the Legendre polynomials $L_n(x)$. These orthogonal polynomials are interrelated to the Jacobi polynomials by the following relations:

$$\begin{aligned} C_n^{(\alpha)}(x) &= R_n^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(x), \quad T_n(x) = R_n^{(-\frac{1}{2}, -\frac{1}{2})}(x), \\ U_n(x) &= (n+1)R_n^{(\frac{1}{2}, \frac{1}{2})}(x), \quad V_n(x) = R_n^{(-\frac{1}{2}, \frac{1}{2})}(x), \\ W_n(x) &= (2n+1)R_n^{(\frac{1}{2}, -\frac{1}{2})}(x), \quad L_n(x) = R_n^{(0,0)}(x). \end{aligned}$$

Ultraspherical Polynomials

The ultraspherical polynomials are a special class of Jacobi polynomials associated with the real parameter $(\lambda > -\frac{1}{2})$. They are orthogonal on the interval $[-1, 1]$, with respect to the weight function $w(x) = (1-x^2)^{\lambda-\frac{1}{2}}$, i.e.,

$$\begin{aligned} \int_{-1}^1 (1-x^2)^{\lambda-\frac{1}{2}} C_m^{(\lambda)}(x) C_n^{(\lambda)}(x) dx \\ = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases} \end{aligned} \quad (15)$$

where

$$h_n = \frac{\sqrt{\pi} n! \Gamma(\lambda + \frac{1}{2})}{(2\lambda)_n (n + \lambda) \Gamma(\lambda)}, \quad (2\lambda)_n = \frac{\Gamma(n + 2\lambda)}{\Gamma(2\lambda)}.$$

We weigh the ultraspherical polynomials so that $C_n^{(\lambda)}(1) = 1$. This is not the usual standardization but has advantage that the Legendre polynomials $L_n(x)$ and the Chebyshev polynomials of the first and second kinds $T_n(x)$ and

$U_n(x)$ can be obtained as three direct special cases. Explicitly

$$\begin{aligned} L_n(x) &= C_n^{(\frac{1}{2})}(x), \quad T_n(x) = C_n^{(0)}(x), \quad U_n(x) \\ &= (n+1)C_n^{(1)}(x). \end{aligned}$$

The following properties of ultraspherical polynomials (see Andrews et al. 1999) are of fundamental importance in the sequel. They are eigenfunctions of the following singular Sturm-Liouville equation

$$(1-x^2)D^2\phi_k(x) - (2\lambda+1)x D\phi_k(x) + k(k+2\lambda)\phi_k(x) = 0,$$

and may be generated by using the recurrence relation

$$\begin{aligned} (k+2\lambda)C_{k+1}^{(\lambda)}(x) &= 2(k+\lambda)x C_k^{(\lambda)}(x) \\ &\quad - k C_{k-1}^{(\lambda)}(x), \\ k &= 1, 2, 3, \dots, \end{aligned}$$

starting from $C_0^{(\lambda)}(x) = 1$ and $C_1^{(\lambda)}(x) = x$, or obtained from the Rodrigue's formula

$$\begin{aligned} C_n^{(\lambda)}(x) &= \left(-\frac{1}{2}\right)^n \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(n + \lambda + \frac{1}{2})} \\ &\quad (1-x^2)^{\frac{1}{2}-\lambda} \frac{d^n}{dx^n} \left[(1-x^2)^{n+\lambda-\frac{1}{2}} \right]. \end{aligned}$$

Now, the derivatives of the ultraspherical polynomials are given in the following theorem.

Theorem 1. *For all $q \geq 1$, the q th derivative of the ultraspherical polynomial $C_k^{(\lambda)}(x)$ is given explicitly by*

$$\begin{aligned} D^q C_k^{(\lambda)}(x) &= \frac{2^q k!}{(q-1)! \Gamma(k+2\lambda)} \times \\ &\quad \sum_{\substack{m=0 \\ (k+m-q) \text{ even}}}^{k-q} \frac{(m+\lambda) \Gamma(m+2\lambda) \left(\frac{k-m+q-2}{2}\right)! \Gamma\left(\frac{k+m+q+2\lambda}{2}\right)}{m! \left(\frac{k-q-m}{2}\right)! \Gamma\left(\frac{k+m-q+2\lambda+2}{2}\right)} C_m^{(\lambda)}(x). \end{aligned} \quad (16)$$

For a proof of Theorem 1, see Doha (1991).

The following integral formula (see Andrews et al. 1999) is useful.

$$\int C_n^{(\alpha)}(x)w(x)dx = -\frac{2\alpha(1-x^2)^{\alpha+\frac{1}{2}}}{n(n+2\alpha)}C_{n-1}^{(\alpha+1)}(x), \quad n \geq 1. \quad (17)$$

Also, the following theorem is very important.

Theorem 2. (Giordano and Laforgia 2003)(Bernstein-type inequality). The following inequality holds for ultraspherical polynomials:

$$(\sin \theta)^\alpha |C_n^{(\alpha)}(\cos \theta)| < \frac{2^{1-\alpha} \Gamma(n + \frac{3\alpha}{2})}{\Gamma(\alpha) \Gamma(n + 1 + \frac{\alpha}{2})}, \quad 0 \leq \theta \leq \pi, \quad 0 < \alpha < 1. \quad (18)$$

The shifted ultraspherical polynomials are defined on $[0, 1]$ by $\tilde{C}_n^{(\lambda)}(x) = C_n^{(\lambda)}(2x - 1)$. All results of ultraspherical polynomials can be easily transformed to give the corresponding results for their shifted ones. The orthogonality relation for $\tilde{C}_n^{(\lambda)}(x)$ with respect to the weight function $(x - x^2)^{\lambda-\frac{1}{2}}$ is given by

$$\int_0^1 (x - x^2)^{\lambda-\frac{1}{2}} \tilde{C}_n^{(\lambda)}(x) \tilde{C}_m^{(\lambda)}(x) dx = \begin{cases} 0, & m \neq n, \\ \frac{4^{-\lambda} \sqrt{\pi} n! \Gamma(\lambda + \frac{1}{2})}{(2\lambda)_n (n + \lambda) \Gamma(\lambda)}, & m = n. \end{cases}$$

First Kind Chebyshev Polynomials

It is well known that the first kind Chebyshev polynomials are defined on $[-1, 1]$ by

$$T_n(x) = \cos n\theta, \quad x = \cos \theta.$$

They are orthogonal on $[-1, 1]$, i.e.,

$$\int_{-1}^1 \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & m \neq n, \\ \pi, & m = n \neq 0, \\ \pi, & m = n = 0. \end{cases} \quad (19)$$

They may be generated by the recurrence relation

$$T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x), \quad i = 1, 2, 3, \dots,$$

starting from $T_0(x) = 1$ and $T_1(x) = x$.

The shifted first kind Chebyshev polynomials $\{T_i^*(x), \quad i = 0, 1, 2, \dots\}$ are a sequence of orthogonal polynomials on $[0, \ell]$ that may be determined with the aid of the following recurrence formula:

$$\begin{aligned} T_{i+1}^*(x) &= 2\left(\frac{2x}{\ell} - 1\right) T_i^*(x) - T_{i-1}^*(x), \\ T_0^*(x) &= 1, \quad T_1^*(x) = \frac{2x}{\ell} - 1. \end{aligned} \quad (20)$$

$T_i^*(x)$ form a complete orthogonal system for $L^2_{\omega(x)}[0, \ell]$, where $\omega(x) = \frac{1}{\sqrt{x(\ell-x)}}$. The orthogonality relation satisfied by $T_i^*(x)$ is

$$\int_0^\ell T_i^*(x)T_j^*(x)\omega(x)dx = \frac{\pi}{2\theta_i} \delta_{ij}, \quad (21)$$

where

$$\theta_i = \begin{cases} \frac{1}{2}, & \text{if } i = 0, \\ 1, & \text{if } i > 0, \end{cases} \quad (22)$$

and

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \quad (23)$$

Moreover, the power form of $T_i^*(x)$ can be expressed as

$$T_i^*(x) = i \sum_{s=0}^i \frac{(-1)^{i-s} 2^{2s} (i+s-1)!}{\ell^s (i-s)! (2s)!} x^s, \quad i > 0. \quad (24)$$

Corollary 1. For every positive integer q , the q th derivative of $T_i^*(x)$ can be expressed in terms of their original polynomials as:

$$D^q T_i^*(x) = \sum_{p=0}^{j-q} B_{j,p,q} T_p^*(x), \quad (25)$$

($j+p+q$) even

where

$$B_{j,p,q} = \frac{j^{2^{2q}} \theta_p(q) \frac{1}{2} (j-p-q)}{\ell^q \left(\frac{1}{2} (j-p-q)\right)! \left(\frac{1}{2} (j+p+q)\right)_{1-q}}, \quad (26)$$

where θ_p defined in Eq. (22).

Proof. The proof of this theorem can be found in Abd-Elhameed et al. (2021).

Second Kind Chebyshev Polynomials

It is well known that the second kind Chebyshev polynomials are defined on $[-1, 1]$ by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta.$$

They are orthogonal on $[-1, 1]$, i.e.,

$$\int_{-1}^1 \sqrt{1-x^2} U_m(x) U_n(x) dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{2}, & m = n. \end{cases} \quad (27)$$

The following properties of second kind Chebyshev polynomials (see Mason and Handscomb 2002) are of fundamental importance in the sequel. They are eigenfunctions of the following singular Sturm-Liouville equation

$$(1-x^2) D^2 \phi_k(x) - 3x D \phi_k(x) + k(k+2) \phi_k(x) = 0,$$

where $D \equiv \frac{d}{dx}$, and may be generated by using the recurrence relation

$$U_{k+1}(x) = 2x U_k(x) - U_{k-1}(x), \\ k = 1, 2, 3, \dots,$$

starting from $U_0(x) = 1$ and $U_1(x) = 2x$, or from Rodrigues formula

$$U_n(x) = \frac{(-2)^n (n+1)!}{(2n+1)! \sqrt{1-x^2}} D^n \left[(1-x^2)^{n+\frac{1}{2}} \right].$$

The following theorem is needed hereafter.

Theorem 3. *The first derivative of second kind Chebyshev polynomials is given by*

$$D U_n(x) = 2 \sum_{\substack{k=0 \\ (k+n) \text{ odd}}}^{n-1} (k+1) U_k(x). \quad (28)$$

For a proof of Theorem 1.1.3, see Mason and Handscomb (2002).

The shifted second kind Chebyshev polynomials are defined on $[0, 1]$ by $U_n^*(x) = U_n(2x-1)$. All results of second kind Chebyshev polynomials can be easily transformed to give the corresponding results for their shifted forms. The orthogonality relation with respect to the weight function $\sqrt{x-x^2}$ is given by

$$\int_0^1 \sqrt{x-x^2} U_n^*(x) U_m^*(x) dx = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{8}, & m = n. \end{cases}$$

The first derivative of $U_n^*(x)$ is given in the following corollary.

Corollary 2. *The first derivative of the shifted second kind Chebyshev polynomials is given by*

$$D U_n^*(x) = 4 \sum_{\substack{k=0 \\ (k+n) \text{ odd}}}^{n-1} (k+1) U_k^*(x), \\ n = 0, 1, 2, \dots \quad (29)$$

Third and Fourth Kinds Chebyshev Polynomials

The Chebyshev polynomials $V_k(x)$ and $W_k(x)$ of third and fourth kinds are polynomials of degree k in x defined, respectively, by (see Mason and Handscomb 2002)

$$V_k(x) = \frac{\cos(k+\frac{1}{2})\theta}{\cos\frac{\theta}{2}} \quad \text{and} \quad W_k(x) = \frac{\sin(k+\frac{1}{2})\theta}{\sin\frac{\theta}{2}},$$

where $x = \cos\theta$; also they can be obtained explicitly as two particular cases of Jacobi polynomials

$P_k^{(\alpha, \beta)}(x)$, for the two nonsymmetric cases correspond to $\beta = -\alpha = \pm \frac{1}{2}$. Explicitly, we have

$$V_k(x) = \frac{(2^k k!)^2}{(2k)!} P_k^{(-\frac{1}{2}, -\frac{1}{2})}(x) \text{ and}$$

$$W_k(x) = \frac{(2^k k!)^2}{(2k)!} P_k^{(\frac{1}{2}, -\frac{1}{2})}(x).$$

It is readily seen that

$$W_k(x) = (-1)^k V_k(-x),$$

hence it is sufficient to establish properties and relations for $V_n(x)$, and then deduce their corresponding properties and relations for $W_n(x)$ (replacing x by $-x$).

The polynomials $V_n(x)$ and $W_n(x)$ are orthogonal on $[-1, 1]$, i.e.,

$$\int_{-1}^1 \omega_1(x) V_k(x) V_j(x) dx = \int_{-1}^1 \omega_2(x) W_k(x) W_j(x) dx$$

$$= \begin{cases} \pi, & k = j, \\ 0, & k \neq j, \end{cases}$$

where

$$\omega_1(x) = \sqrt{\frac{1+x}{1-x}} \text{ and } \omega_2(x) = \sqrt{\frac{1-x}{1+x}}$$

and they may be generated by using the two recurrence relations

$$V_k(x) = 2x V_{k-1}(x) - V_{k-2}(x), \quad k = 2, 3, \dots,$$

with the initial values

$$V_0(x) = 1, \quad V_1(x) = 2x - 1,$$

and

$$W_k(x) = 2x W_{k-1}(x) - W_{k-2}(x),$$

$$k = 2, 3, \dots,$$

with the initial values

$$W_0(x) = 1, \quad W_1(x) = 2x + 1.$$

The shifted Chebyshev polynomials of third and fourth kinds are defined on $[0, 1]$, respectively, by

$$V_n^*(x) = V_n(2x - 1), \quad W_n^*(x) = W_n(2x - 1).$$

All results of Chebyshev polynomials of third and fourth kinds can be easily transformed to give the corresponding results for their shifted ones.

The orthogonality relations of $V_n^*(t)$ and $W_n^*(t)$ on $[0, 1]$ are given by

$$\int_0^1 w_1^* V_m^*(t) V_n^*(t) dt = \int_0^1 w_2^* W_m^*(t) W_n^*(t) dt$$

$$= \begin{cases} \frac{\pi}{2}, & m = n, \\ 0, & m \neq n, \end{cases}$$

where

$$\omega_1^* = \sqrt{\frac{t}{1-t}} \text{ and } \omega_2^* = \sqrt{\frac{1-t}{t}}. \quad (30)$$

For some applications of Chebyshev polynomials on different problems, see Pourbabaei and Saadatmandi (2021), Cao et al. (2021), Abdelhakem et al. (2021), and Ahmadian et al. (2016).

Spectral Methods

Herein, a brief account on the three most commonly used versions of spectral methods, namely, Galerkin, tau, and collocation methods, for solving boundary value problems is given. As an example, consider the following second-order boundary value problem:

$$(\mathcal{L}u)(x) := -\epsilon u'' + p(x)u + q(x)u' = f(x),$$

$$x \in I = (-1, 1), \quad (31)$$

subject to the general boundary conditions

$$B_-(u) := a_- u(-1) + b_- u'(-1) = c_-, \quad (32)$$

$$B_+(u) := a_+ u(1) + b_+ u'(1) = c_+,$$

where $\epsilon > 0$.

The boundary conditions (31) include in particular the Dirichlet boundary conditions ($a_{\pm} = 1, b_{\pm} = 0$), Neumann boundary conditions ($a_{\pm} = 0, b_{\pm} = \pm 1$), and Robin (or mixed) boundary conditions ($a_{-} = b_{+} = 0$ or $a_{+} = b_{-} = 0$). We shall give a brief treatment for some of these boundary conditions. Without loss of generality, we assume that:

$$a_{\pm} \geq 0;$$

$$a_{-}^2 + b_{-}^2 \neq 0, \quad a_{-} b_{-} \leq 0;$$

$$a_{+}^2 + b_{+}^2 \neq 0, \quad a_{+} b_{+} \geq 0;$$

$$q(x) - \frac{p'(x)}{2} \geq 0, \quad \forall x \in I;$$

$$p(1) > 0 \text{ if } b_{+} \neq 0; \quad p(-1) < 0 \text{ if } b_{-} \neq 0.$$

The above conditions are necessary for the well-posedness of Eqs. (31) and (32). Suppose that $\{\phi_k(x); k = 0, 1, \dots\}$ is a sequence of orthogonal polynomials with respect to the weight function w in $L_w^2(I)$, and

$$u_N = \sum_{k=0}^N c_k \phi_k(x), \quad (33)$$

is the approximate solution of Eqs. (31) and (32). Define the residual of Eq. (31) by

$$R(x) = (\mathcal{L}u_N)(x) - f(x), \quad (34)$$

and consider the maximum absolute error

$$e = \max_{-1 \leq x \leq 1} |u(x) - u_N(x)|. \quad (35)$$

Now, the three versions of spectral methods to treat (Eq. 31) subject to the various cases of the boundary conditions (32) will be discussed.

Galerkin Method

Galerkin method is suitable for handling (Eq. 31) subject to the homogenous Dirichlet boundary conditions

$$u(-1) = u(1) = 0. \quad (36)$$

The key idea of Galerkin method is to choose a basis function $\phi_k(x)$ to satisfy the boundary conditions (36), namely

$$\phi_k(-1) = \phi_k(1) = 0, \quad k = 0, 1, \dots, N.$$

The weak formulation of (31) is given by

$$(R(x), \phi_k(x))_w = 0, \quad k = 0, 1, \dots, N. \quad (37)$$

Equation (37) generates $(N + 1)$ linear system of equations in the $(N + 1)$ unknowns c_k , which can be solved by any suitable solver. For example, the authors in Atta et al. (2021) used the Galerkin method to obtain an approximate solution of one-dimensional linear hyperbolic partial differential equations.

Tau Method

In case of problems with complicated boundary conditions, the application of tau method is more appropriate than Galerkin method, since in tau method the basis $\phi_k(x)$ not necessarily satisfies the boundary conditions (32). More precisely, we solve the following linear system of equations

$$(R(x), \phi_k(x))_w = 0, \quad k = 0, 1, \dots, N - 2, \quad (38)$$

$$B_{-}(u_N) = c_{-}, \quad B_{+}(u_N) = c_{+}. \quad (39)$$

For example, the authors in Atta et al. (2020) used the tau method to solve the Bagley-Torvik fractional differential equation.

Collocation Method

Collocation method is usually used for nonlinear equations but may be used for Eqs. (31) and (32) as follows. Solve

$$R(x_i) = 0, \quad i = 1, 2, \dots, N - 1, \quad (40)$$

$$B_{-}(u_N) = c_{-}, \quad B_{+}(u_N) = c_{+}, \quad (41)$$

where $\{x_i; i = 1, \dots, N - 1\}$ are the first $(N - 1)$ distinct roots of $\phi_{N+1}(x)$ called collocation points.

Equations (40) and (41) generate a system of $(N + 1)$ algebraic system of equations in the unknown expansion coefficients which can be solved with the aid of any suitable solver, and hence an approximate semianalytical solution is obtained. For example, the authors in Atta et al. (2019) used the collocation method to solve multi-term fractional differential equations. For more studies about these methods, see Doha et al. (2019), Youssri and Abd-Elhameed (2018), and Türk and Codina (2019).

In this entry, our attention is devoted to the investigation of the third-order nonlinear KdV equation. The KdV equation is a generic equation for the study of weakly nonlinear long waves. The KdV type of equations have been an important class of nonlinear evolution equations with numerous applications in physical sciences and engineering fields. For example, in plasma physics, these equations give rise to the ion acoustic solitons (Das and Sarma 1999). In geophysical fluid dynamics, they describe a long wave in shallow seas and deep oceans (Osborne 1995). Their strong presence is exhibited in cluster physics, superdeformed nuclei, fission, thin films, radar and rheology (Ludu and Draayer 1998), and superconductors (Coffey 1996). However, the physical situations in which the KdV equations arise tend to be highly idealized due to the assumption of constant coefficients. Hence, in recent years, much attention has been paid to the various forms of KdV class of equations with variable coefficients (Singh and Gupta 2006). Many explicit exact methods have been introduced in the literature. Several authors mainly had focused on studying the solutions of nonlinear equations by using various methods, such as the generalized Kudryashov method (Pandir and Sahragül 2021), the extended complex method (Rehman et al. 2021), and the Lie symmetry method (El Bahi and Hilal 2021). On the other hand, a great number of articles were devoted to the numerical treatment of the KdV equation, as in general analytical solutions are not available. For example, in Abrahamsen and Fornberg (2021), Abrahamsen and Fornberg introduced a numerical algorithm using Finite difference and pseudospectral methods to solve KdV equation; in Dehghan and Shokri (2007), the authors solved the KdV equation

using a combination of finite difference and sinc collocation method; and in Başhan (2021), the author presented B-spline differential quadrature method to solve KdV equation.

In accordance with the aforementioned aspects, the main aims of this entry can be summarized in the following threefold:

- Presenting a new technique for solving the KdV equation of the third-order via the shifted Chebyshev polynomials of the first-kind by applying the spectral tau method
- Investigating the convergence and error analysis of the proposed Chebyshev expansion
- Reducing the solution of the equation with its initial and boundary conditions into a system of algebraic equations, then solving it using Newton's procedure

Case Study

Consider the following KdV equation

$$\frac{\partial u(x, t)}{\partial t} + \epsilon u(x, t) \frac{\partial u(x, t)}{\partial x} + \mu \frac{\partial^3 u(x, t)}{\partial x^3} = 0, \\ x \in (0, \ell), \quad t \in (0, \tau], \quad (42)$$

equipped with initial and boundary conditions

$$u(x, 0) = u_0(x), \quad x \in (0, \ell), \\ u(0, t) = u_1(t), \quad u(\ell, t) = u_2(t), \\ u_x(\ell, t) = u_3(t), \quad t \in (0, \tau], \quad (43)$$

where ϵ and μ are positive parameters and $u_0(x)$, $u_1(t)$, $u_2(t)$, and $u_3(t)$ are known functions. Now, consider the following space

$$\mathcal{L}_{\overline{\omega}}^2(\Omega) = \text{span} \left\{ T_i^*(x) T_j^*(t) : i, j = 0, 1, 2, \dots \right\}, \quad (44)$$

where

$$\Omega = (0, \ell] \times (0, \tau], \quad \overline{\omega} = \omega \widehat{\omega}, \\ \omega = \frac{1}{\sqrt{x(\ell - x)}}, \quad \widehat{\omega} = \frac{1}{\sqrt{t(\tau - t)}}.$$

Then the approximate solution $u_N(x, t) \in L^2_{\overline{\omega}}(\Omega)$, of Eq. (42), can be written as

$$u_N(x, t) = \sum_{i=0}^N \sum_{j=0}^N c_{ij} T_i^*(x) T_j^*(t), \quad (45)$$

where

$$c_{ij} = \frac{4\theta_i \theta_j}{\pi^2} \int_0^\tau \int_0^\ell u_N(x, t) T_i^*(x) T_j^*(t) \overline{\omega} dx dt, \quad (46)$$

and θ_i is defined in Eq. (22).

Based on Eq. (45) and Corollary 1, one can write the residual of Eq. (42) in the form

$$\begin{aligned} R_N(x, t) = & \sum_{i=0}^N \sum_{j=0}^N \sum_{p=0}^{j-1} c_{ij} B_{j,p,1} T_i^*(x) T_p^*(t) + \mu \sum_{i=0}^N \sum_{j=0}^N \sum_{p=0}^{i-3} c_{ij} B_{i,p,3} T_p^*(x) T_j^*(t) \\ & + \epsilon \sum_{i=0}^N \sum_{j=0}^N \sum_{r=0}^N \sum_{s=0}^N \sum_{p=0}^{r-1} c_{ij} c_{rs} B_{r,p,1} T_i^*(x) T_j^*(t) T_p^*(x) T_s^*(t). \end{aligned} \quad (47)$$

(j+p+1) even (i+p+3) even (r+p+1) even

The previous equation can be written with the aid of the following property

$$T_i^*(x) T_j^*(x) = \frac{1}{2} \left(T_{i+j}^*(x) + T_{|i-j|}^*(x) \right),$$

as:

$$\begin{aligned} R_N(x, t) = & \sum_{i=0}^N \sum_{j=0}^N \sum_{p=0}^{j-1} c_{ij} B_{j,p,1} T_i^*(x) T_p^*(t) + \mu \sum_{i=0}^N \sum_{j=0}^N \sum_{p=0}^{i-3} c_{ij} B_{j,p,3} T_p^*(x) T_j^*(t) + \frac{\epsilon}{4} \sum_{i=0}^N \sum_{j=0}^N \\ & \sum_{r=0}^N \sum_{s=0}^N \sum_{p=0}^{r-1} c_{ij} c_{rs} B_{r,p,1} \left[T_{i+p}^*(x) T_{j+s}^*(t) + T_{i+p}^*(x) T_{|j-s|}^*(t) + T_{|i-p|}^*(x) T_{j+s}^*(t) + T_{|i-p|}^*(x) T_{|j-s|}^*(t) \right]. \end{aligned} \quad (48)$$

(j+p+1) even (r+p+1) even

The application of tau method leads to

which consequently yields

$$\begin{aligned} ((R_N(x, t), T_m^*(x) T_n^*(t)))_{\overline{\omega}} &= 0, \\ m, n : 0, 1, 2, \dots, N-1, \end{aligned} \quad (49)$$

$$\begin{aligned}
& \sum_{j=0}^N \sum_{\substack{p=0 \\ (j+p+1) \text{ even}}}^{j-1} c_{mj} \frac{B_{jp,1}}{\theta_m \theta_p} \delta_{n,p} + \mu \sum_{i=0}^N \sum_{\substack{p=0 \\ (i+p+3) \text{ even}}}^{i-3} c_{in} \frac{B_{jp,3}}{\theta_p \theta_n} \delta_{m,p} + \frac{\epsilon}{4} \sum_{i=0}^N \sum_{j=0}^N \sum_{r=0}^N \sum_{s=0}^N \sum_{\substack{p=0 \\ (r+p+1) \text{ even}}}^{r-1} \\
& c_{ij} c_{rs} B_{r,p,1} \left[\frac{\delta_{i+p,m} \delta_{j+s,n}}{\theta_{i+p} \theta_{j+s}} + \frac{\delta_{i+p,m} \delta_{|j-s|,n}}{\theta_{i+p} \theta_{|j-s|}} + \frac{\delta_{|i-p|,m} \delta_{j+s,n}}{\theta_{|i-p|} \theta_{j+s}} + \frac{\delta_{|i-p|,m} \delta_{|j-s|,n}}{\theta_{|i-p|} \theta_{|j-s|}} \right] = 0.
\end{aligned} \quad (50)$$

The initial and boundary conditions lead to

$$\begin{aligned}
& \sum_{j=0}^N c_{mj} \frac{\pi(-1)^j}{2\theta_m} = (u_0(x), T_m^*(x))_{\omega}, \quad m : 0, 1, \dots, \left\lfloor \frac{N-2}{2} \right\rfloor, \\
& \sum_{i=0}^N c_{in} \frac{\pi(-1)^i}{2\theta_n} = (u_1(t), T_n^*(t))_{\widehat{\omega}}, \quad n : \left\lfloor \frac{N-2}{2} \right\rfloor + 1, \dots, N, \\
& \sum_{i=0}^N c_{in} \frac{\pi}{2\theta_n} = (u_2(t), T_n^*(t))_{\widehat{\omega}}, \quad n : 0, 1, \dots, \left\lfloor \frac{N-2}{2} \right\rfloor, \\
& \sum_{i=0}^N \sum_{\substack{p=0 \\ (i+p+1) \text{ even}}}^{i-1} c_{in} \frac{\pi B_{ip,1}}{2\theta_n} = (u_3(t), T_n^*(t))_{\widehat{\omega}}, \quad n : \left\lfloor \frac{N-2}{2} \right\rfloor + 1, \dots, N-1.
\end{aligned} \quad (51)$$

Joining the N^2 equations in Eq. (50) with the $(2N+1)$ constraints (51), we get a system of nonlinear algebraic equations of dimension $(N+1)^2$, which may be solved via Newton's iterative technique, and accordingly, the desired approximate solution can be obtained.

Remark 1.

$$\begin{aligned}
((\phi, \psi))_{\widehat{\omega}} &= \int_0^\tau \int_0^\ell \phi \psi \overline{\omega} dx dt \text{ and } (u, v)_{\widehat{\omega}} \\
&= \int_0^\tau u v \widehat{\omega} dt.
\end{aligned}$$

Theorem 5 shows that the given expansion of $u(x, t)$ with a bounded q derivative for some $q > 7$ converges uniformly to $u(x, t)$. Theorem 6 gives an upper bound for the truncation error due to truncating the suggested expansion. Finally, Theorem 7 proves that the global error $|E_N(x, t)| \rightarrow 0$ as $N \rightarrow \infty$.

Lemma 1. (Abumaryam 2018) *The following inequality holds for the first kind Chebyshev polynomials*

Error Estimate

This section is devoted to error analysis of the proposed double expansion. In this respect, three important theorems are stated and proved.

$$\begin{aligned}
|T_n^{(r)}(x)| &\leq \frac{n^2(n^2-1^2)\dots(n^2-(r-1)^2)}{(2r-1)}, \\
&\forall r \geq 1.
\end{aligned} \quad (52)$$

Corollary 3. For all $r \geq 1$, one has

$$|T_n^{*(r)}(x)| \lesssim n^{2r}, \quad (53)$$

where $A \lesssim B$ means that there exists a generic constant C such that $A \leq CB$.

Theorem 4. (Abd-Elhameed et al. 2021) If $u(t) \in C^q(0, \tau)$, such that $u^{(q)}(t)$ is bounded for some $q > 3$ and $u(t)$ is approximated as

$$u(t) \approx u_N(t) = \sum_{i=0}^N a_i T_i^*(t),$$

then the following bound on the expansion coefficients holds

$$|a_i| \lesssim \frac{1}{i^q}, \quad \forall i > 1. \quad (54)$$

Theorem 5. Consider the function $u(x, t) = f_1(x) f_2(t) \in C^q(\Omega)$, where $\Omega = (0, \ell) \times (0, \tau)$, such that $f_1^{(q)}(x)$ and $f_2^{(q)}(t)$ are bounded for some $q > 7$ and $u(x, t)$ is expanded as

$$u(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} T_i^*(x) T_j^*(t). \quad (55)$$

The above series is uniformly convergent to $u(x, t)$. Moreover, the expansion coefficients in Eq. (55) satisfy:

$$|c_{ij}| \lesssim \frac{1}{i^q j^q}, \quad \forall i, j > 1. \quad (56)$$

Proof. From Eq. (46), the expansion coefficients can be written in the following integral form

$$c_{ij} = \frac{4\theta_i \theta_j}{\pi^2} \int_0^\tau \int_0^\ell u(x, t) T_i^*(x) T_j^*(t) \overline{\omega} dx dt. \quad (57)$$

Making use of the identity $|T_i^*(x)| \leq 1$, assumption of $u(x, t)$ and using similar steps as Theorem 4 in Abd-Elhameed et al. (2021), the desired result may be obtained.

Theorem 6. The following truncation error estimate is valid

$$|u(x, t) - u_N(x, t)| \lesssim \frac{1}{N^{q-1}}. \quad (58)$$

Proof. From relations (45) and (55) yields

$$\begin{aligned} |u(x, t) - u_N(x, t)| &\leq \left| \sum_{i=2}^N \sum_{j=N+1}^{\infty} c_{ij} T_i^*(x) T_j^*(t) \right| + \left| \sum_{i=N+1}^{\infty} \sum_{j=2}^{\infty} c_{ij} T_i^*(x) T_j^*(t) \right| \\ &+ \left| \sum_{j=N+1}^{\infty} [c_{0j} T_0^*(x) + c_{1j} T_1^*(x)] T_j^*(t) \right| + \left| \sum_{i=N+1}^{\infty} [c_{i0} T_0^*(t) + c_{i1} T_1^*(t)] T_i^*(x) \right|. \end{aligned} \quad (59)$$

Following similar steps of those given in Abd-Elhameed et al. (2021), one has

$$|c_{0j}| \lesssim \frac{1}{j^q}, \quad |c_{1j}| \lesssim \frac{1}{j^q} \quad (60)$$

and

$$|c_{i0}| \lesssim \frac{1}{i^q}, \quad |c_{i1}| \lesssim \frac{1}{i^q}. \quad (61)$$

Inserting Eqs. (60) and (61) into Eq. (59) and using Theorem 5 along with the identity $|T_i^*(x)| \leq 1$ lead to the estimation.

Lemma 2. Let $u(x, t)$ and $u_N(x, t)$ satisfy the assumptions of Theorem 5; one gets

$$\left| \frac{\partial u(x, t)}{\partial x} - \frac{\partial u_N(x, t)}{\partial x} \right| \lesssim \frac{1}{N^{q-3}}, \quad (62)$$

and

$$\left| \frac{\partial^3 u(x, t)}{\partial x^3} - \frac{\partial^3 u_N(x, t)}{\partial x^3} \right| \lesssim \frac{1}{N^{q-7}}. \quad (63)$$

Proof. The proofs of Eqs. (62) and (63) are similar. Now, the proof (62) may be given as follows: From definition of $u(x, t)$ and $u_N(x, t)$, one gets

$$\begin{aligned} \left| \frac{\partial u(x, t)}{\partial x} - \frac{\partial u_N(x, t)}{\partial x} \right| &\leq \left| \sum_{i=2}^N \sum_{j=N+1}^{\infty} c_{ij} \frac{dT_i^*(x)}{dx} T_j^*(t) \right| + \left| \sum_{i=N+1}^{\infty} \sum_{j=2}^{\infty} c_{ij} \frac{dT_i^*(x)}{dx} T_j^*(t) \right| \\ &+ \left| \sum_{j=N+1}^{\infty} \left[c_{0j} \frac{dT_0^*(x)}{dx} + c_{1j} \frac{dT_1^*(x)}{dx} \right] T_j^*(t) \right| + \left| \sum_{i=N+1}^{\infty} [c_{i0} T_0^*(t) + c_{i1} T_1^*(t)] \frac{dT_i^*(x)}{dx} \right|. \end{aligned} \quad (64)$$

Now, based on Eqs. (60), (61), and Theorem 5 along with the identity $|T_i^*(x)| \leq 1$ and Corollary 3, the desired estimation may be obtained.

Theorem 7. (Global Error) Assume that $u(x, t)$ is the exact solution of Eq. (42), $u_N(x, t)$ being the approximate solution, then $|E_N(x, t)| \rightarrow 0$ as $N \rightarrow \infty$.

Proof. The approximate solution $u_N(x, t)$ satisfies the equation

$$\begin{aligned} \frac{\partial u_N(x, t)}{\partial t} + \epsilon u_N(x, t) \frac{\partial u_N(x, t)}{\partial x} \\ + \mu \frac{\partial^3 u_N(x, t)}{\partial x^3} + E_N(x, t) = 0. \end{aligned} \quad (65)$$

Now, if we subtract Eq. (65) from Eq. (42), which yields

$$\begin{aligned} E_N(x, t) &= \frac{\partial(u(x, t) - u_N(x, t))}{\partial t} + \epsilon(u(x, t) - u_N(x, t)) \frac{\partial(u(x, t) - u_N(x, t))}{\partial x} \\ &+ \mu \frac{\partial^3(u(x, t) - u_N(x, t))}{\partial x^3}. \end{aligned} \quad (66)$$

On other words, Eq. (66) can be written as

$$\begin{aligned} |E_N(x, t)| &\leq \left| \frac{\partial(u(x, t) - u_N(x, t))}{\partial t} \right| + \epsilon |u(x, t) - u_N(x, t)| \left| \frac{\partial(u(x, t) - u_N(x, t))}{\partial x} \right| \\ &+ \mu \left| \frac{\partial^3(u(x, t) - u_N(x, t))}{\partial x^3} \right|. \end{aligned} \quad (67)$$

On the other hand, Lemma 2 and Theorem 6 lead us to deduce that $\left| \frac{\partial(u(x,t)-u_N(x,t))}{\partial x} \right| \rightarrow 0$, $\left| \frac{\partial^3(u(x,t)-u_N(x,t))}{\partial x^3} \right| \rightarrow 0$, and $|u(x,t) - u_N(x,t)| \rightarrow 0$ as $N \rightarrow \infty$. Therefore, it is clear that $|E_N(x,t)| \rightarrow 0$ as $N \rightarrow \infty$.

Applications

In order to show the convenience and validity of the above technique. The present technique will be illustrated throughout the following numerical example.

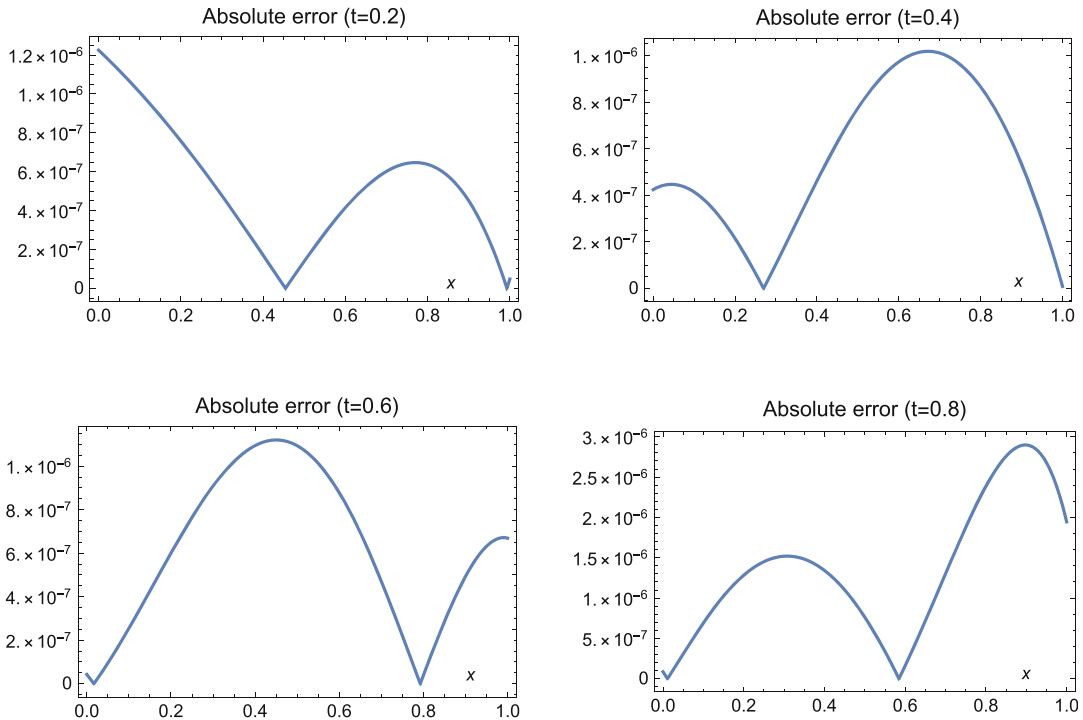
Example 1. Consider the third-order nonlinear KdV equation represented by Eq. (42), with $\epsilon = 1$ and $\mu = 4.84 \times 10^{-4}$. The analytical solution is given in Aksan and Özdeş (2006) and Kong et al. (2019) as $u(x,t) = 3c \operatorname{sech}^2(kx - wt - x_0)$,

where $k = \frac{1}{2} \sqrt{\frac{c\epsilon}{\mu}}$, $w = ck\epsilon$, $x_0 = 6$, and $c = \frac{3}{10}$.

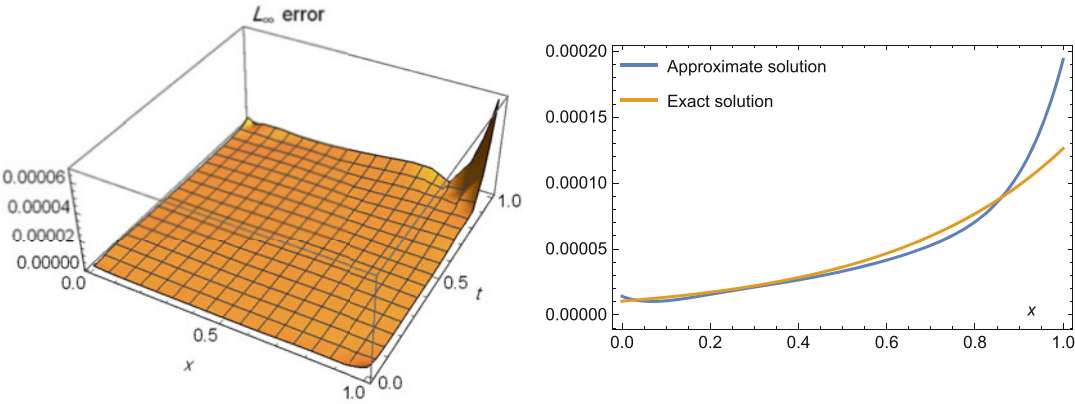
This solution, evaluated at $t = 0$, will be used as the initial condition, and the boundary functions are extracted from the exact solution. Figure 1 shows the absolute error graphs at different values of t when $N = 6$ at $\ell = \tau = 1$, while Fig. 2 illustrates the graphs of the L_∞ error and comparison of exact and approximate solutions at $t = 1$ when $N = 6$ and $\ell = \tau = 1$. For the sake of comparison, in Table 1, a comparison between approximate and exact solutions at different values of x when $N = 4$ at $\ell = 2$ and $\tau = 0.001$, 0.0005 is given. Finally, Fig. 3 sketches the graphs of the approximate solution and the L_∞ error when $N = 3$ at $\ell = 10$ and $\tau = 1$.

Concluding Remarks

This entry provides a discussion of one of the well-known equations named the KdV equation.



Double Tchebyshev Spectral Tau Algorithm for Solving KdV Equation, with Soliton Application, Fig. 1 The absolute error graphs of Example 1

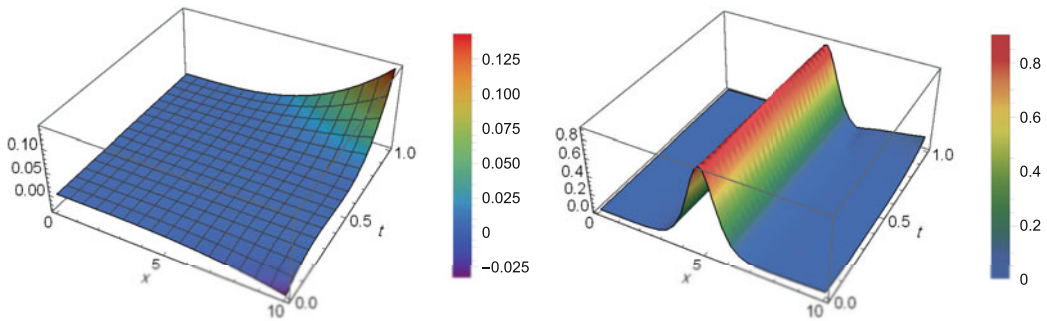


Double Tchebyshev Spectral Tau Algorithm for Solving KdV Equation, with Soliton Application, Fig. 2 The graphs of the L_∞ error (left side) and comparison of exact and approximate solutions at $t = 1$ (right side) for Example 1

Double Tchebyshev Spectral Tau Algorithm for Solving KdV Equation, with Soliton Application, Table 1 Comparison of approximate and exact solutions of Example 1

	$\tau = 0.001$			$\tau = 0.0005$		
x	Approximate	Exact	L_∞ error	Approximate	Exact	L_∞ error
0.1	0.00002541	0.0000281606	2.7471×10^{-6}	0.0000253	0.00002826	2.93743×10^{-6}
0.2	0.00003746	0.0000361213	1.3462×10^{-6}	0.0000374	0.00003625	1.22041×10^{-6}
0.3	0.00004743	0.0000463325	1.1041×10^{-6}	0.0000476	0.00004650	1.16184×10^{-6}
0.4	0.00005446	0.0000594301	4.9666×10^{-6}	0.0000549	0.00005965	4.72349×10^{-6}
0.5	0.00005941	0.0000762301	1.6812×10^{-5}	0.0000600	0.00007651	1.64306×10^{-5}
0.6	0.00006489	0.0000977790	3.2882×10^{-5}	0.0000657	0.00009814	3.23902×10^{-5}
0.7	0.00007522	0.0001254189	5.0195×10^{-5}	0.0000763	0.00012588	4.95338×10^{-5}
0.8	0.00009644	0.0001608712	6.4421×10^{-5}	0.0000980	0.00016147	6.33783×10^{-5}
0.9	0.00013635	0.0002063438	6.9991×10^{-5}	0.0001389	0.00020711	6.81317×10^{-5}
1	0.00020443	0.0002646680	6.0231×10^{-5}	0.0002088	0.0002656	5.68342×10^{-5}
1.1	0.00031193	0.0003394747	2.7540×10^{-5}	0.0003192	0.00034074	2.15303×10^{-5}
1.2	0.00047180	0.0004354200	3.6384×10^{-5}	0.0004835	0.00043704	4.65004×10^{-5}
1.3	0.00069873	0.0005584736	1.4025×10^{-4}	0.0007170	0.0005605	1.56456×10^{-4}
1.4	0.00100913	0.0007162896	2.9284×10^{-4}	0.0010366	0.00071896	3.17641×10^{-4}
1.5	0.00142114	0.0009186790	5.0246×10^{-4}	0.0014611	0.00092211	5.38989×10^{-4}
1.6	0.00195463	0.0011782168	7.7641×10^{-5}	0.0020110	0.00118262	8.28456×10^{-4}
1.7	0.00263119	0.0015110153	1.1201×10^{-3}	0.0027089	0.00151666	1.19224×10^{-3}
1.8	0.00347414	0.0019377145	1.5364×10^{-3}	0.0035787	0.00194495	1.63382×10^{-3}
1.9	0.00450853	0.0024847437	2.0237×10^{-3}	0.0046466	0.00249402	2.15258×10^{-3}

A numerical method is presented to solve the third-order nonlinear KdV equation using spectral tau method and approximating directly the solution using the shifted Chebyshev polynomials of the first-kind as basis functions. The convergence and error analysis were discussed in-depth. The numerical results given in the previous section demonstrate the good accuracy of this method.



Double Tchebyshev Spectral Tau Algorithm for Solving KdV Equation, with Soliton Application, Fig. 3 The graphs of the approximate solution (left side) and the L_∞ error (right side) for Example 1

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